ALBERT A. BENNETT CALCULUS PRIZE EXAM – Dec 7 2014
Here are some possible responses to this semester's Bennett exam.

**1.** Evaluate the integrals:

$$(a) \int_0^1 \frac{x^3 - x^2}{x^2 - 3x + 2}\, dx \qquad\qquad (b) \int_0^{\pi/6} \sin(3x)\, \sin(5x)\, dx$$

Answer: For 1(a), factor and cancel: the integrand equals $x^2/(x-2)$ (except at $x = 1$) so the original integral will equal the integral of this function. Divide and integrate, or use the substitution $u = x - 2$ to get

$$\int_{u=-2}^{u=-1} \frac{(u+2)^2}{u}\, du = \int_{-2}^{-1} \left(u + 4 + \frac{4}{u}\right) du = \left(\frac{u^2}{2} + 4u + 4\ln(|u|)\right)\Big|_{-2}^{-1} = \frac{5}{2} - 4\ln(2)$$

For 1(b) you might want to use the conversion to exponentials: $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$; or use repeated application of integration by parts. But probably easiest is to use the angle-addition formula $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$ to deduce that $\sin(u)\sin(v) = \frac{1}{2}(\cos(u-v) - \cos(u+v))$ so the integral in 1(b) is

$$\int_{x=0}^{\pi/6} \frac{1}{2}\cos(2x) - \frac{1}{2}\cos(8x)\, dx = \left(\frac{\sin(2x)}{4} - \frac{\sin(8x)}{16}\right)\Big|_{x=0}^{\pi/6} = \frac{1}{4}\sin(\frac{\pi}{3}) - \frac{1}{16}\sin(\frac{4\pi}{3})$$
$$= (\frac{1}{4} + \frac{1}{16})\sin(\frac{\pi}{3}) = \frac{5}{16} \cdot \frac{\sqrt{3}}{2}$$

Note: it is very important in Fourier Analysis to show $\int \sin(mx)\sin(nx)dx = 0$ when integrating over an entire period of the sine function, whenever $m \neq n$.

**2.** Find the integer part of $\displaystyle\sum_{n=1}^{40000} \frac{1}{\sqrt{n}}$. (That is, if the sum is evaluated numerically, what are the digits to the left of the decimal point?)

Answer: 398. Indeed, since $f(x) = 1/\sqrt{x}$ is a positive, decreasing function, we learn with the "Integral Test" for series that

$$\sum_{n=a+1}^{n=b} f(n) < \int_{x=a}^{x=b} f(x)\, dx < \sum_{n=a}^{n=b-1} f(n)$$

so since it is easily computed that $\int_1^{40000} x^{-1/2}\, dx = 398$, we see our sum is

$$1 + \sum_{n=2}^{40000} \frac{1}{\sqrt{n}} < 1 + \int_1^{40000} x^{-1/2}\, dx = 399$$

and also that our sum is

$$\sum_{n=1}^{39999} \frac{1}{\sqrt{n}} + \frac{1}{200} > \int_1^{40000} x^{-1/2}\,dx + \frac{1}{200} = 398.05$$

so the integer part of the sum is 398. (To more digits, the sum is about 398.54214548598...)

**3.** For $t > 0$ let $F(t) = \frac{1}{t}\int_0^{\frac{\pi}{2}t} |\cos(2x)|\,dx$ . Compute $\lim_{t\to 0} F(t)$.

Answer: If $f(t)$ is any continuous function and $F(z) = \int_0^z f(x)\,dx$ then the Fundamental Theorem of Calculus asserts that $F'(z) = f(z)$. Computing $F'(0)$ straight from the definition, then, we see $f(0) = \lim_{h\to 0} \frac{1}{h}F(h)$. In particular, for $h = (\pi/2)t$, we deduce

$$\lim_{t\to 0} \frac{1}{(\pi/2)t}\int_0^{(\pi/2)t} |\cos(2x)|\,dx = 1$$

so that our limit is $\pi/2$.

**4.** Find all the critical points of the function $f(x, y) = x^2 + y^2(1-x)^3$, and classify them as local minima, absolute (global) maxima, saddle points, etc.

Answer: As a polynomial, $f$ is differentiable everywhere, so the critical points are only the points $p$ where $\nabla f(p) = 0$. Well,

$$\begin{aligned} f_x &= 2x - 3y^2(1-x)^2 \\ f_y &= 2y(1-x)^3 \end{aligned}$$

so $f_y = 0$ implies $x = 1$ or $y = 0$. In the first case we then see $f_x = 2$ is nonzero so there are no such critical points. When $y = 0$, however, the constraint $f_x = 0$ forces $x = 0$, too. So the origin is the only critical point.

Next we compute the Hessian matrix:

$$\begin{aligned} f_{xx} &= 2 + 6y^2(1-x) \\ f_{xy} &= -6y(1-x)^2 \\ f_{yy} &= 2(1-x)^3 \end{aligned}$$

So $f''(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which is positive-definite ($\det(f'') > 0$ and $\mathrm{Tr}(f'') > 0$) so that the origin is a local minimum.

But despite the fact that there is only one critical point, and that one is a local minimum, it is *not* a global (absolute) minimum, since e.g. $f(2,3) = -5 < f(0,0)$ !

**NOTE:** This problem was scored separately since this topic has now been removed from the syllabus for the calculus sequences. As it turned out, the ranking of the top contestants is the same whether this question is included in the scores or not.

**5.** Points $P$ and $Q$ move together around the parabola $y = x^2$ in such a way that the area cut off from the parabola by the line segment $PQ$ always has area $\frac{4}{3}$. Let $M$ be the midpoint of $PQ$. What curve does $M$ trace out as $P$ and $Q$ vary around the parabola?

Answer: If the $x$-coordinate of $P$ is $p$, then $P = (p, p^2)$ and likewise $Q = (q, q^2)$ for some real $q$. Then $M = \left(\frac{p+q}{2}, \frac{p^2+q^2}{2}\right)$.

Now, the line segment $PQ$ has slope $\frac{q^2-p^2}{q-p} = p + q$, so the line containing $PQ$ has equation $y = p^2 + (p + q)(x - p) = (p + q)x - pq$. Thus the area cut off is

$$\begin{aligned}
A &= \int_{x=p}^{x=q} ((p+q)x - pq) - (x^2)\,dx \\
&= \left((p+q)\frac{x^2}{2} - pq\,x - \frac{x^3}{3}\right)\Bigg|_p^q \\
&= \frac{(p+q)}{2}(q^2 - p^2) - pq(q-p) - \frac{1}{3}(q^3 - p^3) \\
&= (q-p)\left(\frac{(p+q)^2}{2} - pq - \frac{(p^2 + pq + q^2)}{3}\right) \\
&= (q-p)\frac{(p-q)^2}{6} = \frac{(q-p)^3}{6}
\end{aligned}$$

Since we are told $A = 4/3$, this means $(q - p)^3 = 8$, i.e. $q = p + 2$. Thus the midpoints $M$ are the points

$$(x, y) = (p + 1,\ p^2 + 2p + 2)$$

so that each such point lies on the curve $y = x^2 + 1$ (and in fact $M$ traces out the entire parabola as $P$ and $Q$ vary together).