1. Find the number k for which the curves $y = \ln(x)$ and $y = x^k$ have exactly one point of intersection.

ANSWER: The continuous function $f(x) = x^k - \ln(x)$ tends to $+\infty$ both as $x \to 0^+$ and (for $k > 0$) as $x \to +\infty$. So if $f(a) < 0$ for any $a > 0$ then there would be (at least) two points of intersection, on either side of a. But (since f is differentiable) this would have to happen for some a on one side or the other of any point b where $f(b) = 0$, unless $f'(b) = 0$. Hence if the curves have exactly one point of intersection, at $x = b$, then we must have $f(b) = f'(b) = 0$. Since $f'(x) = kx^{k-1} - 1/x$, this requires $k\,b^k = 1$, i.e. $b = (1/k)^{1/k}$. On the other hand when $x = (1/k)^{1/k}$, the y coordinates of the curves are $-\ln(k)/k$ and $1/k$ respectively, so in order for the curves actually to cross there, we must have $-\ln(k) = 1$, i.e. $k = 1/e$. For this k, the curves intersect at (e^e, e), where both curves have slope e^{-e} (that is, they are tangent). This is indeed the unique point of intersection since $f'(x)$ is positive to the right of this point and negative to its left.

2. What is the minimum value of the sum of a convergent geometric series whose third term is 1 and whose other terms are all positive?

ANSWER: If the common ratio between consecutive terms in the series is $r > 0$ then the first two terms are $a = 1/r^2$ and $ar = 1/r$, and the sum of all the terms is $S = a/(1-r) = 1/(r^2 - r^3)$. (Of course we must have $r < 1$ since the series is given to be convergent.) This function S tends to $+\infty$ both as $r \to 0^+$ and as $r \to 1^-$, and is positive on the whole interval $(0,1)$, so it will have a minimum in $(0,1)$, necessarily at a point where the denominator is maximal. That in turn requires a zero derivative: $2r = 3r^2$, so ($r = 0$ or) $r = 2/3$. For this value of r, the sum of the series is $27/4$.

3. Evaluate $\displaystyle\int \frac{dx}{x^7 - x}$.

ANSWER: Since the integrand is a rational function of x, it is reasonable to use its Partial Fractions decomposition; with some effort this can be determined to be

$$-\frac{1}{x} + \frac{\frac{1}{6}}{x-1} + \frac{\frac{1}{6}}{x+1} + \frac{\frac{1}{6}(2x+1)}{x^2+x+1} + \frac{\frac{1}{6}(2x-1)}{x^2-x+1}$$

Amazingly, each of these five summands is easy to antidifferentiate as a single logarithm, and the last four of the logarithms combine exactly to $\frac{1}{6}\log(|x^6-1|)$; with the first summand as well this simplifies to $\frac{1}{6}\log(|1-x^{-6}|)+C$.

But in hindsight the result could have been obtained much faster by using a substitution like $u=x^6$, since then $du=6x^5\,dx=6(u/x)\,dx$ and so $dx=\frac{1}{6}(x/u)\,du$. With this substitution, the integral becomes a much more manageable $\frac{1}{6}\int du/(u(u-1))$; the Partial Fractions Decomposition then leads quickly to the antiderivative $\frac{1}{6}(\ln(u-1)-\ln(u))=\frac{1}{6}\ln(1-u^{-1})=\frac{1}{6}\ln(1-x^{-6})$ as before.

With an abundance of perspicacity one might simply observe

$$\frac{1}{x^7-x}=\frac{x^6-(x^6-1)}{x^7-x}=\frac{1}{6}\frac{6x^5}{x^6-1}-\frac{1}{x}$$

whose antiderivative is then immediate.

4. Compute $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2+2y^2-5xy^2-6y^4}{x^2+2y^2+3x^2y+4x^4}$ or show that the limit does not exist.

ANSWER: The limit does exist and it equals 1. This is most easily seen by writing the input point in polar coordinates; at $(x,y)=(r\cos(\theta),r\sin(\theta))$, the value of the function is

$$1-\frac{r\cos(\theta)\sin(\theta)(5\sin(\theta)+3\cos(\theta))+r^2(6\sin^4(\theta)+4\cos^4(\theta))}{1+\sin^2(\theta)+3r\cos^2(\theta)\sin(\theta)+4r^2\cos^4(\theta)}$$

(where I have written $x^2+2y^2=r^2(1+\sin^2(\theta))$, and have cancelled an r^2 from both numerator and denominator). Now, this denominator is never too small. Indeed for any θ we have $1+\sin^2(\theta)+3r\cos^2(\theta)\sin(\theta)+4r^2\cos^4(\theta)\geq 1+0-3r+0$ and if, say, $r<1/4$ this denominator is larger than $1/4$. On the other hand, we may similarly estimate that $|r\cos(\theta)\sin(\theta)(5\sin(\theta)+3\cos(\theta))+r^2(6\sin^4(\theta)+4\cos^4(\theta))|\leq 8r+10r^2$ for any θ; hence as we approach the origin, no matter what the path, we will have $r\to 0$ and thus the numerator tends to 0, making the whole fraction tend to 0, leaving 1 as the limit of the original function.

