ALBERT A. BENNETT CALCULUS PRIZE EXAM – Dec 8 2013

Here are some possible responses to this semester's Bennett exam.

1. Evaluate the following limit (or explain why the limit does not exist):

$$\lim_{x\to 0^+} \frac{x\sin(\frac{1}{x})}{\ln(1+\sqrt{x})}$$

Answer: As $x \to 0^+$, $\sqrt{x} \to 0^+$ too, and thus $\ln(1+\sqrt{x})$ is approximately as big as $\sqrt{x}$; more precisely

$$\lim_{x\to 0^+} \frac{\ln(1+\sqrt{x})}{\sqrt{x}} = 1$$

so we may compute our limit as

$$\lim_{x\to 0^+} \sqrt{x}\sin(\frac{1}{x}) \cdot \frac{\sqrt{x}}{\ln(1+\sqrt{x})} = \lim_{x\to 0^+} \sqrt{x}\sin(\frac{1}{x}) = 0$$

by the Squeeze Theorem, since $|\sin(u)| \le 1$ for all u.

A small variant: write the function as $\frac{x}{\ln(1+\sqrt{x})} \cdot \sin(\frac{1}{x})$, showing the first part tends to zero and then as above using the Squeeze Theorem.

Note: If you try to use L'Hopital's Rule, you find you must compute the limit of

$$2(1+\sqrt{x}) \cdot \left(\sqrt{x}\sin(1/x) - \frac{\cos(1/x)}{\sqrt{x}}\right)$$

which does not exist (the last term oscillates ever faster and ever larger as $x \to 0$!). But when $f'(x)/g'(x)$ does not have a limit, L'Hopital's Rule is silent — that theorem does *not* guarantee that $f(x)/g(x)$ has no limit, and indeed this example shows that f/g may still have a limit when f'/g' does not.

2. Which is larger — $\ln(2)$ or $\arctan(1)$? You must answer without a calculator of course, and memorized digits are also useless unless you can explain how those digits are computed. Use some calculus to describe these numbers.

Answer:

$$\ln(2) = \int_0^1 \frac{dx}{1+x} < \int_0^1 \frac{dx}{1+x^2} = \arctan(1)$$

since $x > x^2$ on $[0,1]$. (Actually $\ln(2) \approx 0.693$ and $\arctan(1) = \pi/4 \approx 0.785$.)

Equivalently: $f(x) = \ln(1+x) - \arctan(x)$ is 0 when $x = 0$ and easily checked to be decreasing for $0 < x < 1$, so it's negative at $x = 1$. (As it turns out, it's negative iff $x < 2.0633197\ldots$.)

You can instead use the power series for $\ln(1+x)$ and $\arctan(x)$, each evaluated at $x = 1$; taking terms in pairs we have

$$\ln(2) = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots = 1 - 1/6 - 1/20 - \ldots - 2/(8n^2+4n) - \ldots$$

while

$$\arctan(1) = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots = 1 - 2/15 - 2/63 - \ldots - 2/(16n^2 - 1) - \ldots.$$

But for all $n \geq 1$, $16n^2 - 1 > 8n^2 + 4n$. So each negative term in the series for $\ln(2)$ is larger than the corresponding one in $\arctan(1)$, making $\ln(2)$ the smaller of the two numbers.

(Many students knew that $\arctan(1) = \pi/4$, which we can accept as known for this purpose; memorizing the digits $\pi \approx 3.14$ is technically against the rules but even if we allow that, it's still necessary to estimate $\ln(2)$. The Taylor Series above converges slowly, but as one student noted, we can compute $\ln(2) = -\ln(1 - \frac{1}{2}) = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \ldots$ This converges much faster but it is a little harder to estimate the difference between a partial sum and the correct value. We can avoid computing the logarithm: since the exponential function is increasing we need only show that $e^{\arctan(1)} > 2$, which we can do with a few terms of the Taylor series of e^x and $\arctan(x)$; for example, $\arctan(1) > 1 - 1/3 + 1/5 - 1/7 = 76/105$, so $e^{\arctan(1)} > e^{76/105} > \sum_{i=0}^{3} (\frac{76}{105})^i / i! = 7115783/3472875 = 2.04896\ldots$ You can use one fewer term of the series for e^x if you use one more pair of terms for $\arctan(x)$. Or if you assume it known that $\arctan(1) = \pi/4$ and $\pi > 3$ then compute $e > 1+1+1/2+1/6 = 8/3$ and so $e^\pi > (8/3)^3 = 512/27 > 16 = 2^4$ whence $e^{\pi/4} > 2$.)

