ALBERT A. BENNETT CALCULUS PRIZE EXAM **May 5 2012**

1. Determine whether series (a) converges or diverges, and give the radius of convergence for series (b). (Be sure to justify your answer.)

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{\pi n}\right) \qquad\qquad (b)\ \sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$$

ANSWER: Series (a) converges by comparison to the convergent p-series $\sum \frac{1}{n}\frac{1}{\pi n} = \frac{1}{\pi}\sum \frac{1}{n^2}$ because $\sin(x) < x$ for all positive x.

For series (b), the Ratio Test guarantees convergence when $\lim |a_{n+1}/a_n| < 1$, i.e. when

$$1 > \lim\,(n+1)|x|n^n/(n+1)^{n+1} = \lim\,|x|(n/(n+1))^n = |x|/e.$$

That is, the series converges if $-e < x < e$; likewise the series diverges if $x > e$ or $x < -e$. Alternatively, this series is easily handled using *Stirling's Approximation* for $n!$.

Remarks: it's not clear what (a) converges to but we can estimate it as closely as we like with rational multiples of π, using the fact that $\sin(x)$ has a Taylor series which is alternating. For example, for positive $x < 1$, $\sin(x)$ lies between $x - x^3/6$ and $x - x^3/6 + x^5/120$; this gives upper and lower bounds for $\sin(1/(\pi n))$ which we can then sum over all n: writing $\zeta(k)$ for $\sum_{n\geq 1}(1/n^k)$ we see our original series lies between $\zeta(2)/\pi - \zeta(4)/(6\pi^3)$ and $\zeta(2)/\pi - \zeta(4)/(6\pi^3) + \zeta(6)/(120\pi^5)$. But for even integer values of k, it is known that $\zeta(k)$ is a rational multiple of π^k; for example, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$. Thus our original sum lies between $\pi/6 - \pi/540$ and $\pi/6 - \pi/540 + \pi/113400$. In a similar way we can write our original sum as a different infinite series: it's

$$\pi \sum_{i\geq 1} (-1)^{i-1} \frac{\zeta(2i)}{\pi^{2i}(2i-1)!} = \pi \sum_{i\geq 1} \frac{B_{2i} 2^{2i-1}}{(2i)!(2i-1)!}$$

where B_j is the sequence of the *Bernoulli numbers*, a fascinating sequence of rational numbers that appears for example in the Taylor expansion $\frac{t}{e^t-1} = \sum_{j\geq 0} \frac{B_j}{j!} t^j$ I don't know the sum of my series but the first few terms are

$$\pi\left(\frac{1}{6} - \frac{1}{540} + \frac{1}{113400} - \frac{1}{47628000} + \frac{1}{33949238400} - \frac{691}{25487390728800000} + \frac{1}{56800470767040000} \cdots\right)$$

In this way we can rapidly evaluate the original sum to great precision as
0.517808649195312761304466520936669986691589472200515434985657845217366601694 99...
but I don't recognize this number ...

At the endpoints, series (b) will also diverge because the individual terms do not even go to zero — every one is larger (in magnitude) than the one before it. Indeed, as noted above, the ratio of consecutive terms is, in magnitude, equal to $e/(1+\frac{1}{n})^n$, and this ratio is greater than one. (Its logarithm is $1 - n\log(1+1/n)$, but when $0 < x < 1$, $\log(x) < x$ so $\log(1+1/n) < 1/n$ and thus $\log(|a_{n+1}/a_n|) > 0$ so $|a_{n+1}/a_n|) > 1$.)

2. Compute the following limit, or show that it does not exist:

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right)$$

ANSWER: The limit is $\frac{1}{2}$, as shown by the following transformations:

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln(x)}\right) &= \lim_{x\to 1}\frac{x\ln(x) - x + 1}{(x-1)\ln(x)} \\
&= \lim_{x\to 1}\frac{\ln(x)}{\frac{(x-1)}{x} + \ln(x)} \text{ by L'H. Rule} \\
&= \lim_{x\to 1}\frac{x\ln(x)}{(x-1) + x\ln(x)} \\
&= \lim_{x\to 1}\frac{1+\ln(x)}{2+\ln(x)} \text{ by L'H. Rule again}
\end{aligned}$$

and this last limit is 1/2. (This example was taken from Wikipedia.)

