Aspects of the Theory of Non-Relativistic Matter, Coupled to the Quantized Radiation Field

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In section 1.1, we introduce the standard model of non-relativistic quantum electrodynamics. In section 1.2, we present a simplified model which will be used for non-perturbative considerations in later sections.

1.1 The standard model of non-relativistic QED

The object of our analysis is a non-relativistic N-electron system, confined to static nuclei, and coupled to the quantized electromagnetic radiation field. The Hilbert space of pure states of the full system is given by the tensor product space

$$\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{H}_f \ .$$

$\mathcal{H}_{el}$ is the N-electron Hilbert space

$$\mathcal{H}_{el} = \Big(L^2(\mathbf{R}^3, d^3\vec{x}) \otimes \mathbf{C}^2 \Big)^{\otimes_a N} \ , \tag{1}$$

and $\mathcal{H}_f$ is the photon Fock space

$$\mathcal{H}_f := \mathcal{F} = \bigoplus_{n=0}^{\infty} \Big(L^2(\mathbf{R}^3, d^3\vec{k}) \otimes \mathbf{C}^2 \Big)^{\otimes_s n} \ . \tag{2}$$

The subscripts "el" and "f" stand for "electron" and "photon field". The antisymmetrized tensor product $\otimes_a$ in $\mathcal{H}_{el}$, and the symmetrized tensor product $\otimes_s$ in $\mathcal{H}_f$ account for the Fermi statistics of electrons, and the Einstein-Bose statistics of photons, respectively. The factors $\mathbf{C}^2$ account for the spin $\frac{1}{2}$ of electrons, and the two helicity states of photons. We use natural units, for which Planck's constant $\hbar = 1$ and the speed of light $c = 1$.

The transverse modes of the quantized radiation field are described by the vector potential $\vec{A}$ in the Coulomb gauge $\vec{\nabla}\vec{A} = 0$. At time $t = 0$, $\vec{A}$ is given by

$$\vec{A}(\vec{x}) = \sum_{\lambda=1,2} \int_{\mathbf{R}^3} \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega(\vec{k})}} \Big(\vec{\epsilon}_\lambda(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} a_\lambda^\dagger(\vec{k}) + \vec{\epsilon}_\lambda^{\,*}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} a_\lambda(\vec{k}) \Big) \ , \tag{3}$$

where $\omega(\vec{k}) = |\vec{k}|$ is the energy of a photon with momentum $\vec{k}$. $\vec{\epsilon}_\lambda$, $\lambda = 1,2$ are polarization vectors and satisfy

$$\vec{k} \cdot \vec{\epsilon}_\lambda(\vec{k}) = 0 \ , \quad \lambda, \mu = 1,2 \ .$$

due to the Coulomb gauge. The triple $\{\vec{\epsilon}_1, \vec{\epsilon}_2, \frac{\vec{k}}{|\vec{k}|}\}$ constitutes a right-handed orthonormal basis. Consequently,

$$\vec{\epsilon}^{\,*}_\lambda(\vec{k}) \cdot \vec{\epsilon}_\mu(\vec{k}) \;=\; \delta_{\lambda\mu}\,, \quad \lambda,\, \mu \;=\; 1, 2 \;\;.$$

$a^\dagger_\lambda(\vec{k})$ and $a_\lambda(\vec{k})$ are creation and annihilation operators and satisfy the canonical commutation relations

$$[a^\sharp_\lambda(\vec{k}_1), a^\sharp_\mu(\vec{k}_2)] \;=\; 0\,, \quad [a_\lambda(\vec{k}_1), a^\dagger_\mu(\vec{k}_2)] \;=\; \delta_{\lambda\mu}\delta(\vec{k}_1 - \vec{k}_2) \quad , \tag{4}$$

where $a^\sharp = a$ or $a^\dagger$. $\vec{A}(\vec{x})$, $a^\dagger_\lambda(\vec{k})$, and $a_\lambda(\vec{k})$ are unbounded operator-valued distributions on $\mathcal{F}$.

There exists a unique vacuum vector $\Omega_f \in \mathcal{F}$, such that $a_\lambda(\vec{k})\Omega_f = 0$ holds for all $\vec{k}$ and λ. The time evolution of the free radiation field is generated by the Hamiltonian

$$H_f \;=\; \sum_{\lambda=1,2} \int d^3\vec{k}\; \omega(\vec{k})\; a^\dagger_\lambda(\vec{k}) a_\lambda(\vec{k}) \;\;, \tag{5}$$

and the free time evolution of the quantized vector potential is given by

$$\vec{A}(t,\vec{x}) \;=\; e^{itH_f}\; \vec{A}(\vec{x})\; e^{-itH_f} \;\;. \tag{6}$$

The free electric and magnetic fields are defined by

$$\vec{E}(t,\vec{x}) \;=\; \partial_t \vec{A}(t,\vec{x}) \;\;, \quad \vec{B}(t,\vec{x}) \;=\; \vec{\nabla} \wedge \vec{A}(t,\vec{x}) \;\;. \tag{7}$$

Let κ be a smooth function in momentum space with support in a ball of radius $const \cdot \alpha m$, and let $\check{\kappa}$ denote its Fourier transform. $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ is the feinstructure constant. We impose a ultraviolet cutoff on the vector potential by the convolution product

$$\vec{A}_\kappa(\vec{x}) \;=\; (\vec{A} * \check{\kappa})(\vec{x}) \tag{8}$$

in coordinate space. This operation is justified by the non-relativistic nature of the phenomena that are investigated.

The time evolution of the full quantum mechanical system of electrons and radiation field is generated by the Pauli Hamiltonian

$$\begin{aligned} H_{Pauli} \;=\; & \sum_{j=1}^{N} \frac{1}{2m} \left(\frac{1}{i}\, \vec{\nabla}_j \;-\; e\, \vec{A}_\kappa(\vec{x}_j) \right)^2 \;-\; \sum_{j=1}^{N} \frac{e}{2m}\, \vec{\sigma}_j \vec{B}_\kappa(\vec{x}_j) \,\otimes\, \mathbf{1}_f \\ + \;\; & \alpha V(\vec{x}_1, .., \vec{x}_N) \,\otimes\, \mathbf{1}_f \,+\, \mathbf{1}_{el} \,\otimes\, H_f \;\;. \end{aligned} \tag{9}$$

e is the charge, and m is the mass of the electron. The magnetic field which acts on the j-th electron is given by

$$\vec{B}_\kappa(\vec{x}_j) \;=\; \frac{1}{i} \vec{\nabla}_j \wedge \vec{A}_\kappa(\vec{x}_j)\,. \tag{10}$$

$\vec{\sigma}_j = (\sigma^1_j, \sigma^2_j, \sigma^3_j)$ are Pauli matrices which act on the spin space of the j-th electron. The potential $V(\vec{x}_1, .., \vec{x}_N)$ is a sum of two-body Coulomb interactions

$$V(\vec{x}_1, .., \vec{x}_N) \;=\; -\sum_{i=1}^{N}\sum_{j=1}^{M} Z_j V(\vec{x}_i - \vec{R}_j) \,+ \sum_{1 \le i < j \le N} V(\vec{x}_i - \vec{x}_j)\,, \tag{11}$$

where $\vec{R}_j$ are the coordinates, and $-Z_j e$ is the charge of the j-th nucleus, with $j = 1, .., M$, and $V(\vec{x}) := |\vec{x}|^{-1}$.

1.2 Introduction of a simplified model

In sections 2 and 3, we will consider the full, physical Hamiltonian which has been presented above. In section 4 and 5, we will however use a simplified model to focus on the essential difficulties of the analysis. Hence we will introduce dimensionless coordinates in the proposed model to emphasize its perturbative nature, and we will neglect the subleading terms. For this purpose, we dilate the electron coordinates and the photon momenta separately, $(\vec{x}_j, \vec{k}) \mapsto (\eta\vec{x}_j, \mu\vec{k})$, by way of a unitary transformation U_1 on $\mathcal{H}$. One obtains

$$\begin{aligned} \mu H_1 &:= U_1 H_{Pauli} U_1 = \sum_{j=1}^{N} \frac{1}{2m\eta^2} \left[\vec{\sigma}_j \cdot \left(-i\vec{\nabla}_j - \eta\mu e \vec{A}_\kappa(\eta\mu\vec{x}_j) \right) \right]^2 \\ &+ \frac{\alpha}{\eta} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{-Z_j}{|\vec{x}_i - \eta^{-1}\vec{R}_j|} + \sum_{1 \le i < j \le N} \frac{1}{|\vec{x}_i - \vec{x}_j|} \right\} \otimes \mathbf{1}_f + \mu \mathbf{1}_{el} \otimes H_f \ . \end{aligned}$$

We make the choice $\frac{1}{2m\eta^2} = \frac{\alpha}{\eta} = \mu$, and obtain $\eta = \frac{1}{2m\alpha}$, $\mu = 2m\alpha^2$, $\eta\mu = \alpha$ and $\mu\eta e = \alpha e = 2\pi^{1/2}\alpha^{3/2}$. This results in

$$\begin{aligned} H_1 &= \sum_{j=1}^{N} \frac{1}{2m\eta^2} \left[\vec{\sigma}_j \cdot \left(-i\vec{\nabla}_j - 2\pi^{1/2}\alpha^{3/2} \vec{A}_\kappa(\eta\mu\vec{x}_j) \right) \right]^2 \\ &+ \left\{ \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{-Z_j}{|\vec{x}_i - \eta^{-1}\vec{R}_j|} + \sum_{1 \le i < j \le N} \frac{1}{|\vec{x}_i - \vec{x}_j|} \right\} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f \ . \end{aligned}$$

We will interpret η as a parameter independent of α, and for $\alpha = 0$, we obtain

$$H_{\alpha=0} = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f \ ,$$

where

$$H_{el} = \sum_{i=1}^{N} \left\{ -\Delta_j + \sum_{j=1}^{M} \frac{-Z_j}{|\vec{x}_i - \eta^{-1}\vec{R}_j|} \right\} + \sum_{1 \le i < j \le N} \frac{1}{|\vec{x}_i - \vec{x}_j|}$$

is the usual Hamiltonian of Schrödinger quantum mechanics. We assume that the charge distribution of the nuclei is concentrated around the origin in $\mathbf{R}^3$, and unitarily transform the Hamiltonian H, by way of $U_2 := exp[-i\tau\vec{A}_\kappa(\vec{0}) \cdot (\sum_{j=1}^{N} \vec{x}_i)]$. All terms in H are left unchanged by this Pauli-Fierz transformation, except for

$$U_2[-i\vec{\nabla}_j]U_2 = -i\vec{\nabla}_j + \tau\vec{A}_\kappa(\vec{0})$$

and

$$U_2 a_\lambda^\dagger(\vec{k}) U_2 = a_\lambda^\dagger(\vec{k}) + i\tau\vec{\epsilon}_\lambda(\vec{k}) \cdot [\sum_{j=1}^{N} \vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{x}_i] \frac{\kappa(\vec{k})}{\pi\sqrt{2\omega(\vec{k})}} \ .$$

We choose $\tau := 2\pi^{1/2}\alpha^{3/2}$, and H_1 is thus unitarily equivalent to

$$
\begin{aligned}
H_2 &:= U_2 H_1 U_2 \\
&= H_{el} \otimes \mathbf{1}_f + 2\pi^{1/2}\alpha^{3/2}\vec{E}_\kappa(\vec{0}) \cdot (\sum_{j=1}^N \vec{x}_i) + 4\pi c_\kappa \alpha^3 (\sum_{j=1}^N \vec{x}_i)^2 \otimes \mathbf{1}_f \\
&\quad + \sum_{j=1}^N \{ i4\pi^{1/2}\alpha^{3/2}\vec{\nabla}_j \cdot [\vec{A}_\kappa(\alpha\vec{x}_j) - \vec{A}_\kappa(\vec{0})] + 4\pi\alpha^3[\vec{A}_\kappa(\alpha\vec{x}_j) - \vec{A}_\kappa(\vec{0})]^2 \\
&\quad + 2\pi^{1/2}\alpha^{5/2}\vec{\sigma}_j \cdot \vec{B}(\alpha\vec{x}_j) + \mathbf{1}_{el} \otimes H_f \; .
\end{aligned}
$$

The quantity $c_\kappa := \pi^{-2}\int_0^\infty d^3\vec{k}\kappa^2(\vec{k})$ is a cutoff-dependent constant. The operator H_2 can be written in the form

$$
\begin{aligned}
H_2 &= [H_{el} + g^2 c_\kappa (\sum_{j=1}^N \vec{x}_i)^2 + g^2 \sum_{j=1}^N f(\alpha\vec{x}_i)] \otimes \mathbf{1}_f \\
&\quad + \mathbf{1}_{el} \otimes H_f + W_g \\
W_g &= gW_1 + g^2 W_2 \; ,
\end{aligned}
\tag{12}
$$

where $g := 2\pi^{1/2}\alpha^{3/2}$, $f(\vec{x}) := \frac{4}{\pi^2}\int d^3\vec{k}\, \frac{\kappa^2(\vec{k})\sin^2(\vec{k}\cdot\vec{x})}{\omega(\vec{k})}$, and

$$
\begin{aligned}
W_1 &= \sum_\lambda \int d^3\vec{k} [G_{10}(\vec{k},\lambda) \otimes a_\lambda^\dagger(\vec{k}) \; + \; G_{01}(\vec{k},\lambda) \otimes a_\lambda(\vec{k})] \\
W_2 &= \sum_{\lambda,\lambda'} \int d^3\vec{k} d^3\vec{k}' [G_{20}(\vec{k},\lambda;\vec{k}',\lambda') \otimes a_\lambda^\dagger(\vec{k}) a_{\lambda'}^\dagger(\vec{k}') \\
&\quad + \; G_{02}(\vec{k},\lambda;\vec{k}',\lambda') \otimes a_\lambda(\vec{k}) a_{\lambda'}(\vec{k}') \; + \; G_{11}(\vec{k},\lambda;\vec{k}',\lambda') \otimes a_\lambda^\dagger(\vec{k}) a_{\lambda'}(\vec{k}')] \, .
\end{aligned}
$$

The quantities G_{mn} are functions of $(\vec{k},\lambda)$ or $(\vec{k}',\lambda')$ respectively, with values in the operators on $\mathcal{H}_{el}$. One can verify that

$$
\begin{aligned}
& G_{10}(\vec{k},\lambda) := G_{01}(\vec{k},\lambda)^* := \\
& \qquad \frac{\kappa(\vec{k})}{\pi\sqrt{2\omega(\vec{k})}} \sum_{j=1}^N [i\omega(\vec{k})\vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{x}_j + i[e^{-i\alpha\vec{k}\cdot\vec{x}_j}]\vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{\nabla}_j \\
& \qquad\qquad + \frac{i\alpha}{2} e^{-i\alpha\vec{k}\cdot\vec{x}_j}[\vec{\sigma}_j \cdot (\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k}))]] \, ,
\end{aligned}
$$

$$
\begin{aligned}
& G_{20}(\vec{k},\lambda;\vec{k}',\lambda') := G_{02}(\vec{k},\lambda;\vec{k}',\lambda')^* := \\
& \qquad \frac{\kappa(\vec{k})\kappa(\vec{k}')}{2\pi^2\sqrt{\omega(\vec{k})\omega(\vec{k}')}} [\vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{\epsilon}_{\lambda'}(\vec{k}')] \sum_{j=1}^N [[e^{-i\alpha\vec{k}\cdot\vec{x}_j} - 1][e^{-i\alpha\vec{k}'\cdot\vec{x}_j} - 1]]
\end{aligned}
$$

$$
\begin{aligned}
& G_{11}(\vec{k},\lambda;\vec{k}',\lambda') := \\
& \qquad \frac{\kappa(\vec{k})\kappa(\vec{k}')}{2\pi^2\sqrt{\omega(\vec{k})\omega(\vec{k}')}} [\vec{\epsilon}_\lambda(\vec{k}) \cdot \vec{\epsilon}^{\,*}_{\lambda'}(\vec{k}')] \sum_{j=1}^N [[e^{-i\alpha\vec{k}\cdot\vec{x}_j} - 1][e^{-i\alpha\vec{k}'\cdot\vec{x}_j} - 1]] \; .
\end{aligned}
$$

The simplifications imposed on this model that we will use in sections 4 and 5 firstly consist of the frequency cutoff $\kappa(\vec{k})$, which has already been introduced in section 1.2. Then, we will neglect the coupling of the electron to the magnetic field, i.e. we will investigate a system of scalar electrons. Furthermore, we will treat photons as scalar particles, as well, and skip the summation over the polarization states, labelled by λ. Hence, we have to redefine the Hilbert space of the system. It shall now be given by

$$\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{H}_f ,$$

with

$$\mathcal{H}_{el} = L^2(X, dx) ,$$

where $X = \mathbf{R}^{3N}$ is the particle configuration space of the N-electron system, and with the Fock space of scalar photons

$$\mathcal{H}_f \equiv \mathcal{F} = \bigoplus_{n=0}^{\infty} L^2(\mathbf{R}^3, d^3\vec{k}) .$$

We will study Hamiltonians of the form

$$H_g := H_0 + W_g = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + W_g,$$

where H_{el} is a Schrödinger operator on $\mathcal{H}_{el}$

$$H_{el} := -\Delta_x + V(x),$$

and where the photon Hamiltonian is now

$$H_f := \int d^3\vec{k}\, a^\dagger(\vec{k})a(\vec{k}) .$$

The interaction term acquires the form

$$\begin{aligned} W_1 &= \int d^3\vec{k}[G_{10}(\vec{k}) \otimes a^\dagger(\vec{k}) + G_{01}(\vec{k}) \otimes a(\vec{k})] \\ W_2 &= \int d^3\vec{k} d^3\vec{k}'[G_{20}(\vec{k}) \otimes a^\dagger(\vec{k})a^\dagger(\vec{k}') + G_{02}(\vec{k}) \otimes a(\vec{k})a(\vec{k}') \\ &\quad + G_{11}(\vec{k}) \otimes a^\dagger(\vec{k})a(\vec{k}')] . \end{aligned}$$

We will furthermore assume that $V(x)$ is a superposition of Coulomb potentials, and scales like $V(e^\theta x) = e^{-\theta} V(x)$, hence we will drop the terms of order g^2 in (12). We will state the hypotheses on the coupling functions G_{mn} later, in section 4.2.

