ALBERT A. BENNETT CALCULUS PRIZE EXAM May 7 2016

1. Which is larger — e^π or π^e? You must answer without a calculator of course, and memorized digits are also useless unless you can explain how those digits are computed. Use some calculus to describe these numbers.

ANSWER Taking logs (to any base), we see that the question is whether $\pi \log(e) > e \log(\pi)$ or vice versa; dividing by both coefficients, we see we are asking whether $f(x) = \log(x)/x$ takes a larger value at π or e. In fact, f is differentiable on the positive reals, with derivative $(1 - \log(x))/x^2$, which is negative for $x > e$ and positive for $x < e$, so f achieves a maximum at $x = e$. In particular, $f(e) > f(\pi)$, so $\pi \log(e) > e \log(\pi)$, i.e. $e^\pi > \pi^e$. (The numbers are approximately 23.14069264 and 22.45915771, respectively.)

2. Compute the limit

$$\lim_{n\to\infty} \frac{1^4 + 2^4 + 3^4 + \ldots + n^4}{n^5}$$

ANSWER This is

$$\lim_{n\to\infty} \frac{1}{n}\left((\frac{1}{n})^4 + (\frac{2}{n})^4 + (\frac{3}{n})^4 + \ldots + 1^4 \right),$$

which is the Riemann Sum that results from dividing the interval $[0, 1]$ into n equal parts and evaluating the function $f(x) = x^4$ at the right-hand endpoint of each part. So the limit will yield the integral $\int_0^1 x^4\,dx = 1/5$.

REMARK There are formulas for the sum of the first few kth powers. You may already know that $\sum_{i=1}^n i = n(n+1)/2$, and perhaps even that $\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$. It's also true that $\sum_{i=1}^n i^3 = (n(n+1)/2)^2$, which is rather memorable. You are certainly not expected to know the corresponding result for fourth powers, but here it is:

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

3. Compute $\displaystyle\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx$

ANSWER Let $x = u^6$; then $dx = 6u^5\,du$ and the indefinite integral is

$$\int \frac{6u^5}{u^3 + u^2}\,du = \int \frac{6u^3}{u+1}\,du = 6\int \frac{(v-1)^3}{v}\,dv,$$

where $v = u + 1$. Expanding this out gives

$$6 \int (v^2 - 3v + 3 - \frac{1}{v})\, dv = 2v^3 - 9v^2 + 18v - 6\ln(v)$$

$$= 2u^3 - 3u^2 + 6u - 6\ln(u+1) = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1 + \sqrt[6]{x}),$$

up to a constant, of course.

4. Compute

$$F(x) = \int_0^2 \frac{\partial}{\partial y}\left(\frac{x^y - 1}{\ln(x)}\right) dy$$

and

$$\int_0^1 F(x)\, dx = \int_0^1 \int_0^2 \frac{\partial}{\partial y}\left(\frac{x^y - 1}{\ln(x)}\right) dy\, dx$$

ANSWER By the Fundamental Theorem of Calculus,

$$F(x) = \int_0^2 \frac{d}{dy} f(y)\, dy = f(2) - f(0) = \frac{x^2 - 1}{\ln(x)}.$$

It would not be easy to compute $\int_0^1 F(x)\, dx$ directly, but we can view this as a double integral and use Fubini's Theorem. We first observe that the integrand is $\dfrac{1}{\ln(x)} \dfrac{\partial(x^y)}{\partial y} = x^y$, which we easily integrate over $x \in [0,1]$ to be $1/(y+1)$. Then the integral of this over $y \in [0,2]$ is $\ln(3)$.

REMARK This method of computing integrals like $\int_0^1 F(x) dx$ is called the "Feynman Trick". Richard Feynman was adroit at thinking of ways to introduce a second parameter like y into integrals of this type which would allow him to use Fubini's Theorem as shown above.

5. Describe the set of all points which are equidistant between the planes $x + y + 2z = 4$ and $2x + 5y + 5z = 10$.

ANSWER The distance from a point (x_0, y_0, z_0) to a plane $ax + by + cz = d$ is $|ax_0 + by_0 + cz_0 - d|/\sqrt{a^2 + b^2 + c^2}$. So we are looking for the points (x, y, z) for which $|x + y + 2z - 4|/\sqrt{6} = |2x + 5y + 5z - 10|/\sqrt{54}$, i.e. points for which $3(x + y + 2z - 4)$ and $2x + 5y + 5z - 10$ have the same absolute value. Thus they're either equal (i.e., $x - 2y + z = 2$) or negatives of each other (i.e., $5x + 8y + 11z = 22$). The solution set is the union of these two planes.