ALBERT A. BENNETT CALCULUS PRIZE EXAM May 9 2015

1. Let $g(x) = \dfrac{x}{(1-x^2)^2}$. Find $g^{(2015)}(0)$.

ANSWER The Maclaurin series for $1/(1-x^2)$ is $\sum_{n\geq 0} x^{2n}$. (This is a simple geometric series that converges iff $|x| < 1$.) Square this to see

$$1/(1-x^2)^2 = 1 + 2x^2 + 3x^4 + 4x^6 + \ldots + nx^{2n-2} + \ldots$$

Thus the series for $g(x)$ is $\sum_{n\geq 0} nx^{2n-1}$ and in particular the coefficient of x^{2015} is 1008. On the other hand, the coefficient of any x^k in this series is $g^{(k)}(0)/k!$, so $g^{(2015)}(0) = 1008 \cdot 2015!$.

2. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{4x}{x^4+4}\,dx$

ANSWER First find the antiderivative. You could use partial fractions; begin by factoring the denominator. (Find the four (imaginary) roots and multiply conjugate factors in pairs.) This gives $x^4 + 4 = (x^2 - 2x + 2) \times (x^2 + 2x + 2)$. Then the partial fractions decomposition of our integrand can be found to be

$$\frac{4x}{x^4+4} = \frac{1}{x^2-2x+2} - \frac{1}{x^2+2x+2}$$

so that the antiderivative can be found as a difference of two arctangents. But in this particular problem it's easier first to let $u = x^2$ to get a function that can be integrated directly: it's simply $\arctan(x^2/2)$.

Then the value of the improper integral is the limit of the integrals over intervals $[0, N]$, i.e. it's $\lim_{n\to\infty} \arctan(N^2/2) = \pi/2$.

3. Compute the first two coefficients a_0, a_1 of the Maclaurin series $a_0 + a_1x + a_2x^2 + \ldots$ for the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

For Extra Credit, compute the next coefficient a_2.

ANSWER Of course $a_0 = f(0) = 0$.

The "formula" way to compute the derivative of f at a is valid for $a > 0$. To compute $f'(0)$, however, we must use the definition of a derivative:

$$a_1 = f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

Here $f(0) = 0$. To compute this (two-sided) limit we consider both one-sided limits. The limit as h approaches 0 from the left is clearly 0 since $f(h) = 0$ in that case. On the other side we compute the limit with the substitution $h = 1/u$:

$$\lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{u\to+\infty} ue^{-u} = \lim_{u\to+\infty} \frac{u}{e^u} = 0$$

Thus $a_1 = 0$.

The next coefficient a_2 is also zero. We need to differentiate $k(x) = f'(x)$, which is obviously 0 when $x < 0$, has just been shown to equal 0 when $x = 0$, too, and for positive values of x is equal to $e^{-1/x} / x^2$. So to compute $f''(0)$ we must, as above, compute $k'(0)$ using the definition of the derivative: it's $\lim_{h\to 0} k(h)/h$, and we evaluate this limit just as was done in the previous paragraph.

REMARK: It can be shown in this way that ALL the derivatives $f^{(n)}(0)$ are zero, so that the Taylor series is simply $\sum_{n\geq 0} 0\, x^n$, which obviously does not converge to f in any neighborhood of 0. We say that the function f is *infinitely differentiable* but *not analytic.*

4. Does the series $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ converge? Why or why not?

ANSWER No. For all small x, $\sin(x) \approx x$; more precisely, $\lim_{x\to 0} \sin(x)/x = 1$. In particular, $\sin(1/n)$ is roughly $1/n$ and therefore this series will behave like the harmonic series: by the Limit Comparison Test, the two series either both converge or both diverge. As you know, the harmonic series diverges, so this one does too.

5. Does the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^6 + 12y^2}$ exist? Why or why not?

ANSWER The limit does exist; it's zero. It's not sufficient to check that the limit is zero along straight lines (although that is in fact what happens on any curve heading to the origin); indeed if we change the "4" to a "3" or smaller, then the limit will not exist.

Let $h = x^3, k = y/\sqrt{12}$; then $(x, y) \to (0, 0)$ iff $(h, k) \to (0, 0)$, so we may rewrite the limit as

$$\lim_{(h,k)\to(0,0)} \frac{h^{4/3} k}{\sqrt{12}(h^2 + k^2)}$$

Now switch to polar coordinates in the h, k plane: we are simply asking whether

$$\lim_{r\to 0} \frac{r^{7/3} \cos(\theta)^{4/3} \sin(\theta)}{\sqrt{12} r^2}$$

exists (where θ is allowed to vary as $r \to 0$). But $\cos(\theta)$ and $\sin(\theta)$ are never larger than 1 in magnitude, so the whole fraction is less than $r^{1/3}/\sqrt{12}$ and in particular it drops to 0 as we approach the origin.