2 On the structure of the theory, and survey of results

In section 2.1, we will discuss the U(1) gauge invariance of the quantized theory and derive the Ward identities. In section 2.2, we will present the results of this work.

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2.1 Gauge invariance and Ward-Takahashi identities

In this section, we will use path integral methods to derive the Ward-Takahashi identities of non-relativistic quantum electrodynamics, which express the $U(1)$ gauge invariance of the theory on the quantum level. We restrict our analysis to one-electron states coupled to the quantized electromagnetic field, i.e. the case $N = 1$. We note that non-relativistic one-electron states can be described with field theoretical methods because U(1) gauge invariance implies particle number conservation in the low-energy limit. The action functional of this system is given by

$$S[\psi^*, \psi, A^\mu_\kappa, \eta^*, \eta, J^\mu] = \int d^4x \, \{L_{Pauli} \, + \, L_C \, + \, L_S\} \quad ,$$

where L, L_C, L_S are defined as follows:

$$\begin{aligned} L_{Pauli} \;&=\; \psi^*(x)\,(i\partial_t \,-\, eA_{\kappa,0}(x))\,\psi(x) \,-\, \psi^*(x)\frac{1}{2m}\left(\frac{1}{i}\vec{\nabla} \,-\, e\vec{A}_\kappa(x)\right)^2 \psi(x) \\ &+\; \frac{e}{2m}\,\psi^*(x)\,\vec{\sigma}\vec{B}_\kappa(x)\,\psi(x) \,+\, \frac{1}{2}A^\mu_\kappa(x)\left((\partial_t^2 - \Delta)\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)A^\nu_\kappa(x) \end{aligned} \tag{3}$$

is the standard Lagrangian of non-relativistic quantum electrodynamics. $\psi^*(p)$ and $\psi(p)$ are Grassmann field variables which transform like spinors. The bosonic field variables A^μ_κ, $\mu = 0, 1, 2, 3$ account for the ultraviolet cutoff photon field. L_{Pauli} is invariant with respect to the gauge transformations

$$\begin{aligned} \psi(x) \;&\to\; e^{-ie\chi_\kappa(x)}\psi(x)\,, \\ \psi^*(x) \;&\to\; e^{ie\chi_\kappa(x)}\psi^*(x)\,, && (4) \\ A^\mu_\kappa \;&\to\; A^\mu_\kappa + \partial^\mu\chi_\kappa(x)\;. && (5) \end{aligned}$$

The function χ_κ is defined by $\chi_\kappa(x) = (\chi * \check{\kappa})(x)$, where χ is a differentiable function satisfying $\sup_x|\chi(x)|$, $\sup_x|\partial_\mu\chi(x)| \;\leq\; 1$, and where κ is the ultraviolet cutoff function defined in section 1.1. These gauge transformations preserve the ultraviolet cutoff imposed on the radiation field, which follows from the momentum space representation of (5), $A^\mu(k)\kappa(k) \to A^\mu(k)\kappa(k) + i\,k^\mu\,\chi(k)\kappa(k)$. This is necessary to have a finite theory on the perturbative level.

Note that the four summands in L_{Pauli} are *individually* gauge invariant. Thus, they will all obtain different renormalizations, as will be shown in the next section.

As described in section 1, only the transverse part of the electromagnetic vector potential will be quantized. Thus, we choose to work in the *Coulomb gauge* $\vec{\nabla}\vec{A}_\kappa(x) = 0$, which is fixed by the Lagrangian

$$L_C \;=\; \frac{1}{2\,\alpha}(\vec{\nabla}\vec{A}_\kappa(x))^2\,, \tag{6}$$

where α is a finite parameter (not to be confused with the feinstructure constant). Of course, L_C is not gauge invariant. The Faddeev-Popov method cannot be used to fix the Coulomb gauge in abelian theory, because the ghost term in the action is independent of the gauge field and only results in an overall normalization factor of the partition function. The Lagrangian

$$L_S \;=\; \eta^*(x)\psi(x) \;+\; \psi^*(x)\eta(x) \;+\; \vec{A}_\kappa(x)\vec{J}(x)$$

couples the transverse electromagnetic field variable to the external source $\vec{J}(x)$, and the matter field variables to the anticommuting sources $\eta^*(x)$, $\eta(x)$.

The partition function $Z[\eta^*, \eta, \vec{J}]$ is obtained from the functional integral

$$Z[\eta^*, \eta, \vec{J}] \;=\; \int D\psi^*\, D\psi\, D\vec{A}\; e^{i\,S[\psi^*, \psi, A_{\kappa,\mu}, \eta^*, \eta, \vec{J}]}$$

over the field variables ψ^*, ψ, $\vec{A}$, and is the generating functional of the n-point functions of the theory. A mathematically rigorous treatment of fermionic functional integrals would require to define the field variables on a countable space. A standard method to achieve this is to introduce box normalization, i.e. periodic boundary conditions, such that momentum space becomes discrete. After performing functional integration in momentum space, the box normalization can be removed by letting the volume of the boxes go to infinity. However, we will restrict ourselves to the use of functional integrals on a formal level.

Gauge invariance of non-relativistic quantum electrodynamics requires the generating functional $Z[\eta^*, \eta, \vec{J}]$ also to be gauge invariant. This *consistency* condition is the source of powerful nonperturbative identities on the quantum level. As a consequence of the fact that the full action of the theory is not gauge invariant, we will obtain a set of constraints interrelating the n-point functions of the theory, the *Ward-Takahashi identities.* For infinitesimal gauge transformations, (4) reduces to

$$\psi(x) \;\to\; \psi(x) \;-\; ie\chi_\kappa(x)\psi(x) \;\;, \quad \psi^*(x) \;\to\; \psi^*(x) \;+\; ie\chi_\kappa(x)\psi^*(x)\,.$$

The term $\int d^4x\, L_{Pauli}$ in the action is gauge invariant, but $\int d^4x\, L_C \,+\, L_S$ produces an infinitesimal variation of the action functional

$$\delta S \;:=\; \int d^4x \left\{ \frac{1}{\alpha}(\vec{\nabla}\cdot\vec{A})\Delta\chi_\kappa \;-\; \vec{J}\cdot\vec{\nabla}\chi_\kappa \;-\; i\,e\,\chi_\kappa(\eta^*\psi \;-\; \psi^*\eta) \right\}$$

under gauge transformation. The integrand of the partition function $Z[\eta^*, \eta, \vec{J}]$ picks up an additional factor $\exp(i\,\delta S)$ which is approximately $1 \,+\, i\,\delta S$. Gauge invariance of the partition function thus implies that the expectation value of δS vanishes. Using the fact that χ_κ is arbitrary, we obtain

$$\int D\psi^*\, D\psi\, D\vec{A} \left\{ \frac{1}{\alpha}\Delta(\vec{\nabla}\cdot\vec{A}) \;-\; \vec{\nabla}\cdot\vec{J} \;-\; i\,e\,(\eta^*\psi \;-\; \eta\psi^*) \right\} e^{iS} \;=\; 0$$

after partial integration in δS. By definition of the partition function, this can be written as

$$-\frac{i}{\alpha}\Delta\left(\vec{\nabla}\cdot\frac{\delta Z}{\delta\vec{J}(x)}\right) \;-\; \vec{\nabla}\cdot\vec{J}(x)Z \;-\; e\left(\eta^*(x)\frac{\delta Z}{\delta\eta^*(x)} \;+\; \frac{\delta Z}{\delta\eta(x)}\eta(x)\right) \;=\; 0\;\;.$$

The minus sign in the last term is due to the anticommutation relation $\frac{\delta}{\delta\eta_r(x)}\psi_s(y) = -\psi_s(y)\frac{\delta}{\delta\eta_r(x)}$, where r and s are spinor indices. The generating functional of *connected* Feynman graphs is defined by $Z[\eta^*,\eta,\vec{J}] = e^{iW[\eta^*,\eta,\vec{J}]}$, for which

$$\frac{1}{\alpha}\Delta\left(\vec{\nabla}\cdot\frac{\delta W}{\delta\vec{J}(x)}\right) - \vec{\nabla}\cdot\vec{J}(x) - i\,e\left(\eta^*(x)\frac{\delta W}{\delta\eta^*(x)} + \frac{\delta W}{\delta\eta(x)}\eta(x)\right) = 0 \tag{7}$$

holds. The Legendre transform of W is the vertex function Γ, which is the generating functional of the *one-particle irreducible* graphs

$$\Gamma[\psi^*,\psi,\vec{A}] = W[\eta^*,\eta,\vec{J}] - \int d^4x[\eta^*(x)\psi(x) + \psi^*(x)\eta(x) + \vec{A}_\kappa(x)\vec{J}(x)]\,.$$

We substitute the relations

$$\begin{aligned}\tfrac{\delta\Gamma}{\delta\vec{A}(x)} &= -\vec{J}(x)\,, & \frac{\delta W}{\delta\vec{J}(x)} &= \vec{A}(x)\,,\\ \tfrac{\delta\Gamma}{\delta\psi(x)} &= \eta^*(x)\,, & \frac{\delta W}{\delta\eta^*(x)} &= \psi(x)\,,\\ \tfrac{\delta\Gamma}{\delta\psi^*(x)} &= -\eta(x)\,, & \frac{\delta W}{\delta\eta(x)} &= -\psi^*(x)\end{aligned}$$

in Eq. (7) and obtain

$$\frac{1}{\alpha}\Delta\left(\vec{\nabla}\cdot\vec{A}(x)\right) + \vec{\nabla}\cdot\frac{\delta\Gamma}{\delta\vec{A}(x)} - i\,e\frac{\delta\Gamma}{\delta\psi(x)}\psi(x) - i\,e\,\psi^*(x)\frac{\delta\Gamma}{\delta\psi^*(x)} = 0\ . \tag{8}$$

Functional differentiation of this result with respect to $\psi^*(x_1)$ and $\psi(x_2)$, and setting ψ^*, ψ, $\vec{A}$ equal to zero yields

$$-\nabla_x^j\cdot\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)\delta A^j(x)} = i\,e\,\delta(x-x_1)\,\frac{\delta^2\Gamma[0]}{\delta\psi^*(x)\delta\psi(x_2)} - i\,e\,\delta(x-x_2)\,\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x)}\,. \tag{9}$$

The proper one-photon vertex function $\Gamma_j^{(1)}(p,q,p')$ of the interacting system is defined by

$$\int d^4x d^4x_1 d^4x_2 e^{i(p'x_1 - px_2 - qx)}\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)\delta A^j(x)} = i\,e\,(2\pi)^4\delta(p'-p-q)\,\Gamma_j^{(1)}(p,q,p')\ ,$$

and the corresponding inverse electron propagator $G_{el}^{-1}(p)$ is given by

$$\int d^4x_1 d^4x_2 e^{i(p'x_1 - px_2)}\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)} = i\,(2\pi)^4\delta(p'-p-q)\,G_{el}^{-1}(p)\ .$$

The Fourier transform of Eq. (9) with respect to the coordinates x, x_1, x_2 finally yields the *first Ward-Takahashi identity*

$$q^i\Gamma_i^{(1)}(p,q,p+q) = G_{el}^{-1}(p) - G_{el}^{-1}(p+q)\ . \tag{10}$$

which reduces to the *first Ward identity*

$$\Gamma_i^{(1)}(p,0,p) = -\partial_{p_i}G_{el}^{-1}(p).$$

in the limit $q \to 0$. There are two interaction vertices which couple the electron to a single photon line, which originate from the term

$$
\begin{aligned}
S_I^{(1)} &:= \int d^4x \left[\frac{e}{m}\psi^*(x)\,\vec{A}(x)\,\frac{1}{i}\,\vec{\nabla}\,\psi(x) \,+\, \frac{e}{2m}\psi^*(x)\,\vec{\sigma}\,\vec{B}(x)\,\psi(x)\right] \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\left[\frac{e}{m}\psi^*(p+q)\,\vec{A}(q)\,\vec{p}\,\psi(p) \,+\, \frac{e}{2m}\psi^*(p+q)\,\vec{\sigma}\,\vec{B}(q)\,\psi(p)\right] \qquad (11)
\end{aligned}
$$

in the action. Expansion of $e^{iS_I^{(1)}}$ to first order in e shows that the tree level approximation of $ie\Gamma_i^{(1)}(p,q,p+q)$ is the sum of the interaction vertices $i\frac{e}{m}p_k$ and $-\frac{e}{2m}\vec{\sigma}\,\vec{B}$, which we will refer to as the $\vec{p}$-vertex and the $\vec{B}$-vertex for brevity. The $\vec{B}$-vertex can be written in the form $\sigma_{jk}q^jA^k(q)$ with $\vec{B}(q) = i\,\vec{q}\wedge\vec{A}(q)$ and

$$
\sigma_{jk} := \frac{i}{2}[\sigma_j,\sigma_k] \ .
$$

The renormalized vertices are of the form $f_1(p,q)p_k$ and $f_2(p,q)\sigma_{jk}q^j$, therefore we find

$$
ie\,\Gamma_i^{(1)}(p,q,p+q) \,=\, f_1(p,q)p_i \,+\, f_2(p,q)\sigma_{ji}q^j \ ,
$$

where the functions f_1 and f_2 have to be calculated in orders of e^2 using perturbation theory. The lhs of the first Ward-Takahashi identity produces a term $f_2(p,q)\sigma_{ji}q^jq^i$ which vanishes to all orders in e^2 due to the antisymmetry of σ_{ji} in the indices i,j. Hence, we find no implication of the first Ward-Takahashi identity on the correction of the $\vec{B}$-vertex. This is a consequence of the fact that the term $\psi^*(x)\vec{\sigma}\vec{B}(x)\psi(x)$ is by itself gauge invariant. In relativistic quantum electrodynamics, one also finds that the Ward-Takahashi identity does not make any prediction on the correction of the gyromagnetic ratio.

