1. Let T be the triangle in the xy-plane whose vertices are (1,2), (3,3), and (2,5). Find the volume of the solid object obtained by rotating T about the y-axis.

ANSWER: Taking slices perpendicular to the axis of rotation will give annuli ("washers") whose area at height y will be $A(y) = \pi R^2 - \pi r^2$ (the outer and inner radii varying with y), and then the volume of the whole ensemble will be $\int_{y=2}^{y=5} A(y)\,dy$.

The inner radius will, for each y, be the horizontal distance from the y-axis to the left edge of T, which is then the x-coordinate of the point on this edge. A glance at a sketch of T shows that this left edge is the line segment from $(1, 2)$ to $(2, 5)$, which has equation $y = 3x - 1$. So for each y the inner radius will be $r = x = (y + 1)/3$.

The outer radius is similarly the x-coordinate of the right-most point of T having a given height y; unfortunately this is a bit harder to describe. The right edge of T lies along the line $y = 9 - 2x$ that joins $(2, 5)$ to $(3, 3)$, but only for $y > 3$; for $y < 3$ the right edge of T follows the line $y = (x + 3)/2$ that joins the remaining pair of points. So our formula for the outer radius is $R = x =$

$$\begin{cases} 2y - 3 & \text{if } y < 3 \\ (9 - y)/2 & \text{if } y > 3 \end{cases}$$

We may now integrate:

$$V = \int A(y)\,dy = \pi \int_{y=2}^{y=3} (2y-3)^2\,dy + \pi \int_{y=3}^{y=5} \left(\frac{9-y}{2}\right)^2 dy - \pi \int_{y=2}^{y=5} \left(\frac{y+1}{3}\right)^2 dy$$

The first antiderivative is $(2y-3)^3/6$, which evaluates at the endpoints to $9/2-1/6 = 13/3$. The second is $-(9 - y)^3/12$, which evaluates to $(-16/3) - (-18) = 38/3$. The third is $(y + 1)^3/27$, giving $8 - 1 = 7$. Combining these gives the total volume as 10π.

These integrals suggest an alternative, purely-geometric approach. Let Q be the quadrilateral with vertices (1,2), (2,5), (0,5), and (0,2). If this region is spun around the y-axis, it forms a (inverted) truncated cone. Extending to a full cone with a tip at $(0, -1)$ would give a volume of $V = (\pi/3)r^2h = 8\pi$; we obtain the truncated cone by lopping off the bottom half of this, which itself is a similar cone with volume π, leaving the truncated cone to have volume 7π. It is no accident that this equals the third integral computed above: if we attach this truncated cone to the original solid object, it will "plug the hole" and leave the solid whose cross-sections are whole disks instead of annuli.

That enlarged solid is then the union of two truncated cones, glued together along the plane $y = 3$. The top part is part of a cone with apex at $(0, 9)$, a cone of height 6 and radius 3; from this we remove the top cone of height 4 (and radius 2). Hence the upper truncated cone has a volume of $(\pi/3)(54 - 16) = (38/3)\pi$. Similarly the truncated cone below $y = 3$ would have its apex (or nadir!) at $(0, 3/2)$ and so this solid is the difference between a cone of height $3/2$ and radius 3, and a cone of height $1/2$ and radius 1; its volume is then $V = (\pi/3)(27/2 - 1/2) = (13/3)\pi$. The glued-together solid then has volume of 17π, from which we remove the central region of volume 7π to get the volume of 10π. It's not just the same numerical answer as before: it's actually computing the volume in the same way but interpreting the pieces geometrically!

2. Find the sum of each of the following series for $|x| < 1$.

(a) $\displaystyle\sum_{n=1}^{\infty} nx^{n+1}$

(b) $\displaystyle\sum_{n=2}^{\infty} n(n-1)x^{2n}$

ANSWER: The first series equals x^2 times the series $\sum_{n=0}^{\infty} nx^{n-1}$, which is the term-by-term derivative of the series $\sum_{n=0}^{\infty} x^n$, which converges absolutely for all $x \in (0, 1)$ to $1/(1 - x)$. Thus the previous series converges to $1/(1 - x)^2$. Hence series (a) converges to $(x/(1 - x))^2$. (One may also expand $x/(1 - x) = x + x^2 + x^3 + \ldots$, a geometric series, and then square both sides to obtain series (a).)

