ALBERT A. BENNETT CALCULUS PRIZE EXAM Dec 8 2012

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Present Calculus Course:__________ **Instructor:**____________________

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School (Natural Sciences, Engineering, etc.)____________________

Show all work in your solutions; turn in your solutions on the sheets provided. (Suggestion: Do preliminary work on scratch paper that you don't turn in; write up final solutions neatly and in order; write your name on all pages turned in.)

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1. Evaluate the following limit (or explain why the limit does not exist):

$$\lim_{x\to 0} \frac{1}{x^4} \int_0^{x^2} \sin(t^2)\, dt$$

ANSWER: Let $u = t^2$; then we are taking the limit of

$$\frac{1}{x^4} \int_{u=0}^{u=x^4} \frac{\sin(u)}{2\sqrt{u}}\, du$$

where the integrand $f(u)$ is actually continuous on the whole interval $[0, x^4]$ if at $u = 0$ we assign it the value

$$f(0) := \lim_{u\to 0} \frac{\sin(u)}{2\sqrt{u}} = \lim_{u\to 0} \frac{\sin(u)}{u} \cdot \frac{\sqrt{u}}{2} = 1 \cdot 0 = 0.$$

Thus we may apply the Fundamental Theorem of Calculus: if F is any antiderivative of this integrand, the integral is $F(x^4) - F(0)$, so we are taking the limit of $\frac{F(h)-F(0)}{h}$ as $h = x^4 \to 0$. By definition this last limit equals $F'(0) = f(0) = 0$.

Alternative approaches use L'Hôpital's Rule or a careful estimate of the integrand (as in Problem 2).

2. Determine whether this series converges or diverges. (Be sure to explain your reasoning.)

$$\sum_{n=2}^{\infty} \ln(n \sin(\frac{1}{n}))$$

ANSWER: The series converges because the nth term is roughly $-1/6n^2$. This observation can be made precise in several ways.

For $n > 1$ we have $\frac{1}{n} < 1$ and so the Taylor series for sine is a strictly decreasing, strictly alternating series whose terms tend to 0; thus the value of $\sin(1/n)$ lies strictly between any two consecutive partial sums. In particular we have $\frac{1}{n} - \frac{1}{6n^3} < \sin(\frac{1}{n}) < \frac{1}{n}$, and then of course

$$1 - \frac{1}{6} < 1 - \frac{1}{6n^2} < n\sin(\frac{1}{n}) < 1$$

Now, we similarly can estimate the logarithms of such numbers near 1: the Taylor series for $\ln(1-x)$ is $-x - x^2/2 - x^3/3 - \ldots$ and we may estimate the error that results from replacing $\ln(1-x)$ with $-x$. For $x \in [0, \frac{1}{6}]$ the error is by Taylor's Theorem at most $\frac{18}{25}x^2$, that is,

$$-x - \frac{18}{25}x^2 < \ln(1-x) < -x$$

To summarize: the nth term of this series is a negative number and is no larger in magnitude than

$$b_n = \frac{1}{6n^2} + \frac{18}{25}\left(\frac{1}{6n^2}\right)^2$$

The sum of all the b_n is bounded (it's the sum of two convergent p-series) so the original series converges (absolutely).

Since the Taylor Series suggest that the nth term is nearly $1/6n^2$, you can instead swiftly verify convergence using a Limit Comparison Test; to evaluate the limit of $a_n/(1/n^2)$, you may use L'Hopital's Rule together with the facts that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = 1/2$$

NB – our inequalities peg the value of the series (starting at $n = 1$) at somewhere between $-\pi^2/36 = 0.274$ and $-\pi^2/36 - \pi^4/4500 = -0.296$. In fact the actual sum is about -0.2805563362.

3. Evaluate the following limit (or explain why the limit does not exist):

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x) + \frac{1}{2}x^2 - 1}{x^4 + y^4}$$

ANSWER: This limit does not exist. Along the y-axis, where $x = 0$, clearly the function is zero everywhere and thus the limit along this line is zero. But along the x-axis, where

$y = 0$, we have a function of just one variable, whose limit may be obtained from Taylor series or from several interations of L'Hoptial's rule: $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$, so this limit is $1/24$. Since the two directional limits are unequal, the limit (in the sense of functions defined on the plane) does not exist.

