BENNETT CALCULUS PRIZE EXAM — ANSWERS May 8 2018

1. Compute (with explanation) the following limit, or show that it does not exist:

$$\lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{\sin(x)}$$

ANSWER: The limit is zero.

The function may be written as $\frac{x}{\sin(x)} \cdot (x \sin(1/x))$; the first factor approaches 1 as $x \to 0$. The second factor is bounded by $|x|$ and hence approaches zero, and so the product approaches zero too.

Note that L'Hôpital's Rule does not help. It considers the limit of the quotient of the derivatives, that is, it considers

$$\lim_{x\to 0} \frac{2x \sin(1/x) - \cos(1/x)}{\cos(x)}$$

and asserts that *if* this limit exists then the original limit will exist as well and the two limits are equal. But in fact this latter limit does *not* exist and therefore L'Hôpital's Rule is silent. (The denominator approaches 1 and the first part of the numerator approaches zero, but the second part of the numerator has no limit at zero because of oscillation.)

2. Compute the derivative of $f(x) = x^{x^x}$.

ANSWER: Since $f(x) = e^{\ln(x)\cdot x^x}$ we can use the Chain Rule and the Product Rule: $f'(x) = f(x) \cdot \left(\frac{1}{x} \cdot x^x + \ln(x)\frac{d}{dx}(x^x)\right)$. Similarly the derivative of $x^x = e^{x\ln(x)}$ may be computed as $x^x \cdot \left(\frac{1}{x}x + \ln(x)\right)$ so that $f'(x) = x^{x^x+x}\left(\frac{1}{x} + \ln(x) + \ln(x)^2\right)$.

3. Compute $\displaystyle\int \frac{\sin(t)+\cos(t)}{\sqrt{2\sin(t)\cos(t)}}\,dt$. (*Hint*: if $u = \sin(t) - \cos(t)$, what is u^2?)

Extra Credit: Use this idea to evaluate $\displaystyle\int \sqrt{\tan(t)}\,dt$ by first computing

$$\int \sqrt{\tan(t)} + \sqrt{\cot(t)}\,dt \quad \text{and} \quad \int \sqrt{\tan(t)} - \sqrt{\cot(t)}\,dt$$

ANSWER: If $u = \sin(t) - \cos(t)$, then $u^2 = 1 - 2\sin(t)\cos(t)$ and $du = (\sin(t) + \cos(t))\,dt$, so the integral becomes $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) = \arcsin(\sin(t) - \cos(t)) + C$. (Using the angle-addition formula, this may also be written as $\arcsin(\sqrt{2}\sin(t - \pi/4))$.)

If we expand the original integrand into two fractions and multiply by $\sqrt{2}$, we see we have shown that $\int \sqrt{\tan(t)} + \sqrt{\cot(t)}\, dt = \sqrt{2}\arcsin(\sin(t) - \cos(t))$. In the same way (using now $u = \sin(t) + \cos(t)$) we find that $\int \sqrt{\tan(t)} - \sqrt{\cot(t)}\, dt = \sqrt{2}\int \frac{-du}{\sqrt{u^2-1}} = -\sqrt{2}\ln\left(u + \sqrt{u^2-1}\right)$. Now add these two integrals and divide by 2 to see

$$\int \sqrt{\tan(t)}\, dt = \frac{1}{\sqrt{2}}\left(\arcsin(\sin(t) - \cos(t)) - \ln\left(\sin(t) + \cos(t) + \sqrt{2\sin(t)\cos(t)}\right)\right) + C$$

Alternatively, one could let $u = \sqrt{2\cot(x)}$ so that $u^2 = 2\cot(x)$ and $-4u\, du = 4\csc^2(x)\, dx = (4 + u^4)\, dx$; this allows the use of Partial Fractions:

$$\int \sqrt{\tan(x)}\, dx = \sqrt{2}\int \frac{dx}{u} = \frac{1}{\sqrt{2}}\int \frac{(-8)du}{4+u^4} = \frac{1}{\sqrt{2}}\left(\int \frac{u-2}{u^2-2u+2}du - \int \frac{u+2}{u^2+2u+2}du\right)$$

which happens to be the very integral involved in the day's Differential Equations exam!

4. Do these series converge or diverge? Explain.

$$(A)\ \sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}\right)^{-n} \qquad\qquad (B)\ \sum_{n=1}^{\infty}(-1)^n\frac{2+\cos(\pi n)}{n}$$

ANSWER: Both series diverge. In each case the Alternating Series Test does *not* apply because one of the three hypotheses of the theorem is not met.

In series (A), the terms strictly alternate and strictly decrease in magnitude, but the magnitudes do not approach zero: $\lim(1+\frac{1}{n})^n = e$ so our terms' magnitudes approach $1/e > 0$. Of course, a series whose terms do not approach zero cannot converge.

In series (B) the terms strictly alternate and approach zero, but do not steadily decrease in magnitude: since $\cos(\pi n) = (-1)^n$, the magnitudes are

$$1,\ \frac{3}{2},\ \frac{1}{3},\ \frac{3}{4},\ \frac{1}{5},\ \frac{3}{6},\ \frac{1}{7},\ \frac{3}{8},\ \ldots$$

Of course one might hope that the series might converge even if the Alternating Series Test fails to prove this, but no, the series will definitely diverge because the partial sums grow without bound: It's not hard to check that for every integer $n \geq 1$, the inequality $-1/(2n-1) + 3/(2n) \geq (1/2)/n$ holds, so the partial sums grow:

$$\left(\frac{-1}{1}\right) + \left(\frac{3}{2}\right) + \left(\frac{-1}{3}\right) + \left(\frac{3}{4}\right) + \ldots \left(\frac{-1}{2n-1}\right) + \left(\frac{3}{2n}\right) \geq \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

Since the harmonic series diverges (to $+\infty$) our partial sums will also diverge.

5. Find the volume of the intersection of the solid bounded by the cylinders $x^2+z^2=R^2$ and $y^2+z^2=R^2$

ANSWER: We cut the volume in half if we look only at the points for which $z>0$; in half again if we look only where $x>0$ and in half again by restricting to $y>0$. Finally we still have a figure symmetric across the plane $x=y$: half its volume is in the part where $y<x$ and the other half, where $x<y$.

So our volume will be 16 times the volume of the region that lies *above* the half-quadrant of the xy plane where $0<y<x$, and *below* the graph of $f(x,y)=\sqrt{R^2-x^2}$. (Since we are in the lower half of the first quadrant, the other surface $z=\sqrt{R^2-y^2}$ lies above this one, that is, here we are automatically within the second cylinder as soon as we are within the first.)

In short, our volume is $16\int_D \sqrt{R^2-x^2}\,dA$, where D is the triangle bounded by the lines $y=0,\ y=x$, and $x=R$. We compute this integral using Fubini's Theorem:

$$V=16\int_D \sqrt{R^2-x^2}\,dA=16\int_{x=0}^{x=R}\int_{y=0}^{y=x}\sqrt{R^2-x^2}\,dy\,dx=16\int_{x=0}^{x=R}x\sqrt{R^2-x^2}\,dx$$
$$=16(-1/3)(R^2-x^2)^{3/2}|_{x=0}^{x=R}=16R^3/3$$

That is, it's $2/3$ of the volume of the $(2R)\times(2R)\times(2R)$-box that contains our set.