The remaining interaction term of the action is given by

$$
\begin{aligned}
S_I^{(2)} &:= \int d^4x \left[-\frac{e^2}{2m}\psi^*(x)\,\vec{A}(x)\cdot\vec{A}(x)\,\psi(x)\right] \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\left[-\frac{e^2}{2m}\psi^*(p+q)\,\vec{A}(l)\cdot\vec{A}(q+l)\psi(p)\right] \ . \qquad (12)
\end{aligned}
$$

It defines the vertex $-\frac{ie^2}{2m}\delta_{jk}$ which couples the electron propagator to *two* photon lines, and which we will therefore refer to as the "two-photon vertex". The proper two-photon vertex function $\Gamma_j^{(2)}(p,q,l,p')$ is defined by

$$
\int d^4xd^4x_1d^4x_2d^4y e^{i(p'x_1-px_2-qx-ly)}\frac{\delta^4\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)\delta A^j(x)\delta A^k(y)} =
i\,e^2\,(2\pi)^4\delta(p'-p-q-l)\,\Gamma_{jk}^{(2)}(p,q,l,p') \ .
$$

From functionally differentiating (8) with respect to $\psi^*(x_1)$, $\psi(x_2)$ and $\vec{A}(y)$, and setting ψ^*, ψ, $\vec{A}$ equal to zero, we obtain the *second Ward-Takahashi identity* in momentum space representation

$$
q^j\Gamma_{jk}^{(2)}(p,q,l,p+q) \,=\, \Gamma_k^{(1)}(p,l,p+l) \,-\, \Gamma_k^{(1)}(p+q,l,p+q+l) \ .
$$

In contrast to the first Ward-Takahashi identity, the second includes corrections of the $\vec{B}$-vertex. The limit $q \to 0$ determines the second Ward identity

$$\Gamma^{(2)}_{jk}(p, 0, l, p+l) \;=\; -\partial_{p_j}\Gamma^{(1)}_k(p, l, p+l) \;\; .$$

At tree level, this result is trivial to check. The tree level approximation of $\Gamma^{(2)}_{jk}(p, q, l, p+q)$ is $2 \cdot (-\frac{1}{2m}\delta_{jk})$. The factor 2 accounts for the two possible choices to attach the labels j and k to the two photon lines. The tree level approximation of $\Gamma^{(1)}_k(p, l, p+l)$ is the sum of $\frac{1}{m}p_k$ and $-\frac{1}{2im}\sigma_{jk}l^j$. The $\vec{B}$-vertex is independent of the electron momentum $\vec{p}$, hence it is clear that the second Ward identity holds to lowest order.

2.2 Survey of results

In section 3, we will renormalize the parameters of non-relativistic quantum electrodynamics in perturbation theory. Due to the increased number of interaction vertices as compared to relativistic quantum electrodynamics, we have to consider a much larger number of graphs. But due to gauge invariance of the theory, there are many amplitudes that cancel pairwise. A striking difference between non-relativistic and relativistic QED is the absence of charge renormalization in the non-relativistic case, since there is no positron production in the low-energy limit. In conclusion, we will determine the renormalization of the energy scale, of the electron mass, and of the magnetic momentum. In section 3.3, we will discuss the renormalization group flow of the electron mass.

In section 4, we will rigorously prove Fermi's golden rule for the simplified quantum mechanical model introduced in section 1.2. The main result of this analysis will be that the excited eigenstates of an unperturbed system of bounded electrons become unstable, when the interaction with an external quantized electromagnetic field is turned on. They become *resonances* at the presence of the radiation field. One of the main tools for our analysis is the dilation analyticity of the interacting Hamiltonian H_g. The method is based on Balslev-Combes theory [3] which has a long, successful history in the analysis of Schrödinger operators. By definition, resonances are singularities of the analytically continuated resolvent on the second sheet of the associated Riemann surface.

The location of the resonances will be probed by use of the so-called Feshbach map. It maps H_g to an operator which is isospectral to the analytically continuated resolvent for a piece of the spectrum in a small vicinity of an excited eigenstate of the free system. We will prove a sequence of fairly technical lemmata to show the applicability of the Feshbach map, and that it is invertible on a certain region lying in that small vicinity. The result of section 4 is that all resonances of H_g above the ground state are elements of the second sheet of the Riemann surface, spaced at a distance of order g^2 from the real axis. g is the small coupling constant between the electrons and the radiation field.

In section 5, we will use the results of section 4 to prove the exponential decay of resonances. Again, we will apply Balslev-Combes theory, and we will also take advantage of the analyticity properties of the resolvent.

3 Perturbative renormalization of non-relativistic QED

In section 3.1, the collection of Feynman rules of non-relativistic quantum electrodynamics will be completed. Section 3.2 is devoted to the one-loop renormalization of the parameters of the theory. In section 3.3, the renormalization group flow of the electron mass will be discussed.

3.1 Summary of Feynman rules

The interaction vertices of the theory have already been calculated in section 2.1. We will now derive the Feynman propagators of the free electron, and of the free photon field. The action functional of a free electron is given by

$$S_{0,\,el} \;=\; \int \frac{d^4p}{(2\pi)^4}\psi^*(p)\,\left\{p_0\;-\;\frac{|\vec{p}|^2}{2m}\right\}\;\psi(p)\;\;.$$

To obtain its Feynman propagator $\hat{G}_{el,\,0}$ in momentum space representation, we perform an infinitesimal Wick rotation of the time axis $t\;\rightarrow\;t\,e^{-i\delta}$, $\delta>0$. As a consequence, the p_0 axis is infinitesimally rotated to the opposite direction, $p_0\;\rightarrow\;p_0\,e^{i\delta}$, and $S_{0,\,el}$ is mapped to $S^{(\delta)}_{0,\,el}$. The electron propagator is obtained from the (formal) functional integral

$$i\hat{G}_{0,el}(p)\;=\;Z^{-1}_{0,\,el}\int D\psi^* D\psi\,\psi(p)\psi^*(p)\,e^{iS^{(\delta)}_{0,\,el}}\;\;=\;\frac{i}{p_0e^{i\delta}-\frac{|\vec{p}|^2}{2m}}$$

with $Z_{0,\,el}\;:=\;\int\;D\psi^* D\psi\;\;e^{iS^{(\delta)}_{0,\,el}}$. The effect of the factor $e^{i\delta}$ is equivalently obtained by a summand $i\epsilon$, yielding

$$i\hat{G}_{0,el}(p)\;=\;\frac{i}{p_0\;-\;\frac{|\vec{p}|^2}{2m}\;+\;i\,\epsilon}\;\;. \tag{13}$$

In non-relativistic quantum electrodynamics, the electron propagator has only one pole, not two as opposed to its relativistic counterpart. Thus, there exists only a single time-ordering such that the two-point function $\langle 0|T-\psi(x)\psi^*(y)|0\rangle$ does not vanish, $|0\rangle\in\mathcal{H}_{el}$ being the vacuum of the non-interacting system. As it should be, we obtain no positron production in the non-relativistic limit, and particle number conservation is granted.

The Feynman propagator of the radiation field in the Coulomb gauge is calculated from the action

$$S_{f,0}\;=\;\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left\{A^\mu_\kappa(k)\left((k_0^2-|\vec{k}|^2)\eta_{\mu\nu}-k_\mu k_\nu\right)A^\nu_\kappa(k)\;+\;\frac{1}{\alpha}(i\vec{k}\vec{A}_\kappa(k))^2\right\}\;\;.$$

Again, one must Wick rotate the time axis $t\;\rightarrow\;t\,e^{-i\delta}$, $\delta>0$, which results in $k_0\;\rightarrow\;k_0\,e^{i\delta}$. The action $S_{f,0}$ is mapped to $S^{(\delta)}_{f,\,0}$, and the photon propagator is obtained from the (formal) bosonic functional integral

$$i\hat{G}_{0,f}(k) = Z^{-1}_{0,f}\int D\vec{A}\,\vec{A}(k)\otimes\vec{A}(k)\,e^{iS^{(\delta)}_{f,0}}\;=\;-i\,\frac{\mathbf{1}+(\alpha-1)\vec{n}\otimes\vec{n}}{k_0^2e^{2i\delta}\;-\;|\vec{k}|^2}\,\Lambda(\vec{k})\,,$$

where $Z_{0,f} = \int D\vec{A}\, e^{iS^{(\delta)}_{f,0}}$ and $\vec{n} \equiv \frac{\vec{k}}{|\vec{k}|}$, and where $\Lambda(\vec{k}) := \kappa^2(\vec{k})$ imposes an ultraviolet cutoff. Again, we replace the factor $e^{2i\delta}$ by a term $i\epsilon$, which gives

$$i\hat{G}_{0,f}(k) = -i\, \frac{\mathbf{1} - \vec{n}\otimes\vec{n}}{k_0^2 - |\vec{k}|^2 + i\,\epsilon}\, \Lambda(\vec{k})\,. \tag{14}$$

for the choice $\alpha = 0$.

We summarize the Feynman rules of non-relativistic quantum electrodynamics in momentum space representation, before we turn to the perturbative renormalization of the physical parameters of the theory. Note that the raising and lowering of indices is performed by use of δ_{jk}, δ_j^k, and δ^{jk}.

$i\frac{1}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon}$ 

$-i\,\frac{\mathbf{1}-\vec{n}\otimes\vec{n}}{k_0^2 - |\vec{k}|^2 + i\,\epsilon}\,\Lambda(\vec{k})$

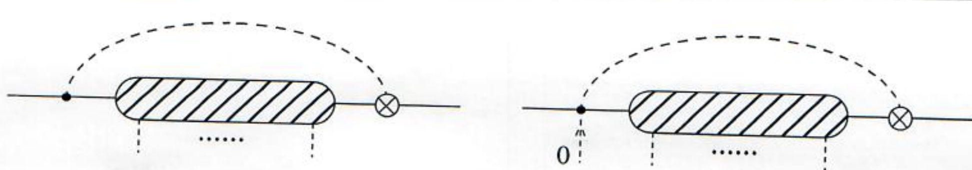
$i\frac{e}{m}p_j$ 

$-\frac{e}{2m}\sigma_{jk}q^j$ 

$-i\frac{e^2}{2m}\delta_{jk}$ 

3.2 Perturbative renormalization to one-loop level

In this section, we will calculate the one-loop radiative corrections of the parameters of non-relativistic quantum electrodynamics. The results derived here will only be relevant to quadratic order with respect to the dimensionless electron velocity $\frac{\vec{p}}{m}$, because the Pauli equation approximates the low energy limit of the Dirac equation only with this accuracy. Due to the absence of positron production, all one-loop graphs contain exactly one inner photon line. The Ward (-Takahashi) identities derived in the previous section will be used repeatedly to classify a number of gauge invariant families of graphs. However, we have shown in the last section that this procedure cannot be applied to the $\vec{B}$-vertex, because it is by itself gauge invariant. Instead, gauge invariance induces the pairwise cancellation of a large number of amplitudes, which we will show now. To be more specific, the amplitudes belonging to the following graphs annihilate the ones obtained from exchanging the outermost vertices (where the external momenta always flow from left to right):



Here "0" denotes vanishing external momentum at the outermost external photon leg. The reason for this cancellation is that the signs of the amplitudes reverse under exchange of the vertices adjacent to the inner photon line, which can be easily demonstrated on the level of Rayleigh-Schrödinger theory. For $kx = k_0 x^0 - \vec{k}\cdot\vec{x}$, the second quantized transverse electromagnetic vector potential is given by

$$\vec{A}(x) = \sum_\lambda \int \frac{d^4k}{(2\pi)^4} \left\{ a_\lambda(\vec{k})\, \vec{\epsilon}_\lambda(\vec{k})\, e^{ikx} \;+\; a^*_\lambda(\vec{k})\, \vec{\epsilon}_\lambda(\vec{k})\, e^{-ikx} \right\} ,$$

and the $\vec{B}$-field operator is obtained from taking its curl

$$\vec{B}(x) = \vec{\nabla} \wedge \vec{A}(x) = \sum_\lambda \int \frac{d^4k}{(2\pi)^4} \left\{ a_\lambda(\vec{k})\, (-i\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k}))\, e^{ikx} \;+\; a^*_\lambda(\vec{k})\, (i\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k}))\, e^{-ikx} \right\} .$$

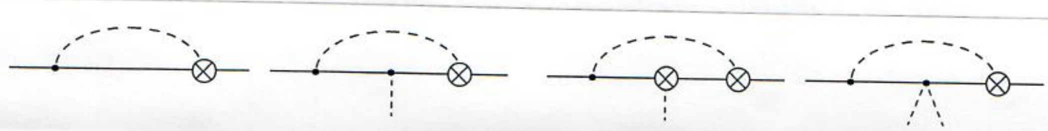
Consider first an arbitrary one-loop graph with the inner photon line adjacing to a $\vec{B}$-vertex and a $\vec{p}$-vertex. The amplitude is determined from summing over a term proportional to one of the expressions

$$\begin{aligned} &\langle \vec{p}, \Omega_f \mid \hat{\vec{A}}(k)\cdot \vec{p}_{el} \mid n \rangle \langle\, n \mid\; \vec{\sigma}\cdot \hat{\vec{B}}(k') \mid \vec{p}, \Omega_f \rangle\, , \\ &\langle \vec{p}, \Omega_f \mid \vec{\sigma}\cdot \hat{\vec{B}}(k) \;\mid n \rangle \langle\, n \mid \hat{\vec{A}}(k')\cdot \vec{p}_{el} \mid \vec{p}, \Omega_f \rangle \end{aligned}$$

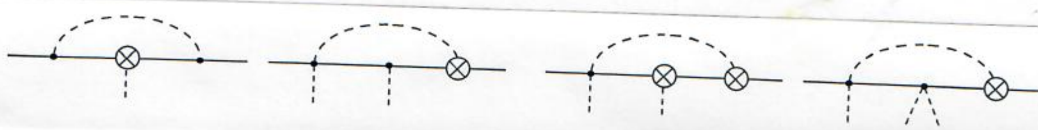
in momentum space, according to the two possible ways to arrange the vertices along the fermion line. $|n\rangle$ denotes an intermediate state in $\mathcal{H}_{el} \otimes \mathcal{H}_f$ with non-zero photon number. $\vec{p}_{el}$ stands for the momentum operator on the electron Hilbert space $\mathcal{H}_{el}$, and the vector $|\vec{p}, \Omega_f\rangle$ is the tensor product of a $\vec{p}_{el}$-eigenstate with the photon vacuum. The only nonzero Fourier components in these expressions are given by the terms

$$\begin{aligned} &\langle \vec{p}, \Omega_f \mid a_\lambda(\vec{k})\, (\vec{\epsilon}_\lambda(\vec{k})\cdot \vec{p}_{el}) \mid n \rangle \langle\, n \mid \vec{\sigma}\cdot (i\vec{k}' \wedge \vec{\epsilon}_\lambda(\vec{k}'))\, a^*_\lambda(\vec{k}') \mid \vec{p}, \Omega_f \rangle\, , \\ &\langle \vec{p}, \Omega_f \mid a_\lambda(\vec{k})\, \vec{\sigma}\cdot (-i\vec{k} \wedge \vec{\epsilon}_\lambda(\vec{k})) \mid n \rangle \langle\, n \mid\; (\vec{\epsilon}_\lambda(\vec{k}')\cdot \vec{p}_{el})\, a^*_\lambda(\vec{k}') \mid \vec{p}, \Omega_f \rangle\, , \end{aligned}$$

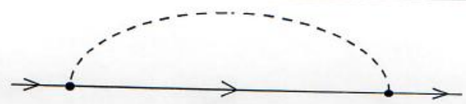
respectively. All other terms vanish since they include annihilation operators that act on the photon vacuum. Considering this last expression, it is obvious that the relative sign between the amplitudes in question is negative. The same argument can be applied if the $\vec{B}$-vertex couples over the inner photon line and the two-photon vertex to an external classical electromagnetic potential with zero momentum. For nonvanishing external momentum $\vec{q}$, gauge invariance implies that the resulting amplitudes must be proportional to $\sigma_{ik} q^i$, as has been explained in the previous section. Thus, they must vanish for momentum $\vec{q} = 0$, which is the explanation of the pairwise cancellations in question. In fact, many of these graphs produce pathological terms proportional to $\sigma_{ik} p^i$, where $\vec{p}$ is the external fermion momentum, which vanish when the amplitudes are grouped in pairs. In conclusion, the following one-loop graphs and their mirror pictures can be discarded.



Furthermore, the following graphs and their eventual mirror pictures can be verified to be negligible for small external electron momentum $\vec{p}$.



