ALBERT A. BENNETT CALCULUS PRIZE EXAM Dec 09, 2020

Submit your solutions *with all work shown*, **by 5pm (Austin time) as an email attachment to rusin@math.utexas.edu. During the exam you must abide the rules previously sent via email.**

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1. Find the limit of the sequence $\{a_1, a_2, a_3, \ldots\}$ where

$$a_n = \left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} \right)$$

2. There is a function f whose graph lies in the first quadrant. When the graph is rotated around the x-axis we obtain a surface S which has the following unusual feature: the volume of the region that is inside S and lies between the planes $x = 0$ and $x = b$ is b^2, for every constant $b > 0$. What is the function f ?

3. Compute an antiderivative: $\displaystyle\int \frac{2\tan(x)}{\sqrt{1-\sin^4(x)}}\,dx$

4. Let $f(x) = 21/(x^2+x+1)$ so that $f(2) = 3$ and $f(4) = 1$. For any value of d between 1 and 3, the horizontal line $y = d$ crosses the graph of f exactly once to form a region R_d bounded by this horizontal line, the graph, and the vertical lines $x = 2$ and $x = 4$. (You might call it a "butterfly" or "bow-tie" shape.) For what value of d is the area of R_d smallest?

5. Caasi Notwen is traveling around the first quadrant of the x, y plane; the x- and y-coordinates of her position at time t are denoted $x(t)$ and $y(t)$, respectively. She starts at the point (1,1). She notices that at each moment t her velocity maintains a rigid relationship to her position: the x- and y- components of her velocity are given, respectively, by

$$x'(t) = x(t)\,(-6 + 2y(t)) \qquad \text{and} \qquad y'(t) = y(t)\,(7 - 3x(t))$$

At how many different points can Caasi cross the line $y = 1$?

Answers will soon appear at `http://www.math.utexas.edu/users/rusin/Bennett/` .