3. Compute the first four terms $a_0 + a_1x + a_2x^2 + a_3x^3$ of the Maclaurin series (i.e. the Taylor series at 0) for

$$f(x) = \ln(1 - x + x^2)$$

ANSWER: Since $1 - x + x^2 = \frac{1+x^3}{1+x}$ we may write $f(x) = \ln(1+x^3) - \ln(1+x)$. But the Maclaurin series for natural logarithms is $\ln(1+x) = x - x^2/2 + x^3/3 - \ldots$ so also $\ln(1+x^3) = x^3 - x^6/2\ldots$ and thus our function expands as

$$f(x) = 0 - x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \ldots$$

The answer may also be obtained fairly easily by substituting $-x + x^2$ for x in the series for $\ln(1+x)$.

4. Find the equation of a plane that contains the points $P_1 = (1,3,4)$ and $P_2 = (1,2,3)$ and also forms a $60°$ angle with the plane $x + y - 2z = 6$. (There are two correct answers; you need find only one.)

ANSWER: The angle between two planes equals the angle between their normal vectors, so our normal vector v must be at a 60-degree angle to $u = \langle 1, 1, -2\rangle$. That means $\frac{u \cdot v}{||u||\,||v||} = \cos(60) = 1/2$. Our normal vector is also perpendicular to every vector in our plane, including the vector $P_2P_1 = \langle 0, 1, 1\rangle$.

So if our normal vector is, say, $v = \langle A, B, C\rangle$ then $B + C = 0$ and $A + B - 2C = (1/2)(\sqrt{6})(\sqrt{A^2 + B^2 + C^2})$. The first equation says $C = -B$; then squaring the second equation shows $(A + B - 2C)^2 = (6/4)(A^2 + B^2 + C^2)$, i.e. $(A + 3B)^2 = (3/2)(A^2 + 2B^2)$ or simply $A^2 - 12AB - 12B^2 = 0$. Thus we must have $A = (6 \pm 4\sqrt{3})B$ so that our normal vector $\langle A, B, C\rangle$ is parallel to $\langle 6 \pm 4\sqrt{3}, 1, -1\rangle$.

Then the plane is $(6 \pm 4\sqrt{3})x + y - z = D$ for some constant D, and we can compute the value of D since the points P_i are to be on the plane: $D = (6 \pm 4\sqrt{3}) + 2 - 3 = 5 \pm 4\sqrt{3}$. That is, the planes are $(6 \pm 4\sqrt{3})x + y - z = 5 \pm 4\sqrt{3}$.

N.B.: My apologies: I intended that the algebra would work out nicely for you, which it does when the cosine is $\sqrt{3}/2$ –i.e. when the angle is 30°, not 60°. I goofed! Sorry...

5. Find the point (x, y) on the ellipse $x^2 + 4y^2 = 74$ where the function $F(x, y) = (x + 12y) + (x + 12y)^3$ is largest.

ANSWER: First note that $F(x, y) = g(f(x, y))$ where $g(u) = u + u^3$ and $f(x, y) = x + 12y$. The reason to make this observation is that g is an increasing function of one variable (its derivative is $g'(u) = 1 + 3u^2 > 0$) and therefore the value of g is at its largest when its input is at its largest. Therefore we need only maximize $f(x, y)$ on the ellipse. But that is standard Lagrange Multipliers: you might want first to observe that f is continuous and the ellipse is "compact", which guarantees that there *is* some point on the ellipse where f will be maximized; then Lagrange Multipliers tells us where such a point must be. We want to maximize f on the set where $k(x, y) = 0$, where $k(x, y) = x^2 + 4y^2 - 74$, and so (since there are no places where f' or k' fail to exist) at the critical point, f' and k' will be parallel. These derivatives are $\langle 1, 12\rangle$ and $\langle 2x, 8y\rangle$ respectively, so at a critical point, $8y = 24x$. But if $y = 3x$ and $x^2 + 4y^2 = 74$ then $37x^2 = 74$, so $x = \pm\sqrt{2}$ and $y = 3x = \pm 3\sqrt{2}$. At these points, $f(x, y) = x + 12y = \pm 37\sqrt{2}$, so clearly the maximum value of f, and thus of F, is at $(x, y) = (\sqrt{2}, 3\sqrt{2})$.

You can also use Lagrange Multipliers directly on F; the vector $F'(x, y)$ is simply the vector $f'(x, y)$ stretched by a factor of $g'(f(x, y))$ and in particular F' and f' always point in the same direction. Using f instead of F' merely simplifies the algebra.