From the remaining, dominant graphs, we first consider



which contributes to the renormalization of the electron mass. Using the Feynman rules derived in the previous section, we find

$$i(-i)\left(i\frac{e}{m}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(\delta_{jk} - n_j\, n_k)p^j\, p^k}{(p_0 + k_0 - \frac{|\vec{p}+\vec{k}|^2}{2m} + i\epsilon)(k_0^2 - |\vec{k}|^2 + i\,\epsilon)}\, \Lambda(\vec{k})\ ,$$

where we have used the notation $\vec{n} \equiv \frac{\vec{k}}{k}$. The LSZ reduction formula in scattering theory states that the on-shell transition amplitude between two asymptotic one-particle states is obtained from fixing the external electron lines of the two-point function to the mass shell, yielding $p_0 = \frac{|\vec{p}|^2}{2m}$. It is tempting to do so now, especially in view of the amplitudes that are familiar from Rayleigh-Schrödinger theory. However, note that the Ward (-Takahashi) identities apply to the *general n-point functions*, which are defined for *arbitrary* tuples $(p_0, \vec{p})$. Applying the Ward (-Takahasi) identities to the on-shell amplitudes corresponds to the same error as stating that $\partial_y f(x,y)|_{x=g(y)}$ equals $\partial_y f(g(y),y)$ for arbitrary differentiable functions $f : I\!R^2 \to I\!R$ and $g : I\!R \to I\!R$. Thus, we have to fix the external electron lines to the mass shell *after* application of the Ward (-Takahashi) identities. In relativistic quantum electrodynamics, one is much less tempted to make this error due to relativistic covariance of the theory: One is less likely to treat p_0 any differently from $\vec{p}$.

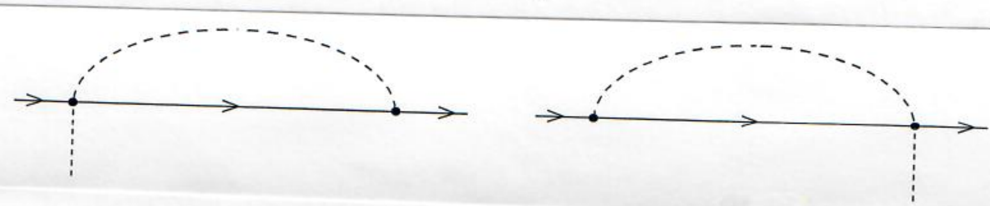
We can now continue our calculation. For simplicity, we assume $\Lambda(\vec{k})$ to be a step function that imposes a sharp ultraviolet cutoff at momentum λ. The integrand has two poles at positive $Re(k_0)$ in the lower half-plane, and one pole at negative $Re(k_0)$ in the upper half-plane. We choose a loop in the upper half-plane for the k_0-integration contour, which yields $\frac{2\pi i}{2|\vec{k}|}$ times the nonsingular factor of the integrand at $k_0 = -|\vec{k}|$. With a slight abuse of notation, where we write k for $|\vec{k}|$, we thus find

$$\frac{ie^2}{m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{(\delta_{jk} - n_j\, n_k)p^j\, p^k}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})}\, \Lambda(\vec{k})\ . \tag{15}$$

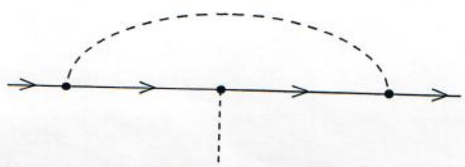
All amplitudes that will be computed in this section include the same k_0-contour integral that has just been demonstrated. Thus, we will always start directly at this step of the calculation in what follows. We can now apply the first Ward identity do derive the corresponding one- and two-photon amplitudes for zero external photon momentum. This is achieved by taking the derivative of (15) with respect to the external electron momentum $\vec{p}$, and multiplying the result by a factor $(-e)$. We obtain a sum of two terms

$$\frac{ie^3}{m^2}\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{2\,(\delta_{jk}-n_j\,n_k)\,p^k}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})}\,\Lambda(\vec{k}) \;+\; \frac{-ie^3}{m^3}\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{(\delta_{jk}-n_j\,n_k)p^j\,p^k\;(\vec{p}+\vec{k})}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})^2}\,\Lambda(\vec{k})\;.(16)$$

The first term is identical to the sum of the amplitudes of



and the second term is the amplitude of

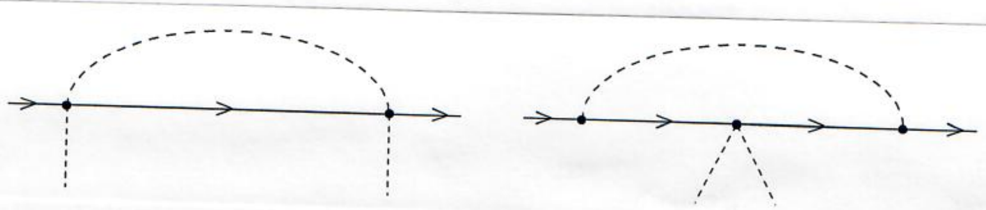


as can be checked by direct calculation, where a factor 2 must be accounted for each two-photon vertex, since there are two possibilities of labelling the adjacing photon lines. Because time reflection invariance is broken in the nonrelativistic limit, the flux direction of the incoming photon momentum (which we assume to be zero wherever we apply the Ward identities) is uniquely defined, and indicated by the arrows. From power counting, we conclude that the first term in (16) is logarithmically divergent, whereas the second one is finite, and of higer order in $\frac{\vec{p}}{m}$.

Using the second Ward identity for zero external photon momentum (i.e. deriving (16) with respect to $\vec{p}$, and multiplying by $(-e)$), we obtain four additional amplitudes, two of which can be discarded, since they correspond to higher order corrections in e. The two remaining terms

$$\frac{-ie^2}{m^2}\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{2\,(\delta_{jk}-n_j\,n_k)}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})}\,\Lambda(\vec{k}) \;+\; \frac{ie^2}{m^3}\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{(\delta_{jk}-n_j\,n_k)p^j\,p^k}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})^2}\,\Lambda(\vec{k})$$

are associated to the graphs



respectively. Again, the first term is logarithmically divergent, and the second term is finite.

We can now evaluate (15) for small external fermion momentum, setting $\vec{p}$ and $p_0 = \frac{|\vec{p}|^2}{2m}$ equal to zero, where the latter is now fixed to the mass shell. In this limit, the inverse mass of the electron is renormalized by the term

$$\frac{-i|\,\vec{p}\,|^2}{2m}\;\frac{2e^2}{3\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa}{1+\kappa} \;=\; \frac{-i|\,\vec{p}\,|^2}{2m}\;\frac{2e^2}{3\pi^2}\,log\left(1+\frac{\lambda}{2m}\right)\,, \tag{17}$$

which must be added to the inverse electron propagator $-ip_0 + i\frac{|\vec{p}|^2}{2m}$. In this calculation, we have substituted the new variable $\kappa := \frac{k}{2m} \in [0, \frac{\lambda}{2}]$, and have used the formula

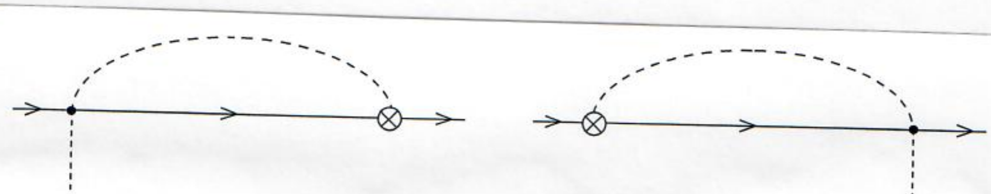
$$\int_{S^2} d\Omega_{\vec{k}}\,(\delta_{jk} - n_j\,n_k) \;=\; \frac{8\pi\delta_{jk}}{3}$$

in the numerator of the integrand. S^2 is the two dimensional unit sphere, $d\Omega_{\vec{n}}$ is the integration measure on S^2, and the subscript corresponds to the point located at the tip of $\vec{n}$. It can be shown in the same manner that the corresponding radiative corrections of the $\vec{p}$- and the two-photon vertex are given by

$$\frac{-ie\vec{p}}{m}\;\frac{2e^2}{3\pi^2}\,log\left(1+\frac{\lambda}{2m}\right)\quad,\qquad \frac{ie^2\delta_{jk}}{2m}\;\frac{2e^2}{3\pi^2}\,log\left(1+\frac{\lambda}{2m}\right)\,, \tag{18}$$

for small $\vec{p}$, which have to be added to the tree level vertices $\frac{ie\vec{p}}{m}$ and $\frac{-ie^2\delta_{jk}}{2m}$ respectively.

As explained in the previous section, it is not possible to derive the amplitudes associated to the graphs



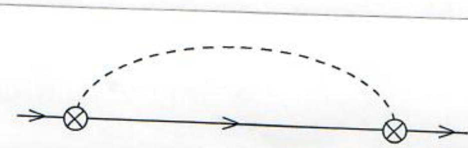
from the Ward (-Takahashi) identities. In order to calculate them directly from the Feynman rules, it is appropriate to consider pairs of graphs with opposite relative positions of the interaction vertices adjacing to the inner photon line, to obtain the desired radiative corrections proportional to $\sigma_{ik}q^i$. Moreover, a factor 2 arises due to the two-photon vertices, as explained above. Consequently, the total amplitude associated to these graphs is given by

$$2\times\frac{-e^3}{4m^2}\int\frac{d^3\vec{k}}{(2\pi)^3}\left\{\frac{-\vec{\sigma}\cdot(\vec{k}\wedge\vec{A}_{cl})}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})}+\frac{\vec{\sigma}\cdot(\vec{k}\wedge\vec{A}_{cl})}{2k(p_0-k-\frac{(\vec{p}+\vec{k}+\vec{q})^2}{2m})}\right\}\Lambda(\vec{k})\,.$$

After some algebra, and setting p_0 and $\vec{p}$ in the denominator equal to zero, we arrive at

$$\frac{e}{2m}\,\vec{\sigma}\cdot(\vec{q}\wedge\vec{A}_{cl})\,\frac{e^2}{3\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\;\;\kappa}{(1+\kappa)^2}\,. \tag{19}$$

The second dominant family of one-loop graphs is characterized by an inner photon line that joins two $\vec{B}$-vertices. The corresponding radiative correction of the electron propagator is given by



and is associated to the amplitude

$$i(-i)[-\frac{e}{2m}]^2 i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\sigma_{ri}k^r(\delta^{ij} - n^i n^j)\sigma_{sj}k^s}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})} \Lambda(\vec{k}) .$$

The expression in the numerator reduces to $\sigma_{ri}k^r(\delta^{ij} - n^i n^j)\sigma_{sj}k^s = -2k^2$, which simplifies the integral to

$$\frac{-ie^2}{4m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\ k^2}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})} \Lambda(\vec{k}) . \tag{20}$$

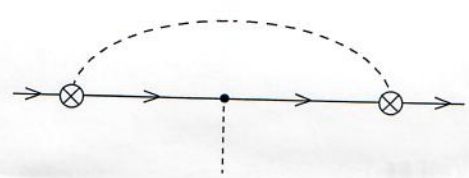
From power counting, we conclude that the result diverges *quadratically.* In the limit of small $\vec{p}$, where $\vec{p}$ and p_0 are set equal to zero in the denominator, we see immediately that the quadratically divergent term is independent of $\vec{p}$. Its value in this limit is given by $\frac{-ime^2}{\pi^2}\left(\frac{\lambda}{2m}\right)^2$, which is a shift of the electron energy scale denoted by p_0. However, Taylor expansion of (20) up to second order with respect to $\frac{\vec{p}}{m}$ generates a logarithmically divergent term,

$$\frac{i|\vec{p}|^2}{2m}\ \frac{e^2}{4m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\ k^2}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})^2} \Lambda(\vec{k}) ,$$

which contributes to the radiative correction of the inverse electron mass. Applying the first Ward identity, i.e. taking the derivative of (20) with respect to $\vec{p}$, and multiplying by $(-e)$, we find

$$\frac{ie^3}{4m^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\ k^2\ \ (\vec{p}+\vec{k})}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})^2} \Lambda(\vec{k}) ,$$

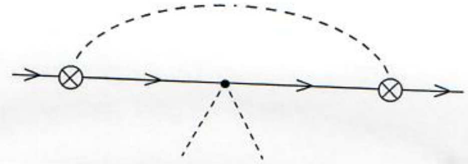
which is the amplitude of the graph



Application of the second Ward identity yields

$$\frac{-ie^4\delta_{jk}}{4m^3} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{2\ k^2}{2k(p_0 - k - \frac{(\vec{p}+\vec{k})^2}{2m})^2} \Lambda(\vec{k}) ,$$

which is the amplitude of

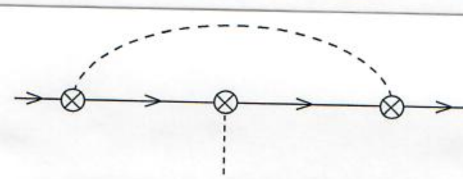


The second term that arises from the $\vec{p}$-derivative corresponds to a contribution of higher order in e, and has been dropped. After setting the external electron momentum equal to zero, and fixing its energy to the mass shell, we obtain the radiative corrections of the mass, the $\vec{p}$-vertex, and the two-photon vertex

$$\frac{i|\,\vec{p}\,|^2}{2m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ ,\quad \frac{ie\vec{p}}{m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ ,\quad \frac{-ie^2\delta_{jk}}{2m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ ,$$

respectively.

The correction of the $\vec{B}$-vertex belonging to this family of Feynman graphs



is "invisible" in regard of the Ward-Takahashi identities. Its amplitude is given by

$$\frac{-e^3\sigma_{jk}q^j}{8m^3}\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2\,k^2}{2k(p_0-k-\frac{(\vec{p}+\vec{k})^2}{2m})^2}\Lambda(\vec{k})\ \rightarrow\ \frac{-e\sigma_{jk}q^j}{2m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2} \tag{21}$$

in the limit of small external electron momentum. In contrast to (19), this correction of the $\vec{B}$-vertex is given by the *same* logarithmically divergent factor as the other graphs in its family. Thus, all logarithmically divergent one-loop corrections with two $\vec{B}$-vertices adjacing to the inner photon line contribute solely to the renormalization of the inverse electron mass $\frac{1}{m}$, yielding the contribution

$$\frac{1}{m}\frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}}\frac{d\kappa\ \kappa}{(1+\kappa)^2}\ =\ \frac{1}{m}\frac{e^2}{2\pi^2}\left(log\left(1+\frac{\lambda}{2m}\right)-\frac{1}{1+\frac{2m}{\lambda}}\right)\ .$$

For the total renormalization of the inverse electron mass, we thus obtain from this and (17)

$$\begin{aligned}\frac{1}{m}\ &\rightarrow\ \frac{1}{m}\left\{1\ -\ \frac{2e^2}{3\pi^2}\left[log\left(1+\frac{\lambda}{2m}\right)\right]\ +\ \frac{e^2}{2\pi^2}\left[log\left(1+\frac{\lambda}{2m}\right)-\frac{1}{1+\frac{2m}{\lambda}}\right]\right\}\\ &=\ \frac{1}{m}\left\{1\ -\ \frac{e^2}{6\pi^2}log\left(1+\frac{\lambda}{2m}\right)-\frac{e^2}{2\pi^2}\frac{1}{1+\frac{2m}{\lambda}}\right\}\ .\end{aligned} \tag{22}$$

The $\vec{p}$- and the two-photon vertex are renormalized by the same value, which is in agreement with the Ward (-Takahashi) identities. For this reason, the term $\frac{1}{2m}(\vec{p}-e\vec{A})^2$ in the Hamiltonian is renormalized by replacing $\frac{1}{m}$ with (22). The radiative correction of the gyromagnetic ratio of the electron is implicit in the renormalization of the $\vec{B}$-vertex, given by

$$\begin{aligned}\frac{-e\sigma_{jk}q^j}{2m} &\rightarrow \frac{-e\sigma_{jk}q^j}{2m}\left\{1 - \frac{e^2}{3\pi^2}\int_0^{\frac{\lambda}{2m}} \frac{d\kappa\ \kappa}{(1+\kappa)^2} + \frac{e^2}{2\pi^2}\int_0^{\frac{\lambda}{2m}} \frac{d\kappa\ \kappa}{(1+\kappa)^2}\right\} \\ &= \frac{-e\sigma_{jk}q^j}{2m}\left\{1 + \frac{e^2}{6\pi^2}log\left(1+\frac{\lambda}{2m}\right) - \frac{e^2}{6\pi^2}\frac{1}{1+\frac{2m}{\lambda}}\right\},\end{aligned}$$

due to (19) and (21). We observe that the difference between the renormalization of the $\vec{B}$-vertex and the inverse electron mass is a *finite* additive term

$$\frac{e^2}{3\pi^2}\frac{1}{1+\frac{2m}{\lambda}},$$

which originates from the renormalization of the gyromagnetic ratio of the electron. If we choose the photon energy cutoff to be given by the electron rest energy, $\lambda = m$, as Bethe did in his original derivation of the Lamb shift, we obtain the value

$$\frac{e^2}{9\pi^2}.$$

This is a fairly good approximation of the famous relativistic result derived by Schwinger, $\frac{\alpha}{2\pi} = \frac{e^2}{8\pi^2}$. It shows, as Bethe's calculation of the Lamb shift also does, that the seemingly ad-hoc choice of the cutoff $\lambda = m$ does have physical significance in non-relativistic quantum electrodynamics. The usual argument for choosing this value is that the relevant part of the energy spectrum in the non-relativistic limit is located close to the point on the mass shell which corresponds to the electron rest mass. Thus, all energies involved in the theory can exceed the electron rest energy only by a small fraction, even in case of *virtual* states, since otherwise, the non-relativistic limit would simply not be valid. The cutoff imposed due to this heuristic argument leads to results which are in very good agreement with experimental data, i.e. the Lamb shift and the anomalous magnetic moment.

