ALBERT A. BENNETT CALCULUS EXAM ANSWERS 5/3/08

1. Let $f(x) = \int_0^x \cos(t^2)\,dt$. Write the Maclaurin series (Taylor series centered at 0) for each of the following functions of x.

(i)	$\cos(x)$	(iii)	$f(x)$
(ii)	$\cos(x^2)$	(iv)	$g(x) = f(x^2)$

ANSWER: The first of these is standard fare: $\cos(x) = \sum_{n\geq 0}(-1)^n x^{2n}/(2n)!$. Substitute x^2 for x to obtain (ii): $\cos(x^2) = \sum_{n\geq 0}(-1)^n x^{4n}/(2n)!$.

Now, $f(x)$ defined as an antiderivative of this function — specifically it's the antiderivate which also has $f(0) = 0$. We can obtain an antiderivative of a function defined by a power series by antidifferentiating term-by-term; the new series will converge on the interior of the interval of convergence of the previous one. In our case, the power series converges everywhere, so

$$f(x) = \sum_{n\geq 0} \frac{(-1)^n x^{4n+1}}{(4n+1)\cdot(2n)!}$$

(Observe that this is indeed the antiderivative having $f(0) = 0$.)

Finally we simply replace x by x^2 to obtain the Maclausin series

$$g(x) = \sum_{n\geq 0} \frac{(-1)^n x^{8n+2}}{(4n+1)\cdot(2n)!} = x^2 - \frac{x^{10}}{10} + \frac{x^{18}}{216} - \dots$$

2. Find the equations of all lines which are tangent to the curve $y = x^3 - x$ and are perpendicular to the line $y = 4x + 5$.

ANSWER: A line is perpendicular to another (nonvertical) line if the product of the slopes is -1, so the problem simply asks us to find tangent lines to the curve which happen to have a slope of $-1/4$. The slope of the line that's tangent at a point (x, y) is $dy/dx = 3x^2 - 1$, and $3x^2 - 1 = -1/4$ iff $x = \pm 1/2$. At these points $y = x(x^2 - 1) = \mp 3/8$, so the tangent line passing through the point will be

$$(y \pm \frac{3}{8}) = -\frac{1}{4}(x \mp \frac{1}{2}), \qquad \text{or} \qquad x + 4y \pm 1 = 0$$

3. Let a curve be given by the parametric equations

$$x = e^t \sin t - e^t \cos t$$
$$y = e^t \sin t + e^t \cos t$$

Find the arclength of the curve from $t = 0$ to $t = \ln(2)$.

ANSWER: The length of a curve presented parametrically is

$$\int_{t_1}^{t_2} \sqrt{dx^2 + dy^2} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In this case we compute the derivatives dx/dt and dy/dt as, respectively

$$(e^t \cos(t) + e^t \sin(t)) \pm (e^t \sin(t) - e^t \cos(t)) = 2e^t \sin(t) \quad \text{resp. } 2e^t \cos(t)$$

The sum of the squares of these is simply $(2e^t)^2$, so the arclength integral is $\int 2e^t\, dt = 2e^t$. Evaluating at the given endpoints shows the length to be $4 - 2 = 2$.

4. Suppose that x and y are given as functions of s and t by the equations

$$x = e^{st}\ , \qquad y = s^2 t^3$$

Suppose also that s is a function of t such that $ds/dt = (1 + t^3)^{-1}$ Then y can be regarded as a function of x. Compute dy/dx in terms of s and t.

ANSWER: Geometrically, the problem asserts that there is a transformation carrying the s,t-plane to the x,y-plane; and that we will be restricted to curve in the s,t-plane. Thus the transformation will give us a curve in the x,y-plane. We are asked to find the slope of the line tangent to each point of the latter curve.

There are several ways to organize the calculations here. For example, instead of presenting the curve in the s,t plane explicitly (s as function of t), we may present it parametrically:

$$s = f(u), \qquad t = u \qquad \text{where } f'(u) = (1 + u^3)^{-1}$$

Then the curve in the xy plane is also presented parametrically simply as the composite of two transformations $\mathbf{R}^1 \to \mathbf{R}^2 \to \mathbf{R}^2$:

$$x = e^{uf(u)}, \qquad y = f(u)^2 u^3$$

Then the tangent vector points in the direction of the derivative vector:

$$\left\langle \frac{dx}{du}, \frac{dy}{du} \right\rangle = \langle\, e^{uf(u)}(f(u)+uf'(u)),\ 3u^2f(u)^2+2u^3f(u)f'(u)\,\rangle$$

We were given the derivative of f' so this may be written

$$\langle\, e^{uf(u)}(f(u)+u/(1+u^3)),\ 3u^2f(u)^2+2u^3f(u)/(1+u^3)\,\rangle$$

By the Chain Rule, dy/dx will be the ratio $(dy/du)/(dx/du)$. We can abbreviate this somewhat since the problem asked for the slope dy/dx to be written in terms of s and t; simply replace each u with t and each $f(u)$ with s:

$$\frac{dy}{dx} = \frac{3t^2s^2+2t^3s/(1+t^3)}{e^{ts}(s+t/(1+t^3))}$$

This can be massaged a bit but not made much simpler.

5. Let $f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0\ , \\ 0 & \text{if } x = 0\ . \end{cases}$

(i) Show that $\lim_{x\to 0} f'(x)$ does not exist.

(ii) Using the definition of the derivative, show that $f'(0) = 0$.

ANSWER: At nonzero values of x, the derivative of f may be computed by the usual rules: $f'(x) = 2x\sin(1/x) - \cos(1/x)$. Since the sine function is bounded, the first summand of f' tends to zero as $x \to 0$. But the second summand has no limit: whenever $x = 1/(n\pi)$ for some integer n, the value of $\cos(1/x)$ is $(-1)^n$, so that the summand oscillates between $+1$ and -1 on our approach toward zero. Hence $f'(x)$ does not converge to a limit either.

On the other hand, right *at* $x = 0$, the usual rules for differentiation do not apply but we still have recourse to the very definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} h\sin(1/h) = 0$$

since the sine function is bounded but the factor h itself is tending to zero.

In other words, f is a continuous function everywhere; it *has* a derivative at every point, meaning that f is everywhere differentiable and that f' is everywhere defined; but this new function f' is not continuous. (We say that f is *differentiable* but not *continuously differentiable*.)