4. Find all functions $f(x, y)$ for which $\nabla f(x, y) = \langle y, -x \rangle$.

ANSWER: There are no such functions.

Certainly a contradiction arises if you assume f is defined on the whole plane. In that case, as we traverse around the circle $p(t) = (R\cos(t), R\sin(t))$, we would see that the function should steadily decrease with t: $\frac{d}{dt} f(p(t)) = \nabla f \cdot p'(t) = yp_1'(t) - xp_2'(t) = -\cos(t)^2 - \sin(t)^2 = -1 < 0$; but of course by the time we return to our starting point (at $t = 2\pi$) the value of the function has to be the same as it was when we started (at $t = 0$). This argument seems to show, more generally, that a gradient field cannot show a flow that forms loops.

But ... sometimes it does! Consider the function given simply in polar coordinates as $f(r, \theta) = \theta$, that is, $f(x, y) = \arctan(y/x)$. The direction of greatest increase is always along circles! You can even compute this directly: $\nabla f = \frac{1}{x^2+y^2}\langle y, -x \rangle$. Any function of θ alone will do; for instance you can show ∇f is parallel to $\langle y, -x \rangle$ when $f(x, y) = \ln(x) - \ln(y)$ (and explain why this is indeed a function of θ alone!)

What these examples show is that it is quite possible for ∇f to point along circles — but only when f is defined on a *subset* of the plane. (For example, $f = \theta$ is not well-defined along the positive x-axis — at least, not if we want f to be differentiable.)

Since the domain of f was not specified in the problem, it is better to show that there is no function satisfying the given condition even locally (i.e. on open subsets of the plane). But that's also easy to do, by a different argument: if such an f existed, we would have $f_{xy} = 1$ and $f_{yx} = -1$, contradicting Clairaut's Theorem. (Observe that these second-order partial derivatives are clearly continuous — a hypothesis of Clairaut's theorem.)

5. Consider the surface

$$S = \{(x, y, z) \,|\, xyz = 27,\ x > 0,\ y > 0,\ z > 0\}$$

Show that all pyramids formed by the three coordinate planes and a plane tangent to the surface S have the same volume.

ANSWER: At a point $P_0 = (x_0, y_0, z_0)$ on the surface S, the vectors normal to the surface are parallel to the gradient of $xyz - 27$, that is, they point in the same (or opposite) direction as $\langle y_0 z_0, x_0 z_0, x_0 y_0 \rangle$. Therefore the plane tangent at P_0 is

$$(y_0 z_0)(x - x_0) + (x_0 z_0)(y - y_0) + (x_0 y_0)(z - z_0) = 0$$

The same plane can be represented by the more elegant equation

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 3$$

after expanding and dividing by $x_0 y_0 z_0$. From this equation it is clear where the plane meets each of the coordinate axes: the intercepts are $(3x_0, 0, 0)$, $(0, 3y_0, 0)$, and $(0, 0, 3z_0)$; that is, this is a pyramid formed by three perpendicular planes and a fourth plane that leaves legs of lengths $3x_0$, $3y_0$, and $3z_0$.

There are several ways to determine the volume of such a pyramid; for example you may integrate the function

$$z = z_0 \left(3 - \frac{x}{x_0} - \frac{y}{y_0} \right)$$

over the appropriate triangle, or you may use a change of variables of the form $(x', y', z') = (x/x_0, y/y_0, z/z_0)$ in the volume integral $V = \int\int\int_D 1 \, dx \, dy \, dz$ to compare this volume to that of the pyramid having leg lengths 3, 3, and 3. Or you may use the fact that the volume of a "cone" over any planar region can be shown (by integrating over a line perpendicular to the plane) to be 1/3 the height times the area on the bottom. In particular, this tells us that the volume of such a pyramid is always one-sixth the volume of the enclosing box — in our case,

$$V = \frac{1}{6}(3x_0)(3y_0)(3z_0) = \frac{9}{2}(x_0 y_0 z_0).$$

Since the point (x_0, y_0, z_0) lies in S, this is exactly 243/2, irrespective of which point of tangency P_0 is chosen.

You can generalize this problem a bit: the surface S can be defined by any equation of the form F =constant, where $F(x, y, z) = f(\, xyz\,)$ for any smooth function f of one variable. Indeed, you may find that the proof is clearer and easier to understand in the general case than in the special case $f(u) = u$!