3.3 Renormalization group flow of the electron mass

Using the results derived in the previous section, we will now analyze the renormalization group flow of the electron mass. Note that due to the absence of charge renormalization in the non-relativistic limit, it is the mass renormalization that accounts for the qualitative behaviour of the interaction for varying energy scales. According to the standard renormalization procedure, the renormalized mass (22) is given the (constant) physical value m_R of the electron mass, hence

$$\frac{1}{m_R} = \frac{1}{m_B(\lambda)}\left\{1 - \frac{3\alpha}{4\pi}\log\left(1+\frac{\lambda}{2m_B(\lambda)}\right)\right\}. \tag{23}$$

This equation implicitly defines the dependency of the *bare mass* $m_B(\lambda)$ on the cutoff frequency λ. Put more precisely, the invariance of this equation with respect to redefinitions of the renormalization point λ induces the *renormalization group flow* of the bare mass. We obtain the Callan-Symanzik equation for the flow of the bare mass by applying the operator ∂_λ to both sides of (23), which yields

$$0 = -\frac{\partial_\lambda m_B(\lambda)}{m_B(\lambda)} \left\{ 1 - \frac{3\alpha}{4\pi} log \left(1 + \frac{\lambda}{2m_B(\lambda)} \right) - \frac{3\alpha}{4\pi} \frac{1}{2m_B(\lambda) + \lambda} \right\} - \frac{3\alpha}{4\pi} \frac{1}{2m_B(\lambda) + \lambda}$$

after some rearrangement of the summands. Since (23) implies that $m_B(\lambda) = m_R + O(\alpha)$, we observe that this equality is of the form

$$0 = -\frac{\partial_\lambda m_B(\lambda)}{m_B(\lambda)} (1 + O(\alpha)) - \frac{3\alpha}{4\pi} \left(\frac{1}{2m_R + \lambda} + O(\alpha) \right) .$$

Only keeping the leading order terms of order $O(1 = \alpha^0)$ in the brackets, we obtain the approximate flow equation for the mass

$$\frac{\partial_\lambda m_B(\lambda)}{m_B(\lambda)} = -\frac{3\alpha}{4\pi} \frac{1}{2m_R + \lambda} .$$

Integration with respect to λ results in

$$log(m_B(\lambda)) = -\frac{3\alpha}{4\pi} log\,(2m_R + \lambda) + const ,$$

which finally yields

$$m_B(\lambda) = m_R \left(\frac{3m_R}{2m_R + \lambda} \right)^{\frac{3\alpha}{4\pi}}$$

for the initial condition $m_B(\lambda = m_R) := m_R$. The renormalization group equation for the bare mass is a consequence of the invariance of the theory with respect to redefinitions of the renormalization point. The *physical* implication is that it also expresses the scale transformation invariance of the theory. This is because a change of the momentum scale by a factor s results in a shift of the renormalization point by s. Since m_R is invariant with respect to this scale transformation and to the corresponding shift of the renormalization point, one recovers the Callan-Symanzik equation derived above. Assuming that a certain value has been attributed to $m_B(\lambda_0)$ at some fixed renormalization point λ_0, the *running mass* $m(s) \equiv m_B(s\lambda_0)$ shows how the electron mass of the interacting system varies effectively for a change of the momentum scale. In our case, $\lambda_0 = m_R$ implies that $s \in [0, 1]$.

The scaling limit of $m(s)$ for $s \to 0$ is given by

$$m(0) \;=\; m_R \left(\frac{3}{2} \right)^{\frac{3\alpha}{4\pi}} \approx m_R \left\{ 1 + \frac{3\alpha}{4\pi} log \left(\frac{3}{2} \right) \right\} , \qquad (24)$$

which is (23), solved for $m_B(\lambda)$ with the substitution $\lambda \to m_R = m_B(\lambda = m_R)$. The limit $s \to 0$ in momentum space is equivalent to the scaling limit $\frac{1}{s}\vec{x} \to \infty$ in the Euclidean $\mathbf{R}^3$, i.e. the limit of *macroscopic* dimensions. Thus, (24) corresponds to the *macroscopically* observed, effective mass of an electron that is spatially restricted to a *microscopic* system. As an example, consider bound state electrons confined to atomic orbits. The reference length scale at which $m(s=1) = m_R$ is fixed, is the Bohr radius, whereas the macroscopic experimenter makes his observation at a (relative) scale $s \rightarrow 0$. Therefore, the experimentally observed effective mass in this case is the one given in (24), and not the classical electron mass m_R, as is exemplified by the Lamb shift.

As a contrasting example, measurement of the Compton process involves free electrons upon which no intrinsic microscopic length scale is imposed, in the contrary, the electrons follow paths of macroscopic extension. Accordingly, the experimenter does not measure any electron mass that differs from its classical value (in the Compton experiment, the momentum and the velocity of the electron are both measurable, hence the mass is known).

In conclusion, we summarize some phenomenological issues on the observation of quantum processes and the renormalization group. The prescription of attributing physical values to the renormalized parameters of a quantum field theory shows that the relevant scale of the quantum process of interest defines an *intrinsic* scale of the problem. The correspondence to the length scale that is intrinsic to the observer is induced by the renormalization group flow of these parameters. Similar to active and passive transformations of coordinate systems, the renormalization group equations express the physical self-similarity of a quantum field theory under "active" scale transformations, as well as the invariance of the theory with respect to "passive" changes of the renormalization point. Here one finds a "principle of relative scales" which is very similar to the principle of relativistic covariance. In special relativity, the only sensible way to specify the intrinsic length of an object or the intrinsic duration of a process is to do so with respect to the *rest frame* of the given system. In all other frames, the observed values vary with the ratio of relative velocities. In analogy, the parameters of a quantum field theoretic system are given the classical, *"physical"* value at the intrinsic or "proper" scale of the problem. The observed values are functions of the relative scale difference between the quantum system and the observer system. In case of large discrepancies between these scales, the observed parameter values are *scaling limits* under the renormalization group flow.

4 Radiation theory of atoms and molecules

In section 4.1, we introduce conceptual facts about resonances and dilation analyticity. In section 4.2, we verify that Balslev-Combes theory can be applied to the given problem. In section 4.3, we introduce the Feshbach map, and in section 4.4, we prove Fermi's golden rule. The presented proof is an extended version of the one given in [5]. In the following, we will consider the simplified model introduced in section 1.2.

4.1 Resonances and dilation analyticity

Our intention is to study the fate of excited energy states of the N-electron Hamiltonian H_{el} from section 1.2, if the coupling to the radiation field is turned on. These eigenvalues are embedded in the continuous spectrum of $H_0 = H_{el} + H_f$, but become resonances at the presence of W_g, as we will prove in section 4.4. The location of the resonance energies are predicted by Bethe's calculation of the Lamb shift, and by Fermi's golden rule.

For the spectral analysis of H_g, we will use the framework of Balslev-Combes theory [3]. Let $U_{el}(\theta)$ be the one-parameter group of unitary dilations on the N-electron Hilbert space $\mathcal{H}_{el} = (L(\mathbf{R}, d^3x))^{\otimes_a N}$, given by

$$(U_{el}(\theta)\phi)(\vec{x}_1, .., \vec{x}_N) = e^{\frac{3N\theta}{2}} \phi(e^{\theta}\vec{x}_1, .., e^{\theta}\vec{x}_N), \quad \phi \in \mathcal{H}_{el}, \quad \theta \in \mathbf{R}.$$

Due to the factor $e^{\frac{3N\theta}{2}}$, $U_{el}(\theta)$ becomes unitary. Let $C_0(\mathbf{R}^3)$ denote the one-photon space of continuous and compactly supported functions. One defines the one-parameter group of unitary dilations $U_f(\theta)$ on $C_0(\mathbf{R}^3)$ by

$$(U_f(\theta)f)(\vec{k}) = e^{-\frac{3\theta}{2}} f(e^{-\theta}\vec{k}), \quad f \in C_0(\mathbf{R}^3), \quad \theta \in \mathbf{R} ,$$

and the spectral deformation on the Fock space $\mathcal{F}$ given by

$$U_f(\theta)\, a^\dagger(f)\, U_f(\theta)^{-1} = a^\dagger(U_f(\theta)f) .$$

Accordingly, we define the one-parameter group of unitary dilations $U(\theta)$ on the full Hilbert space $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{H}_f$ by

$$U(\theta) = U_{el}(\theta) \otimes U_f(\theta) . \tag{40}$$

$U(\theta)$ can be represented as $e^{\theta A_{dil}}$, where

$$A_{dil} = A_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes A_f, \tag{41}$$

is the anti-self-adjoint infinitesimal generator of dilations, and in which

$$\begin{aligned} A_{el} &= \frac{1}{2} \sum_{j=1}^{n_{el}} \left(\vec{p}_j \nabla_{\vec{p}_j} + \nabla_{\vec{p}_j} \vec{p}_j \right) \\ A_f &= \frac{1}{2} \sum_{\lambda} \int d^3\vec{k}\, a^\dagger_\lambda(\vec{k}) \left(\vec{k} \nabla_{\vec{k}} + \nabla_{\vec{k}} \vec{k} \right) a_\lambda(\vec{k}) \end{aligned}$$

are the dilation generators on $\mathcal{H}_{el}$ and $\mathcal{H}_f$, respectively. A vector $\psi \in \mathcal{H}$ is dilation analytic if there exists a vector valued continuation of $U(\theta)\psi$ to a strip $|Im(\theta)| < a$. To use dilations for spectral analysis, one considers the family of operators $A(\theta) = U(\theta)AU(\theta)^{-1}$, associated to a given operator A on $\mathcal{H}$. An operator A is said to be dilation analytic, or of class $\mathcal{F}_a$, if $A(\theta)$ can be analytically continuated to $|Im(\theta)| < a$.

We will now give a definition of resonances by use of Balslev-Combes theory. It will be shown in section 4.2 that H_g is dilation analytic. Anticipating this result, we consider

$$\langle \psi_2, (H_g - z)^{-1}\psi_1 \rangle = \langle U(\theta)\psi_2, (H_g(\theta) - z)^{-1} U(\theta)\psi_1 \rangle \,, \; \psi_1, \psi_2 \in \mathcal{H} \,, \; \theta \in \mathbf{R} \tag{42}$$

which holds due to the unitarity of $U(\theta)$. Assuming dilation analyticity for the vectors ψ_1, ψ_2, we continuate θ into a complex neighborhood $\mathcal{O} \subset \mathbf{C}^+$ of $\{0\}$ (for a discussion of domain problems, see [5]). The identity (42) holds for $\theta \in \mathcal{O}$, since it holds for all real θ. The rhs of (42) can be analytically extended from $\mathbf{C}^+$ over the real axis to the part of the resolvent set of $H_g(\theta)$ which lies in $\mathbf{C}^-$. Hence, one obtains an analytic continuation of the lhs of (42). Since the thresholds and eigenvalues of $H_g(\theta)$ do not depend on θ, resonances can be defined as the complex eigenvalues of $H_g(\theta)$.

The imaginary parts of these eigenvalues are specified by Fermi's golden rule, which will be proved in section 4.4. It states that all eigenvalues of $H_g(\theta)$ except for the ground state energy have non-vanishing imaginary parts.

4.2 Dilation analyticity of the Hamiltonian H_g

We consider the Hamiltonian $H_g = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f + W_g$, and show dilation analyticity of its three constituents individually.

The kinetic energy operator $T = -\Delta_x$, $x := (\vec{x}_1, .., \vec{x}_N)$, in $H_{el} = T + V(x)$ is dilation analytic, with $T(\theta) = e^{-2\theta}T$. The potential V has been assumed to be a sum of two-body Coulomb interactions in section 1.2. $V(x)$ thus scales like $V(\theta) = e^{-\theta}V$, and is dilation analytic. The discrete spectrum $\sigma_d^{(el)}$ of $H_{el}(\theta)$, the thresholds, and the resonances are independent of θ. The continuous spectrum $\sigma_{ac}^{(el)}(H_{el})$ is modified by complex dilation, and will consist of branches of continuous spectrum which emanate from the thresholds, and which are rotated by an angle 2ϕ into $\mathbf{C}^-$, where $\phi := Im(\theta) > 0$. Thus one has

$$H_{el}(i\phi) = -e^{-2i\phi}\Delta_x + e^{-i\phi}V(x) \tag{43}$$

for $\theta = i\phi \in \mathbf{C}^+$. Spectral analysis of N-body Schrödinger operators by use of Balslev-Combes theory has been subject to intensive study ever since the original paper [3] appeared.

The dilation of the Hamiltonian H_f of the radiation field requires the choice of a one-photon basis $\{f_i\}$ on $\mathbf{C}_0(\mathbf{R}^3)$, and to represent H_f in terms of $a^\dagger(f_i)$, $a(f_j)$ by way of

$$H_f = \sum_{i,j} \langle f_i, \omega f_j \rangle \, a^\dagger(f_i) a(f_j) \ .$$

Using $\langle f_i, \omega f_j \rangle = \int d^3\vec{k} \bar{f}_i(\vec{k}) \omega(\vec{k}) f_j(\vec{k})$ and the usual definition of $a^\sharp(f)$, one obtains that

$$H_f(\theta) = \sum_{i,j} \langle f_i, \omega_\theta f_j \rangle \, a^\dagger(f_i) a(f_j) \ ,$$

where $\omega_\theta(\vec{k}) := \omega(e^{-\theta}\vec{k})$. For the dispersion $\omega(\vec{k}) = |\vec{k}|$, one finds that $H_f(\theta) = e^{-\theta}H_f$. Note that since the modulus function $|\cdot| : \mathbf{R}^3 \to \mathbf{R}^+$ is not analytic, one has to extract the factor $e^{-\theta}$ from $|e^{-\theta}\vec{k}|$ before analytic continuation. The spectrum of H_f consists of a single eigenvalue $\{0\}$ at the bottom of the continuous spectrum, corresponding to the vacuum state Ω_f. There is no separation between the point spectrum and the continuous spectrum of H_f because photons are massless particles. For $\theta = i\phi \in \mathbf{C}^+$, the continuous spectrum of $H_f(i\phi)$ which emanates from $\{0\}$ is rotated by ϕ into $\mathbf{C}^-$.

The spectrum of the complex dilated, free Hamiltonian $H_{g=0}(i\phi)$ thus consists of branches of continuous spectrum of $H_f(i\phi)$, which emanate from each element of the spectrum of $H_{el}(i\phi)$, from each threshold, and from each resonance, cf. the following figure.

We have introduced the interaction Hamiltonian W_g at the end of section 1.2. The dilated $W_g(\theta) = gW_1(\theta) + g^2W_2(\theta)$ is characterized by the coupling functions $G^{(\theta)}_{m,n}$, $1 \leq m+n \leq 2$, which are defined by

$$\begin{aligned} W_1(\theta) &= \int d^3\vec{k}[G^{(\theta)}_{10}(\vec{k}) \otimes a^\dagger(\vec{k}) + G^{(\theta)}_{01}(\vec{k}) \otimes a(\vec{k})] \\ W_2(\theta) &= \int d^3\vec{k}d^3\vec{k}'[G^{(\theta)}_{20}(\vec{k}) \otimes a^\dagger(\vec{k})a^\dagger(\vec{k}') + G^{(\theta)}_{02}(\vec{k}) \otimes a(\vec{k})a(\vec{k}') \\ &\quad + G^{(\theta)}_{11}(\vec{k}) \otimes a^\dagger(\vec{k})a(\vec{k}')] \,. \end{aligned}$$

For $m+n=1$, one finds the scaling behaviour $G^{(\theta)}_{m,n}(\vec{k}) = e^{3\theta/2}G_{m,n}(e^{-\theta}\vec{k})$, and $G^{(\theta)}_{m,n}(\vec{k},\vec{k}') = e^{3\theta}G_{m,n}(e^{-\theta}\vec{k}, e^{-\theta}\vec{k}')$ for $m+n=2$. The coupling functions are dilation analytic, and hence have analytic continuations for $\theta \in \mathbf{C}^+$. We choose $\theta = i\phi \in \mathbf{C}^+$ in the following.