3. Evaluate the following series, or explain why the series does not converge:

$$\frac{1}{1} + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots = \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i}$$

Answer: The denominator of the nth term is a finite sum typically treated in calculus books during the introduction to Integration; it is known to sum to $n(n+1)/2$. So we are summing $\sum 2/(n(n+1))$. But this particular series is a familiar example of a telescoping series: by Partial Fractions we may write the n term as $2(1/n - 1/(n+1))$ so our series is

$$2 \sum_{n \geq 1} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 2 \left((1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots \right) = 2$$

4. Where does this function attain its maximum value?

$$F(x,y) = \int_x^{x+4} \int_y^{y+6} e^{-(u^2+t^2)}\, dt du$$

Answer: $F(x,y)$ is the volume of the region under the graph of this exponential function (whose graph is the famous "bell curve" rotated around its central axis) and lying over $[x, x+4] \times [y, y+6]$, a rectangle in the plane having width 4 and height 6. By symmetry we should position the center of the rectangle under the highest point of the surface (at (0,0)), so we should take $x = -2$ and $y = -3$.

Note: We could also maximize by finding the point where $\nabla F(x,y) = (0,0)$. We can compute the partial derivatives using the Fundamental Theorem of Calculus (and, for $\partial F/\partial y$, Fubini's Theorem):

$$\partial F/\partial x = \int_y^{y+6} e^{-((x+4)^2+t^2)}\,dt - \int_y^{y+6} e^{-(x^2+t^2)}\,dt = (\int_y^{y+6} e^{-t^2}\,dt) \cdot \left(e^{-(x+4)^2} - e^{-x^2}\right)$$

which can only be zero when $e^{-(x+4)^2} = e^{-x^2}$, i.e. when $(x+4)^2 = x^2$, which requires $x = -2$. Likewise we must have $y = -3$. (Actually this isn't really a multivariable calculus problem because $F(x,y) = f(x)g(y)$ where each of f and g is a function defined by a single integral; we simply choose x to maximize f and y to maximize g.)

5. Find all vectors v in $\mathbf{R}^3$ for which

$$v \cdot u_1 = 10, \qquad v \cdot u_2 = 11, \qquad v \cdot u_3 = 12,$$

where

$$u_1 = \langle 1,2,3 \rangle \qquad u_2 = \langle 4,5,6 \rangle \qquad u_3 = \langle 7,8,9 \rangle$$

Answer: These three equations are actually redundant, since $u_2 = (u_1 + u_3)/2$. Now, the solution set to any one equation of the form $v \cdot u = c$ is a plane perpendicular to u. Thus the solution set to any two (and thus all three) of our equations is an intersection of two non-parallel planes, i.e. a line. Indeed, that line is perpendicular to both u_i and hence is parallel to $u_1 \times u_2 = \langle -3, 6, -3 \rangle$. Then all we need to describe the solution set completely is one point on the line, for example the point where this line pierces the x, y plane. Well, the vector $v = \langle x, y, 0 \rangle$ is determined by the equations

$$x + 2y = 10 \qquad 4x + 5y = 11$$

from which we determine that $3y = 29$ and thus $y = 29/3$ and $x = -28/3$. So the entire solution set is the collection of vectors

$$\langle -28/3, 29/3, 0 \rangle + t \langle 1, -2, 1 \rangle.$$