Similarly we may deduce that $\sum_{n=0}^{\infty} n(n-1)r^{n-2}$ is the second derivative of $1/(1-r)$, hence converges to $2/(1 - r)^3$ for each $r \in (-1, 1)$; multiply by r^2, then substitute $r = x^2$, to see that series (b) converges to $2x^4/(1 - x^2)^3$.

3. Suppose that $f(x)$ and $g(x)$ are 3-times differentiable functions for all x. Let $h(x) = g(f(x))$. Suppose that $f'(0) = f''(0) = f'''(0) = 0$. Show that $h'(0) = h''(0) = h'''(0) = 0$.

ANSWER: The Chain Rule and Product Rule allow us to calculate, for any x, that

$$\begin{aligned} h'(x) &= g'(f(x))f'(x), \\ h''(x) &= g''(f(x))(f'(x))^2 + g'(f(x))f''(x), \\ h'''(x) &= g'''(f(x))(f'(x))^3 + 3g''(f(x))f'(x)f''(x) + g'(f(x))f'''(x) \end{aligned}$$

Each of these is a sum of products, and in every product we see at least one factor of $f'(x)$, $f''(x)$, or $f'''(x)$. Assuming that all three of these vanish at some point x, we conclude that h', h'', and h''' do, too.

4. Find the equation of the plane which contains the points $(1, 0, 0)$ and $(0, 0, 1)$ and is tangent to the curve $(x, y, z) = (2, t, 2t^2)$ at some point. (There are actually two such planes. Find them both.)

ANSWER: Every plane is the solution set to an equation of the form $Ax+By+Cz = D$ for some numbers A, B, C, D; in each case, the vector $\langle A, B, C\rangle$ is perpendicular to the plane.

For any particular plane, we can determine the points where the curve meets that plane: this happens iff t takes on a value t_0 which makes (x, y, z) satisfy the equation of the plane, that is, iff $A \cdot 2 + B \cdot t_0 + C \cdot 2t^2 = D$. This gives a constraint on the possible values of t_0.

The tangent vector to the curve at the point where $t = t_0$ is the vector $(x', y', z') = (0, 1, 4t_0)$. In order for a plane to be tangent to the curve, this tangent vector would have to be perpendicular to the plane's normal vector, and so the dot product would be zero. This gives us another constraint on t_0: the curve can only be tangent to a particular plane there if $0 = 0 \cdot A + 1 \cdot B + 4t_0 \cdot C = B + 4t_0C$.

Combining these two paragraphs we see that the curve is only tangent to any plane if there is a single value of t_0 that makes both these conditions hold at once. We can eliminate $t_0 = -B/(4C)$ from these equations and with a bit of algebra we see that this happens iff $(8C)(2A - D) = B^2$.

Now, we are also given two points that must lie in the plane; these points force $A = D$ and $C = D$, respectively. Together with the previous paragraph this shows $8D^2 = B^2$, i.e. $B = \pm 2\sqrt{2}D$. Hence our plane must be one of

$$x \pm 2\sqrt{2}y + z = 1$$

which meets the curve when $t = t_0 = \mp\sqrt{2}/2$, i.e. at the point $(2, \mp\sqrt{2}/2, 1)$

5. Let $f(x) = x^2 \sin(\pi/x)$. Show that there are infinitely many values of x between 0 and 1 such that $f'(x) = 0$.

ANSWER: Since $f(1/n) = 0$ for every natural number n, we may apply Rolle's Theorem on each interval $[1/(n+1), 1/n]$ to conclude that $f'(x) = 0$ at least once in each such interval. (In fact, there is only one critical point in each of these intervals; the critical points occur when $u = 1/x$ satisfies $\tan(\pi u) = \pi u/2$, and the function $g(u) = \tan(\pi u) - \pi u/2$ is easily shown to be increasing between any two odd multiples of $1/2$.)