Hypothesis 1 *For $m+n=1$ and $\vec{k} \in \mathbf{R}^3$, $G^{(i\phi)}_{m,n}(\vec{k})$ are analytic functions with values in the quadratic forms on the domain of $(-\Delta_x)^{\frac{1}{2}}$.*
$(\vec{k},\vec{k}') \in \mathbf{R}^3 \otimes \mathbf{R}^3$, $G^{(i\phi)}_{m,n}(\vec{k},\vec{k}')$ are analytic functions with values in the bounded operators on $\mathcal{H}_{el}$.

The smallest non-negative function J which satisfies

$$\sup_{\phi \leq \phi_0} \| (-\Delta_x+1)^{-\frac{1}{4}} G^{(i\phi)}_{m,n}(\vec{k}) (-\Delta_x+1)^{-\frac{1}{4}} \| \leq J(\vec{k})$$

for $m+n=1$, and

$$\sup_{\phi \leq \phi_0} \| G^{(i\phi)}_{m,n}(\vec{k},\vec{k}') \| \leq J(\vec{k})J(\vec{k}')$$

Spectral analysis of N-body Schrödinger operators by use of Balslev-Combes theory has been subject to intensive study ever since the original paper [3] appeared.

Dilation of the radiation field Hamiltonian H_f requires the choice of a one-photon basis $\{f_i\}$ on $\mathbf{C}_0(\mathbf{R}^3)$ to represent H_f in terms of $a^\dagger(f_i)$, $a(f_j)$

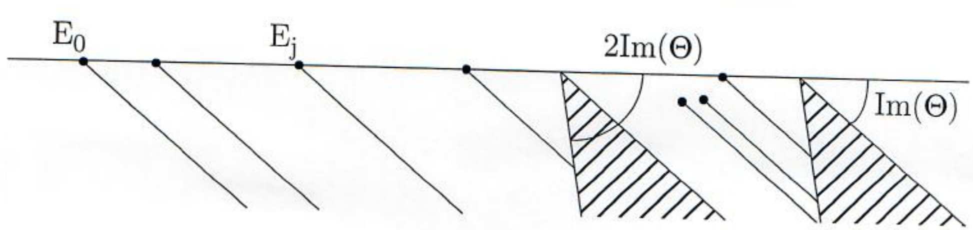
$$H_f = \sum_{i,j} \langle f_i, \omega f_j \rangle \, a^\dagger(f_i) a(f_j) \ .$$

Using $\langle f_i, \omega f_j \rangle = \int d^3\vec{k} \bar{f}_i(\vec{k}) \omega(\vec{k}) f_j(\vec{k})$ and the usual definition of $a^\sharp(f)$, it follows that

$$H_f^{(\theta)} = \sum_{i,j} \langle f_i, \omega_\theta f_j \rangle \, a^\dagger(f_i) a(f_j) \ ,$$

where $\omega_\theta(\vec{k}) := \omega(e^{-\theta}\vec{k})$. For the dispersion relation $\omega(\vec{k}) = |\vec{k}|$, one finds that $H_f^{(\theta)} = e^{-\theta} H_f$. Note that since the modulus function $|\cdot| : \mathbf{R}^3 \to \mathbf{R}^+$ is not analytic, the factor $e^{-\theta}$ must be extracted from $|e^{-\theta}\vec{k}|$ before analytic continuation. The spectrum of H_f consists of a single eigenvalue $\{0\}$ at the bottom of its continuous spectrum, corresponding to the vacuum state Ω_f. There is no separation between the point spectrum and the continuous spectrum of H_f, since photons are massless particles. For $\theta = i\phi \in \mathbf{C}^+$, the continuous spectrum of $H_f^{(i\phi)}$ that emanates from $\{0\}$ is rotated by ϕ into $\mathbf{C}^-$.

The spectrum of the complex dilated, free Hamiltonian $H_{g=0}^{(i\phi)}$ thus consists of branches of continuous spectrum of $H_f^{(i\phi)}$, which emanate from each element of the spectrum of $H_{el}^{(i\phi)}$, from each threshold, and from each resonance, cf. the following figure.



We have introduced the interaction Hamiltonian W_g at the end of section 1.2. The unitarily dilated $W_g^{(\theta)} = gW_1^{(\theta)} + g^2 W_2^{(\theta)}$ is characterized by the coupling functions $G_{m,n}^{(\theta)}$, $1 \le m+n \le 2$, which are defined by

$$\begin{aligned} W_1^{(\theta)} &= \int d^3\vec{k} [G_{10}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) \, + \, G_{01}^{(\theta)}(\vec{k}) \otimes a(\vec{k})] \\ W_2^{(\theta)} &= \int d^3\vec{k} d^3\vec{k}' [G_{20}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) a^\dagger(\vec{k}') \, + \, G_{02}^{(\theta)}(\vec{k}) \otimes a(\vec{k}) a(\vec{k}') \\ &\quad + G_{11}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) a(\vec{k}')] \, . \end{aligned}$$

We have $G_{m,n}^{(\theta)}(\vec{k}) = e^{3\theta/2} G_{m,n}(e^{-\theta}\vec{k})$ for $m+n = 1$, and $G_{m,n}^{(\theta)}(\vec{k}, \vec{k}') = e^{3\theta} G_{m,n}(e^{-\theta}\vec{k}, e^{-\theta}\vec{k}')$ for $m+n = 2$. Thus, the coupling functions are dilation analytic, and we choose $\theta = i\phi \in \mathbf{C}^+$. The properties of the coupling functions are specified in the following hypothesis.

for $m+n=2$, and all $\vec{k},\vec{k}'\in\mathbf{R}^3$, obeys

$$\Gamma(\beta) := [\int d^3\vec{k}\,(1+\omega(\vec{k})^{-(1+\beta)})\,|J(\vec{k})|^2]^{\frac{1}{2}} \leq \infty$$

for some $\beta > 0$.

4.3 The Feshbach map

It will be our intention in section 4.4 to investigate the location of the spectrum of $H_g(i\phi)$ in the vicinity of the excited energy eigenvalues of H_{el}, with $\phi > 0$. Consequently, we will consider an arbitrary excited eigenvalue E_j of H_{el}, and the corresponding eigenspace with projector P_j. We will furthermore introduce an auxiliary ultraviolet cutoff ρ_0 on photon energies and consider the complex dilated projector $P_0(i\phi) = P_j(i\phi)\otimes\chi_{H_f<\rho_0}$ on $\mathcal{H}$, where χ_I is the characteristic function on the interval I. The Feshbach map will be introduced to map $H_g(i\phi)-z$ isospectrally to an operator $f_{P_0}(H_g(i\phi)-z)$ on the strongly reduced Hilbert space $P_0(i\phi)\mathcal{H}$, for z in a complex neighborhood of E_j. Fermi's golden rule is proved by showing that $f_{P_0(i\phi)}(H_g(i\phi)-z)$ is invertible for z in a specific complex neighborhood of E_j.

To give a general definition of the Feshbach map, we assume that a bounded, but not necessarily orthogonal projector P on a Hilbert space $\mathcal{H}$ is given. Furthermore, we consider a densely defined, closed operator H on $\mathcal{H}$, whose domain contains the range of P. Consequently, one obtains the complementary projector $\bar{P} = \mathbf{1} - P$, and the operators $H_P = PHP$ and $H_{\bar{P}} = \bar{P}H\bar{P}$. $H_{\bar{P}}$ is an operator on $\bar{P}\mathcal{H}$, and for $z\in\rho(H_{\bar{P}})$, one can define the resolvent

$$R_{\bar{P}}(H-z) = \bar{P}(H_{\bar{P}}-z)^{-1}\bar{P} \tag{44}$$

on $\bar{P}\mathcal{H}$. $\rho(a)$ denotes the resolvent set of an operator A. The Feshbach map is defined by

$$f_P(H-z) = P(H-z-HR_{\bar{P}}(H-z)H)P \tag{45}$$

provided that $z\in\rho(H_{\bar{P}})$, and provided that the additional assumptions

$$\| R_{\bar{P}}(H-z)HP \| < \infty\ ,\quad \| PHR_{\bar{P}}(H-z) \| < \infty \tag{46}$$

hold. Moreover, $S_P(z) ;= P - R_{\bar{P}}(H-z)HP$ is a family of operators which satisfies $Ker\{S_P(z)\} = Ran\{\bar{P}\}$. The Feshbach map has the following isospectrality property.

Thm. 1 (Feshbach map) *Let the assumptions (46) hold, and let $z\in\rho(H_{\bar{P}})$. Then, with $\sigma_\sharp = \sigma, \sigma_{pp}$,*

$$z\in\sigma_\sharp(H) \iff z\in\{z\,|\,0\in\sigma_\sharp(f_P(H-z))\}\,.$$

The eigenfunctions of the operators $H_{\bar{P}}$ and $f_P(H-z)$ are related by

$$Ker\{(H-z)S_P(z)\} = Ker\{f_P(H-z)\}$$

$$P\,Ker\{(H-z)\} = Ker\{f_P(H-z)\}\,.$$

These relations imply $dimKer\{(H-z)\} = dimKer\{f_P(H-z)\}$. •

A proof can be found in [5]. In our case, the projector P will be given by $P_0(i\phi) = P_j(i\phi)\otimes\chi_{H_f\leq\rho_0}$, and H will be represented by $H_g(i\phi)$.

4.4 Proof of Fermi's golden rule

The Hamiltonian H_g has a single eigenvalue, the ground state energy E_0, which lies at the bottom of its spectrum, located at the tip of the continuous spectrum. The existence and uniqueness of a ground state in systems of confined electrons which are coupled to the radiation field have been proved in [5, 6]. In this section, we will prove that all embedded eigenvalues of the non-interacting system above the ground state become resonances when the interaction is turned on. According to the definition of resonances in terms of Balslev-Combes theory given in section 4.1, we will try to find the locations of complex eigenvalues of the complex dilated Hamiltonian

$$H_g(i\phi) = H_{el}(i\phi) \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes e^{-i\phi} H_f + W_g(i\phi) \tag{47}$$

in the vicinity of excited eigenvalues E_j of H_{el}, for $\phi > 0$. Our proof is different from the one given in [5], in that we use dilation analyticity of the full Hamiltonian, and not only of the field Hamiltonian. It was not necessary in our case to introduce an artificial spatial cutoff on the electron wave functions in $W_g(i\phi)$. This advantage is however traded for the loss of self-adjointness of the N-electron Hamiltonian. Many arguments in [5] based on the spectral theorem will be substituted by considerations about the numerical range of an operator. The numerical range $Num(A)$ of an operator A on $\mathcal{H}$ is defined as the set

$$Num(A) = \{(\psi, A\psi) \,|\, \psi \in \mathcal{H}, \, (\psi, \psi) = 1\} \subset \mathbf{C}\,. \tag{48}$$

We will use the following lemma.

Lemma 4.1 *For $z \in \rho(A)$, the bound $\| (A - z)^{-1} \| \leq [dist(Num(A), z)]^{-1}$ holds.*

Proof Due to the Cauchy-Schwarz theorem, one has $\| (A - z)\psi \| \geq |(\psi, (A - z)\psi)|$ for $(\psi, \psi) = 1$, and $z \in \mathbf{C}$. Then it follows from the definition of the numerical range that $|(\psi, (A - z)\psi)| \geq dist(Num(A), z)$, where $dist$ is the distance function on $\mathbf{C}$. Consequently, one has the bound $\| (A - z)^{-1}\eta \| \leq [dist(Num(A), z)]^{-1}$ for all η with $(\eta, \eta) = 1$, and the assertion of the lemma follows. •

We will extensively use the fact that the potential V in H_{el} is a superposition of Coulomb potentials and satisfies a Kato bound.

Lemma 4.2 *Let $T = -\Delta_x$ be the kinetic energy operator of the N-electron system. To each $\epsilon < \frac{1}{2}$, there exists a constant $b(\epsilon) < \infty$ such that*

$$\| V\psi \| \leq \epsilon \| T\psi \| + b(\epsilon) \| \psi \| \,. \quad \bullet \tag{49}$$

For a proof, see for instance [8].

We consider an excited, N_j-fold degenerate energy eigenstate E_j of H_{el}. Its corresponding eigenspace $\mathcal{E}_j \subset \mathcal{H}_{el}$ is spanned by the N_j eigenvectors $\psi_{j,l}$, $l = 1, .., N_j$, which satisfy $H_{el}\psi_{j,l} = E_j\psi_{j,l}$. We denote the orthogonal projection $\mathcal{H}_{el} \to \mathcal{E}_j$ by P_j, which has a finite range. After complex dilation, E_j remains an eigenvalue of $\mathcal{H}_{el}(i\phi)$. The complex dilated projector $P_j(i\phi)$ is not orthogonal anymore, but its range is still finite, and for two distinct eigenvalues E_j and

E_k, the relation $P_j(i\phi)P_k(i\phi) = \delta_{jk}P_j(i\phi)$ is still valid. We note that the components of the $N_j \times N_j$ matrix T_j

$$[T_j]_{ll'} = \int_{-\infty}^{E_j-0} dE \int d^3\vec{k}\, \langle G_{10}(\vec{k})\psi_{j,l}, d\chi_{H_f\leq E} G_{10}(\vec{k})\psi_{j,l'}\rangle\, \delta[\omega(\vec{k}) - E_j + E] \tag{50}$$

are usually calculated in order to find the life-time of resonances, according to Fermi's golden rule. We will refer to T_j as the transition matrix.

Hypothesis 2 *For $j \geq 1$ we assume that the transition matrix T_j associated to the eigenvalue E_j is self-adjoint and positive definite.*

It is our intention to study the spectrum of $H_g(i\phi)$ in the vicinity of E_j by use of the Feshbach map. Hence, we define the projector

$$P_0(i\phi) \;=\; P_j(i\phi) \otimes \chi_{H_f\leq\rho_0}$$

on $\mathcal{H}$, where ρ_0 is a cutoff on photon energies, and χ_I is the characteristic function of the interval I. Its complementary projector yields

$$\bar{P}_0(i\phi) = P_j(i\phi) \otimes \chi_{H_f\leq\rho_0} + \bar{P}_j(i\phi) \otimes \mathbf{1}_f$$

with $\bar{P}_j(i\phi) := \mathbf{1} - P_j(i\phi)$. Thus we can define the Feshbach map

$$f_{P_0(i\phi)}(H_g(i\phi) - z) \;=\; P_0(i\phi)[H_g(i\phi) - z - W_g(i\phi)R_{\bar{P}_0(i\phi)}((H_g(i\phi) - z)W_g(i\phi)]P_0(i\phi)$$

provided that the assumptions (46) hold.

Applicability of the Feshbach map

We will investigate the spectrum of $H_g(i\phi)$ in a ball $D(E_j, \frac{\rho_0}{2})$ of radius $\frac{\rho_0}{2}$ around E_j. Hence we have to verify that the Feshbach map is defined for $z \in D(E_j, \frac{\rho_0}{2})$ by showing that the conditions (46) hold. For this purpose, we will prove the following lemmata 4.4 $\sim$ 4.11, and use the notation $H_0 \equiv H_{g=0}$. As a consequence of lemmata 4.1 and 4.2, we obtain an estimate on the numerical range of $H_{g=0}(i\phi)$.