A solution to this problem is incomplete if it does not demonstrate that the numerators are bounded by a diminishing function of r irrespective of the values of θ, or if it does not demonstrate that the denominators are bounded away from 0. (In fact the denominator of the original function is $(46x^4+16x^2+2(3x^2+4y)^2)/16$, which is clearly never zero except at the origin.) Comparable examples like $f(x,y)=xy^2/(x^2+y^4)$ show that it is insufficient to check for convergence only along the axes, or along straight lines, or by considering only lowest-degree terms. Finally note that there is no theorem comparable to L'Hôpital's Rule for functions of more than one variable.

5. Evaluate the integral $\int_S f\,dA$ where $f(x,y) = 1/(1-xy)$ and $S = [0,1]\times[0,1]$ is the unit square in the first quadrant. This is an improper integral; interpret this as

$$\lim_{t\to 1^-} \int_0^t \int_0^t \frac{1}{1-xy}\,dx\,dy$$

(Hint: you will probably need either to find an alternative description of the integrand f or to transform the domain S with a change of variables.)

ANSWER: The integrand is the sum of a geometric series $\sum_{k\geq 0} x^k y^k$; for a fixed $y \leq t < 1$ this may be viewed as a power series in x, with radius of convergence $1/|y| > 1$, so in particular, we may compute an antiderivative of this function of x termwise, and then use the Fundamental Theorem of Calculus to compute the integral with respect to x:

$$\int_0^t \frac{1}{1-xy}\,dx = \sum_{k\geq 0} y^k \frac{t^{k+1}}{k+1}$$

Again we may view this as a power series in y whose radius of convergence (from the Ratio Test) is $1/t > 1$, so again an antiderivative may be computed termwise for $y < 1$, allowing us to compute the integral as

$$\int_0^t \int_0^t \frac{1}{1-xy}\,dx\,dy = \sum_{k\geq 0} \frac{t^{2(k+1)}}{(k+1)^2}$$

Letting t increase to 1 now shows the improper integral converges to $\sum_{k\geq 0} 1/(k+1)^2 = 1 + (1/4) + (1/9) + \ldots$.

You may recognize this as a (convergent) p-series; the value of the sum of this series is fairly well known, but we will determine it by evaluating the integral another way.

Attempting to compute the integral using Fubini's Theorem (without series) looks easy: the inner integral is $\ln(1-ty)/y$; but the antiderivative of this function of y (even if we take $t = 1$) cannot be expressed in terms of familiar functions. (This integral is essentially the definition of the "dilogarithm" function.)

One could attempt to use polar coordinates: it is clear that the integrals over the two triangles on either side of the line $y = x$ are equal, so we need only compute

$$2\int_0^{\pi/4} \int_0^{t\,\sec(\theta)} \frac{r}{1-r^2\cos(\theta)\sin(\theta)}\,dr\,d\theta$$

Again the inner integral is not hard: it evaluates to $\ln(1-t^2\tan(\theta))/(-2\cos(\theta)\sin(\theta))$, but once again the antiderivative cannot be expressed in terms of the usual elementary functions. (Substituting $u=\tan(\theta)$ leads again to the integral defining the dilogarithm.)

A different coordinate system is more helpful. Let $T:\mathbf{R}^2\to\mathbf{R}^2$ be $T(u,v)=(u+v,u-v)$, a mapping that rotates and expands the plane. Specifically (by computing the Jacobian matrix or otherwise) we find that for any set A in the u,v plane, its image $T(A)$ has double the area that A has. This applies in particular to the triangle D bounded by the points $(u,v)=(0,0)$, $(1/2,1/2)$, and $(1,0)$; the image $T(D)$ is the lower half-square described in the previous paragraph. So the integral we wish to compute can be re-expressed in u,v coordinates using the multivariate Change of Variables theorem:

$$\int_S f\,dA = 2\int_{T(D)} f\,dxdy = 2\int_D (f\circ T)\,|\det(T')|\,du\,dv$$

Here the jacobian determinant is 2 as noted above, and the composite function is $f(x,y)=f(u-v,u+v)=1/(1-(u^2-v^2))$. In order to complete the computation using Fubini's Theorem we will split D into two parts along the line $u=1/2$ to express our answer as

$$4\int_D (f\circ T)\,du\,dv = 4\int_0^{1/2}\int_0^u \frac{1}{1-u^2+v^2}\,dv\,du + 4\int_{1/2}^1\int_0^{1-u}\frac{1}{1-u^2+v^2}\,dv\,du$$

The inner antiderivative in each summand is $(1/s)\arctan(v/s)$ where $s=\sqrt{1-u^2}$. Evaluating this for $v=u$ (in the first summand) and $v=1-u$ (in the second) gives integrals in u which appear to be difficult but in fact are easily changed by a substitution: in the first case let $u=\sin(\theta)$ and in the second use $u=\cos(2\theta)$; each of the integrals then reduces simply to $\int\theta\,d\theta=\frac{1}{2}\theta^2$. So the first summand evaluates to $\pi^2/9$. and the second to $\pi^2/18$, making the total integral over both halves of D come out to $\pi^2/6$.

Comparing this computation of the original integral with the one obtained using series proves the famous formula

$$\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}$$

(One can similarly describe $\zeta(3)=\sum_{n\geq 1}\frac{1}{n^3}$ as an integral over a cube in $\mathbf{R}^3$, but alas that integral has no other known easy description.)