Lemma 4.3 *Let $\epsilon < \frac{1}{2}$ and $b(\epsilon)$ be the constants specified in lemma 4.2, and let $\epsilon < \frac{\sin 2\phi}{\sin\phi}$. Then, $Im(z) \leq b(\epsilon)\sin\phi$ for all $z \in Num(H_{g=0}(i\phi))$.*

Proof For any vector $\psi = \psi_{el} \otimes \psi_f \in \mathcal{H}$, one has

$$\begin{aligned}\langle\psi, H_{g=0}(i\phi)\psi\rangle \;&=\; e^{-2i\phi}\langle\psi_{el}, T\psi_{el}\rangle_{el} + e^{-i\phi}\langle\psi_{el}, V\psi_{el}\rangle_{el} \\ &+\; e^{-i\phi}\langle\psi_f, H_f\psi_f\rangle_f\,,\end{aligned}$$

where $\langle\cdot\rangle_{el}$ and $\langle\cdot\rangle_f$ are the scalar products on $\mathcal{H}_{el}$ and $\mathcal{H}_f$, respectively. One finds that $\langle\psi_{el}, T\psi_{el}\rangle_{el} = \int dx|\nabla_x\psi_{el}(x)|^2 > 0$, where $x := (\vec{x}_1, .., \vec{x}_N)$, and $\langle\psi_{el}, V\psi_{el}\rangle_{el} = \int dx V(x)|\psi_{el}(x)|^2$

for the contributions on $\mathcal{H}_{el}$. Furthermore, H_f is self-adjoint and non-negative, hence $\langle\psi_f, H_f\psi_f\rangle_f > 0$.

Using the Cauchy-Schwarz theorem, the Kato bound for V can be shown to imply $|\langle\psi_{el}, V\psi_{el}\rangle| \leq \epsilon\langle\psi_{el}, T\psi_{el}\rangle_{el} + b(\epsilon)\langle\psi_{el}, \psi_{el}\rangle_{el}$, where we have used the positivity of the scalar products on the rhs. For $\langle\psi_{el}, \psi_{el}\rangle_{el} = 1$, one thus finds that

$$\langle\psi_{el}, V\psi_{el}\rangle \geq -\epsilon\langle\psi_{el}, T\psi_{el}\rangle_{el} - b(\epsilon) .$$

The imaginary part of $\langle\psi, H_{g=0}(i\phi)\psi\rangle$ is given by

$$-\sin 2\phi\,\langle\psi_{el}, T\psi_{el}\rangle_{el} \;-\; \sin\phi\,\langle\psi_{el}, V\psi_{el}\rangle_{el} \;-\; \sin\phi\,\langle\psi_f, H_f\psi_f\rangle_f ,$$

which is bounded from above by

$$-\sin 2\phi\,\langle\psi_{el}, T\psi_{el}\rangle_{el} + \epsilon\sin\phi\,\langle\psi_{el}, T\psi_{el}\rangle_{el} + \sin\phi\, b(\epsilon) \;-\; \sin\phi\,\langle\psi_f, H_f\psi_f\rangle_f .$$

The sum of the first two terms is negative, due to the assumption on ϵ. Because of the non-negativity of H_f, the last term is also negative. Thus we arrive at the assertion of the lemma. We note that the upper bound $b(\epsilon)\sin\phi$ is purely ϕ-dependent, due to the restriction on ϵ by the values of ϕ. •

Lemma 4.4 *Let $\bar{P}_{el}(i\phi,\lambda) := P_j(i\phi)\chi_{\lambda\in[0,\rho_0]} + \chi_{\lambda\geq\rho_0}$. There exists a constant Γ_1, and a function $c(\lambda)$ with values in $[0,\Gamma_1]$, such that for all $z \in D(E_j, \frac{\rho_0}{2})$,*

$$\| \, R_{\bar{P}_{el}(i\phi,\lambda)}(H_{el}(i\phi) + \lambda e^{-i\phi} - z) \, \|_{\mathcal{H}_{el}} \leq c(\lambda) , \tag{51}$$

and $c(\lambda) = O(\lambda^{-1})$ for $\lambda \to \infty$.

Proof In the following, we drop the subscript $\mathcal{H}_{el}$. By definition of $\bar{P}_{el}(i\phi,\lambda)$, the lhs of (51) yields

$$\| \, R_{\bar{P}_j(\theta)}(H_{el}(i\phi) + \lambda e^{-i\phi} - z) \, \| \, \chi_{\lambda\in[0,\rho_0]} \\ + \| \, [H_{el}(i\phi) + \lambda\, e^{-i\phi} - z]^{-1} \, \| \, \chi_{\lambda\geq\rho_0} .$$

For each $\lambda \in [0,\rho_0]$, $\| \, R_{\bar{P}_j(\theta)}(H_{el}(i\phi) + \lambda e^{-i\phi} - z) \, \| =: c_\lambda(z)$ is bounded, because $z \in D(E_j, \frac{\rho_0}{2})$ is not in the spectrum of $\bar{P}_j(i\phi)[H_{el}(\theta) + \lambda e^{-i\phi}]$, and for each finite $\lambda \geq \rho_0$, $\| \, [H_0(i\phi) + \lambda e^{-i\phi} - z]^{-1} \, \| =: c'_\lambda(z)$ is bounded, due to the same reason. For large λ, we use lemma 4.1 to derive that

$$\| \, [H_{el}(i\phi) + \lambda e^{-i\phi} - z]^{-1} \, \| \leq [dist(Num(H_{el}(i\phi) + \lambda e^{-i\phi}, z))]^{-1} .$$

Lemma 4.2 implies that the rhs is bounded by $[\lambda\sin\phi - c - \frac{\rho_0}{2}]^{-1}$ for some constant c, and tends to zero like $O(\lambda^{-1})$ for $\lambda \to \infty$. Now we define $c_\lambda := \sup_{z\in D(E_j,\frac{\rho_0}{2})}\{c_\lambda(z)\}$ and $c_\lambda := \sup_{z\in D(E_j,\frac{\rho_0}{2})}\{c'_\lambda(z)\}$ for all $\lambda \in \mathbf{R}^+$. The function $c(\lambda) := c_\lambda\chi_{\lambda\in[0,\rho_0]} + c'_\lambda\chi_{\lambda\geq\rho_0}$ satisfies the assertion of the lemma, and we set $\Gamma_1 := \sup_\lambda\{c(\lambda)\}$. •

Lemma 4.5 *For all* $z \in D(E_j, \frac{\rho_0}{2})$, $\| R_{\bar{P}_0(i\phi)}(H_0(i\phi) \ - \ z) \| \leq \Gamma_1$ *holds.*

Proof Let dQ_λ be the spectral measure on $\mathcal{H}_f$ associated to the photon Hamiltonian H_f. Spectral decomposition of H_f yields

$$R_{\bar{P}_0(i\phi)}(H_0(i\phi) \ - \ z) \ = \ \int_{\lambda \geq 0} R_{\bar{P}_{el}(i\phi,\lambda)}(H_{el}(i\phi) + \lambda e^{-i\phi} - z) \otimes dQ_\lambda \ ,$$

where $\bar{P}_{el}(i\phi, \lambda) := P_j(i\phi)\chi_{\lambda\in[0,\rho_0]} + \chi_{\lambda\geq\rho_0}$. This expression is bounded from above by

$$sup_{\lambda\geq 0} \| R_{\bar{P}_{el}(i\phi,\lambda)}(H_{el}(i\phi) + \lambda e^{-i\phi} - z) \|_{\mathcal{H}_{el}}$$

which is smaller than Γ_1, due to lemma 4.4. This concludes the proof. •

Lemma 4.6 *Let* $\epsilon < \frac{1}{2}$ *and* $b(\epsilon)$ *be the constants specified in lemma 4.2. Then, the inequality*

$$\| [H_{el} - w]\psi \|_{\mathcal{H}_{el}} \leq \ 2 \ \| [H_{el}(i\phi) - we^{-2i\phi}]\psi \|_{\mathcal{H}_{el}} + \frac{2[b(\epsilon) + |w|\epsilon]}{1-\epsilon} \| \psi \|_{\mathcal{H}_{el}} \tag{52}$$

holds for all $\psi \in \mathcal{H}_{el}$ *and all* $w \in \mathbf{C}$.

Proof In the following, we drop the subscripts $\mathcal{H}_{el}$. Using $H_{el}(i\phi) \ = \ e^{-2i\phi}T \ + \ e^{-i\phi}V$ and lemma 4.2, we obtain the bound

$$\| [T + V - w]\psi \| \leq \| [T + e^{i\phi}V - w]\psi \| \ + |1 - e^{i\phi}| \ \| V\psi \| \ \ , \tag{53}$$

where $T \ = \ -\Delta_x$ is the kinetic energy operator on $\mathcal{H}_{el}$. Due to (49), we find the inequality

$$\| V\psi \| \leq \epsilon \ \| [T + e^{i\phi}V - w]\psi \| \ + \epsilon \ \| V\psi \| \ + [b(\epsilon) \ + \ \epsilon|w|] \ \| \psi \| \ \ ,$$

which we solve for $\| V\psi \|$. Inserting the resulting estimate for $\| V\psi \|$ in (53) yields

$$\| [T + V - \mu]\psi \| \leq \frac{1+\epsilon}{1-\epsilon} \| e^{-2i\phi}[T + e^{i\phi}V - \mu]\psi \| \ + \frac{2[b(\epsilon) + |\mu|\epsilon]}{1-\epsilon} \| \psi \| ,$$

where we have used that $|1 - e^{i\phi}| \leq 2$. Since $\frac{1+\epsilon}{1-\epsilon} \leq 2$ for $\epsilon < \frac{1}{2}$, we arrive at the assertion of the lemma. •

Lemma 4.7 *Let* $c \in \mathbf{R}$ *satisfy* $c < E_0$. *Then there is a constant* Γ_c, *such that for all* $z \in D(E_j, \frac{\rho_0}{2})$,

$$\| [H_0 - c]R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \| \ \leq \ \Gamma_c \, , \tag{54}$$

$$\| R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z)[H_0 - c] \| \ \leq \ \Gamma_c \, , \tag{55}$$

and

$$\| [H_0 - c]^{-\frac{1}{2}} R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z)[H(0) - c]^{-\frac{1}{2}} \| \leq 2\Gamma_c \, . \tag{56}$$

Proof We begin with the proof of (54). (55) is proved exactly in the same way. Applying lemma 4.6, and using spectral decomposition of the photon Hamiltonian, the lhs of (54) is bounded from above by

$$\sup_{\lambda>0}\{ \parallel [H_{el}(i\phi) + e^{2i\phi}[\lambda - c]] R_{\bar{P}_{el}(i\phi,\lambda)}(H_{el}(i\phi) + \lambda e^{i\phi} - z) \parallel_{\mathcal{H}_{el}} + \frac{2[b(\epsilon) + \epsilon|\lambda - c|]}{1-\epsilon} \parallel R_{\bar{P}_{el}(i\phi,\lambda)}(H_{el}(i\phi) + \lambda e^{i\phi} - z) \parallel_{\mathcal{H}_{el}} \} ,$$

where $\bar{P}_{el}(i\phi,\lambda) := P_j(i\phi)\chi_{\lambda\in[0,\rho_0]} + \chi_{\lambda\geq\rho_0}$. Due to lemma 4.4, this expression is smaller than

$$2 + \sup_{\lambda>0}\{ (|e^{2i\phi}[\lambda - c] + \lambda e^{i\phi} - z| + \frac{2[b(\epsilon) + \epsilon|\lambda - c|]}{1-\epsilon}) \cdot c(\lambda) \}$$

The term in brackets diverges like $O(\lambda)$ for $\lambda \to \infty$, but $c(\lambda)$ converges to zero like $O(\lambda^{-1})$. Thus, the supremum $s(z)$ exists for all $z \in D(E_j, \frac{\rho_0}{2})$, and the constant $\Gamma_c := \sup_z\{s(z)\}$ satisfies (54).

To prove (56), we use the spectral decomposition of the undilated operators on the lhs. Let dQ_λ be the spectral measure associated to H_f, and let dP_t be the spectral measure defined by H_{el}. For notational convenience, we write $\bar{P}_0(i\phi)[H_0(i\phi) - z]^{-1} =: \int K(\lambda) \otimes dQ_\lambda$. The lhs of (56) is then given by

$$\begin{aligned} &\parallel \int [t + \lambda - c]^{\frac{1}{2}}[t' + \lambda - c]^{\frac{1}{2}} dP_t K(\lambda) dP_{t'} \otimes dQ_\lambda \parallel \\ &= \parallel \int_{t\geq t'\geq E_0} \sqrt{\frac{t' + \lambda - c}{t + \lambda - c}} dP_t[t + \lambda - c]K(\lambda)dP_{t'} \otimes dQ_\lambda \\ &\quad + \int_{t'\geq t\geq E_0} \sqrt{\frac{t + \lambda - c}{t' + \lambda - c}} dP_t K(\lambda)[t' + \lambda - c]dP_{t'} \otimes dQ_\lambda \parallel . \end{aligned}$$

Now we use the identity $dP_t[t + \lambda - c] = dP_t[H_{el} + \lambda - c]$ and find the upper bound

$$\sup_{\lambda>0}\{ \sup_{t\geq t'\geq E_0} \sqrt{\frac{t' + \lambda - c}{t + \lambda - c}} \parallel [H_{el} + \lambda - c]K(\lambda) \parallel_{\mathcal{H}_{el}} + \sup_{t'\geq t\geq E_0} \sqrt{\frac{t + \lambda - c}{t' + \lambda - c}} \parallel K(\lambda)[H_{el} + \lambda - c] \parallel_{\mathcal{H}_{el}} \}$$

which is smaller than $2\Gamma_c$, due to (54) and (55). •

Lemma 4.8 *Let $c \in \mathbf{R}$ satisfy $c < E_0$. Then, there is a constant Γ'_c, such that*

$$\parallel [H_0 - c]^{\frac{1}{2}}[H_0(i\phi) - c]^{-1} \parallel \ \leq \ \Gamma'_c[E_0 - c]^{-\frac{1}{2}} , \tag{57}$$

$$\parallel [H_0(i\phi) - c]^{-1}[H_0 - c]^{\frac{1}{2}} \parallel \ \leq \ \Gamma'_c[E_0 - c]^{-\frac{1}{2}} . \tag{58}$$

Proof The argument is the same as the one used to prove (56). The lhs of (57) is bounded by

$$\sup_{\lambda>0} \parallel \int_{t\geq E_0} [H_{el}(i\phi) + \lambda - c]^{-1}][t + \lambda - c]^{-\frac{1}{2}}[H_{el} + \lambda - c]\, dP_t \parallel_{\mathcal{H}_{el}} ,$$

which is smaller than $\sup_{t \geq E_0}[t + \lambda - c]^{-\frac{1}{2}} \parallel [H_0(i\phi) - c]^{-1}[H_0 - c] \parallel$. One can show that $\parallel [H_0(i\phi) - c]^{-1}[H_0 - c] \parallel \leq \Gamma'_c$ for some constant Γ'_c, in the same way as in lemma 4.7. The same applies to (58). This concludes the proof. •

Lemma 4.9 *Let $c \in \mathbf{R}$ satisfy $c < E_0$. Then, there is a constant $\Gamma_c^{(W)}$ which satisfies*

$$
\begin{aligned}
\parallel [H_0 - c]^{-1} W_g(i\phi) \parallel &\leq g\Gamma_c^{(W)} , \\
\parallel W_g(i\phi)[H_0 - c]^{-1} \parallel &\leq g\Gamma_c^{(W)} ,
\end{aligned}
$$

and

$$
\parallel [H_0 - c]^{-\frac{1}{2}} W_g(i\phi)[H(0) - c]^{-\frac{1}{2}} \parallel \leq g\Gamma_c^{(W)} .
$$

The proof will be given after lemma 4.19, among a series of other lemmata that require the same argumentation technique.

Lemma 4.10 *Let $c \in \mathbf{R}$ satisfy $c < E_0$. Then, the following bounds are satisfied.*

$$
\begin{aligned}
\parallel [H_0 - c]^{-\frac{1}{2}} W_g(i\phi) P_0(i\phi) \parallel &\leq g\Gamma_c^{(W)}\Gamma'_c[E_0 - c]^{-\frac{1}{2}}[E_j - c]^{-1} , &(59)\\
\parallel P_0(i\phi) W_g(i\phi)[H_0 - c]^{-\frac{1}{2}} \parallel &\leq g\Gamma_c^{(W)}\Gamma'_c[E_0 - c]^{-\frac{1}{2}}[E_j - c]^{-1} . &(60)
\end{aligned}
$$

Proof Both inequalities are proved in the same way. The lhs of (59) is bounded by

$$
\begin{aligned}
\parallel [H_0 - c]^{-\frac{1}{2}} W_g(i\phi)[H_0 - c]^{-\frac{1}{2}} \parallel \cdot \parallel [H_0 - c]^{\frac{1}{2}}[H_0(i\phi) - c] \parallel \cdot \\
\parallel [H_0(i\phi) - c]^{-1} P_0(i\phi) \parallel ,
\end{aligned}
$$

which is smaller than $g\Gamma_c^{(W)}\Gamma'_c[E_j - c]^{-1}$, due to lemmata 4.8 and 4.9. It is straightforward to show that $\parallel [H_0(i\phi) - c]^{-1} P_0(i\phi) \parallel \leq [E_j - c]^{-1}$. •

Using these lemmata, it is now easy to prove that the assumptions (46) hold, and that the Feshbach map is thus defined for $z \in D(E_j, \frac{\rho_0}{2})$. We will use the *second resolvent identity* repeatedly in the following. Let A and B be operators on $\mathcal{H}$, such that the products AB and BA are well-defined. Then, one has the second resolvent identity

$$
\begin{aligned}
[A + B]^{-1} &= A^{-1} + A^{-1}B[A + B]^{-1} \\
&= A^{-1} + [A + B]^{-1} B A^{-1} .
\end{aligned}
$$

We can now show that

Lemma 4.11 *Let $c \in \mathbf{R}$ satisfy $c < E_0$, and let $z \in D(E_j, \frac{\rho_0}{2})$. Moreover, let the coupling constant g satisfy $g < [\Gamma_c^{(W)}\Gamma_c]^{-1}$. Then, one has the bound*

$$
\parallel R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \parallel \leq \Gamma_1[1 - g\Gamma_c^{(W)}\Gamma_c]^{-1} .
$$

Proof Due to the second resolvent identity, one finds

$$\| R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \leq \| R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \|$$
$$+ \| R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z)\bar{P}_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \| ,$$

which is bounded by

$$\Gamma_1 + \| R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \cdot \| W_g(i\phi)[H_0 - c]^{-1} \| \cdot$$
$$\| [H_0 - c][H_0(i\phi) - z]^{-1} \| ,$$

due to lemma 4.5, and where $c \in \mathbf{R}$ and $c < E_0$. Using lemmata 4.7 and 4.9, we obtain the bound

$$\| R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \leq \Gamma_1 + g\Gamma_c^{(W)}\Gamma_c \| \bar{P}_0(i\phi)[H_g(i\phi) - z]^{-1} \| \ .$$

The assertion of the lemma follows. •

The assumptions (46) can now be verified.

Proposition 1 *Let $c \in \mathbf{R}$ satisfy $c < E_0$, and let $z \in D(E_j, \frac{\rho_0}{2})$. Furthermore, let the coupling constant g satisfy $g < [\Gamma_c^{(W)}\Gamma_c]^{-1}$. Then, one has the bounds*

$$\| P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \ < \ g\Gamma_c^{(W)}\Gamma_c[1 - g\Gamma_c^{(W)}\Gamma_c]^{-1} ,$$
$$\| R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z)\bar{P}_0(i\phi)W_g(i\phi)P_0(i\phi) \| \ < \ g\Gamma_c^{(W)}\Gamma_c[1 - g\Gamma_c^{(W)}\Gamma_c]^{-1} .$$

Proof Both inequalities are proved in the same way. By use of the second resolvent identity, we obtain

$$\| P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \leq$$
$$\| P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \|$$
$$+ \| P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \cdot$$
$$\| W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \| \ .$$

For $c \in \mathbf{R}$ and $c < E_0$, we have that

$$\| W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \| \leq \| W_g(i\phi)[H_0 - c]^{-1} \| \cdot$$
$$\| [H_0 - c]R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \| ,$$

which is smaller than $g\Gamma_c^{(W)}\Gamma_c$, due to lemmata 4.7 and 4.9. Hence, we obtain

$$\| P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| \leq g\Gamma_c^{(W)}\Gamma_c$$
$$+ g\Gamma_c^{(W)}\Gamma_c \| P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi) - z) \| ,$$

and the assertion of the proposition follows immediately. •

Spectrum of the Feshbach map

We will now investigate the spectrum of the Feshbach map, which is in our case given by

$$\begin{aligned} f_{P_0(i\phi)}(H_g(i\phi)-z) &= P_0(i\phi)\otimes\chi_{H_f<\rho_0}[E_j - z + e^{-i\phi}H_f] \\ &+ gP_0(i\phi)W_1(i\phi)P_0(i\phi) + g^2P_0(i\phi)W_2(i\phi)P_0(i\phi) \\ &- P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi)-z)\bar{P}_0(i\phi)W_g(i\phi)P_0(i\phi)\,. \end{aligned} \tag{61}$$

The last term in (61) can be decomposed into a sum of two terms $\Sigma_1+\Sigma_2$ by use of the second resolvent identity, where

$$\Sigma_1(z) := P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi)-z)\bar{P}_0(i\phi)W_g(i\phi)P_0(i\phi)\,,$$

and

$$\Sigma_2(z) := P_0(i\phi)W_g(i\phi)\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi)-z)\bar{P}_0(i\phi)W_g(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi)-z)W_g(i\phi)P_0(i\phi)\,.$$

We will now show that $\Sigma_2(z)$ is very small.

Lemma 4.12 *Let $z\in D(E_j,\frac{\rho_0}{2})$, and let $c\in\mathbf{R}$ satisfy $c<E_0$. Then,*

$$\| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi)-z)\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\| \le 2\Gamma_c[1-2g\Gamma_c\Gamma_c^{(W)}]^{-1}\,. \tag{62}$$

Proof By use of the second resolvent identity, the lhs of (62) is bounded by

$$\begin{aligned} &\| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi)-z)\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\| + \\ &\qquad \| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi)-z)\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\| \cdot \\ &\| \, [H_0-c]^{-\frac12}W_g(i\phi)[H_0-c]^{-\frac12} \,\| \cdot \| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi)-z)\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\|\,, \end{aligned}$$

which is smaller than

$$2\Gamma_c + 2g\Gamma_c\Gamma_c^{(W)} \, \| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi))\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\|\,,$$

due to lemmata 4.7 and 4.9. The assertion of the lemma follows. •

Lemma 4.13 *Let $z\in D(E_j,\frac{\rho_0}{2})$. Then, $\| \, \Sigma_2(z) \, \| = O(g^3)$.*

Proof We introduce an auxiliary operator $[H_0-c]$, with $c\in\mathbf{R}$, and $c<E_0$. Then, $\| \, \Sigma_2(z) \, \|$ is bounded by

$$\begin{aligned} &\| \, P_0(i\phi)W_g(i\phi)[H_0-c]^{-\frac12} \,\| \cdot \| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_0(i\phi)-z)\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\| \cdot \\ &\qquad \| \, [H_0-c]^{-\frac12}W_g(i\phi)[H_0-c]^{-\frac12} \,\| \cdot \\ &\| \, [H_0-c]^{\frac12}\bar{P}_0(i\phi)R_{\bar{P}_0(i\phi)}(H_g(i\phi)-z)\bar{P}_0(i\phi)[H_0-c]^{\frac12} \,\| \cdot \| \, [H_0-c]^{-\frac12}W_g(i\phi)P_0(i\phi) \,\| \end{aligned}$$

The assertion of the lemma follows from lemmata 4.8, 4.9, and 4.12. •

We further decompose $\Sigma_1(z)$ into a dominant and a small term, $\Sigma_1(z) = \Sigma_{11}(z) + \Sigma_{12}(z)$, where

$$\Sigma_{11}(z) := g^2 P_0(i\phi) W_1(i\phi) \bar{P}_0(i\phi) R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \bar{P}_0(i\phi) W_1(i\phi) P_0(i\phi) ,$$

and

$$\Sigma_{12}(z) := \sum_{l+l' \geq 3} g^{l+l'} P_0(i\phi) W_l(i\phi) \bar{P}_0(i\phi) R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \bar{P}_0(i\phi) W_{l'}(i\phi) P_0(i\phi) .$$

Lemma 4.14 *Let $z \in D(E_j, \frac{\rho_0}{2})$. Then, $\| \Sigma_{12}(z) \| = O(g^3)$.*

Proof We introduce an auxiliary operator $[H_0 - c]$, with $c \in \mathbf{R}$, and $c < E_0$. The expression in the sum in $\Sigma_{12}(z)$ is bounded by

$$\| P_0(i\phi) W_l(i\phi) [H_0 - c]^{-\frac{1}{2}} \| \cdot \| [H_0 - c]^{\frac{1}{2}} \bar{P}_0(i\phi) R_{\bar{P}_0(i\phi)}(H_0(i\phi) - z) \bar{P}_0(i\phi) [H_0 - c]^{\frac{1}{2}} \| \cdot \\ \| [H_0 - c]^{-\frac{1}{2}} W_{l'}(i\phi) P_0(i\phi) \| ,$$

and is of order $O(1 = g^0)$, which follows from lemmata 4.9 and 4.10. •

We will now show that the term $P_0(i\phi) W_g(i\phi) P_0(i\phi)$ can also be made small. The following two lemmata will be used repeatedly in what follows.

Lemma 4.15 (Pull-through formula) *Let $F : \mathbf{R}^+ \to \mathbf{C}$ be a borel function with $F[r] = O(1 + r)$. Then, $F[H_f]$ is defined on the domain of H_f by its spectral decomposition, and the following intertwining relations hold*

$$\begin{aligned} F[H_f] a^\dagger(\vec{k}) &= a^\dagger(\vec{k}) F[H_f + \omega(\vec{k})] , \\ a(\vec{k}) F[H_f] &= F[H_f + \omega(\vec{k})] a(\vec{k}) . \end{aligned}$$

Proof The assertions follow from $F[H_f] \prod_{i=1}^n a^\dagger(\vec{k}_i) \Omega = F[\sum_{i=1}^n] \prod_{i=1}^n a^\dagger(\vec{k}_i) \Omega$. •

Lemma 4.16 *Let $\psi \in \mathcal{H}_f$, and $f \in C_0(\mathbf{R}^3)$. Then, the following bounds hold.*

$$\| a(f)\psi \|_{\mathcal{H}_f} \leq \| \omega^{-\frac{1}{2}} f \|_{L^2} \| H_f^{\frac{1}{2}} \psi \|_{\mathcal{H}_f} \tag{63}$$

$$\| a^\dagger(f)\psi \|^2_{\mathcal{H}_f} \leq \| \omega^{-\frac{1}{2}} f \|_{L^2} \| H_f^{\frac{1}{2}} \psi \|^2_{\mathcal{H}_f} + \| f \|^2_{L^2} \| \psi \|^2_{\mathcal{H}_f} \tag{64}$$

Proof In the following, we will omit the integration variables and the subscripts $\mathcal{H}_f$. Due to the Cauchy-Schwarz theorem, we have

$$\| a(f)\psi \| \leq \int |f| \, \| a(f)\psi \| \leq \left(\int \frac{|f|^2}{\omega} \right)^{\frac{1}{2}} \left(\int \omega \, \| a\psi \|^2 \right)^{\frac{1}{2}} ,$$

We further decompose $\Sigma_1(z)$ into a dominant and a small term, $\Sigma_1(z) = \Sigma_{11}(z) + \Sigma_{12}(z)$, where

$$\Sigma_{11}(z) := g^2 P_0(i\phi) W_1(i\phi) R_{\bar{P}_0(i\phi)}(H_0(i\phi)) W_1(i\phi) P_0(i\phi) ,$$

and

$$\Sigma_{12}(z) := \sum_{l+l' \geq 3} g^{l+l'} P_0(i\phi) W_l(i\phi) R_{\bar{P}_0(i\phi)}(H_0(i\phi)) W_{l'}(i\phi) P_0(i\phi) .$$

Lemma 4.14 *Let $z \in D(E_j, \frac{\rho_0}{2})$. Then, $\| \Sigma_{12}(z) \| = O(g^3)$.*

Proof We introduce an auxiliary operator $[H_0 - c]$, with $c \in \mathbf{R}$, and $c < E_0$. The expression in the sum in $\Sigma_{12}(z)$ is bounded by

$$\| P_0(i\phi) W_l(i\phi) [H_0 - c]^{-\frac{1}{2}} \| \cdot \| [H_0 - c]^{\frac{1}{2}} R_{\bar{P}_0(i\phi)}(H_0(i\phi)) [H_0 - c]^{\frac{1}{2}} \| \cdot \\ \| [H_0 - c]^{-\frac{1}{2}} W_{l'}(i\phi) P_0(i\phi) \| ,$$

and is of order $O(1 = g^0)$, which follows from lemmata 4.9 and 4.10. •

We will now show that the term $P_0(i\phi) W_g(i\phi) P_0(i\phi)$ can also be made small. The following two lemmata will be used repeatedly in what follows.

Lemma 4.15 (Pull-through formula) *Let $F : \mathbf{R}^+ \to \mathbf{C}$ be a borel function with $F[r] = O(1 + r)$. Then, $F[H_f]$ is defined on the domain of H_f by its spectral decomposition, and the following intertwining relations hold*

$$\begin{aligned} F[H_f] a^\dagger(\vec{k}) &= a^\dagger(\vec{k}) F[H_f + \omega(\vec{k})] , \\ a(\vec{k}) F[H_f] &= F[H_f + \omega(\vec{k})] a(\vec{k}) . \end{aligned}$$

Proof The assertions follow from $F[H_f] \prod_{i=1}^n a^\dagger(\vec{k}_i) \Omega = F[\sum_{i=1}^n] \prod_{i=1}^n a^\dagger(\vec{k}_i) \Omega$. •

Lemma 4.16 *Let $\psi \in \mathcal{H}_f$, and $f \in C_0(\mathbf{R}^3)$. Then, the following bounds hold.*

$$\| a(f)\psi \|_{\mathcal{H}_f} \leq \| \omega^{-\frac{1}{2}} f \|_{L^2} \| H_f^{\frac{1}{2}} \psi \|_{\mathcal{H}_f} \tag{63}$$

$$\| a^\dagger(f)\psi \|^2_{\mathcal{H}_f} \leq \| \omega^{-\frac{1}{2}} f \|_{L^2} \| H_f^{\frac{1}{2}} \psi \|^2_{\mathcal{H}_f} + \| f \|^2_{L^2} \| \psi \|^2_{\mathcal{H}_f} \tag{64}$$

Proof In the following, we will omit the integration variables and the subscripts $\mathcal{H}_f$. Due to the Cauchy-Schwarz theorem, we have

$$\| a(f)\psi \| \leq \int |f| \, \| a(f)\psi \| \leq \left(\int \frac{|f|^2}{\omega} \right)^{\frac{1}{2}} \left(\int \omega \, \| a\psi \|^2 \right)^{\frac{1}{2}} ,$$

which implies (63), since

$$\int \omega \parallel a(f)\psi \parallel^2 = \langle \psi, H_f \psi \rangle \, .$$

(64) is obtained from

$$a(f)a^\dagger(f) = \int\int \bar{f}(p)f(k)a(p)a^\dagger(k) = \int\int \bar{f}(p)f(k)a^\dagger(k)a(p) + \int |f(p)|^2 \, ,$$

and from using that

$$\int\int \bar{f}(p)f(k)\langle \psi, a^\dagger(k)a(p)\psi\rangle \leq \left(\int f(p) \parallel a(p)\psi \parallel^2\right)^2 \leq \int \frac{|f|^2}{\omega}\langle \psi, H_f\psi\rangle \, . \bullet$$

Lemma 4.17 *Let β and $\Gamma(\beta)$ be the constants specified in hypothesis 1. Then,*

$$\parallel g^2 P_0(i\phi)W_2(i\phi)P_0(i\phi) \parallel \leq 3g^2\rho_0^{1+\beta}\Gamma(\beta)^2 \tag{65}$$

Proof $P_0(i\phi)W_2(i\phi)P_0$ is a sum of three contributions which have to be estimated separately. However, the estimates all have the same structure, and we demonstrate the proof for the term $\int d^3\vec{k}d^3\vec{k}' G_{11}^{(i\phi)}(\vec{k},\vec{k}') \otimes a^\dagger(\vec{k})a(\vec{k}')$. Let ζ and ψ be arbitrary vectors in $\mathcal{H}$. Then, due to hypothesis 1, we get

$$\begin{aligned} &|\langle P_0(i\phi)\zeta, \int d^3\vec{k}\, d^3\vec{k}'\, G_{11}^{(i\phi)}(\vec{k},\vec{k}') \otimes a^\dagger(\vec{k})a(\vec{k}')P_0(i\phi)\psi\rangle| \\ &\qquad \leq \int d^3\vec{k}\, d^3\vec{k}'\, J(\vec{k})J(\vec{k}') \parallel a(\vec{k})P_0(i\phi)\zeta \parallel \parallel a(\vec{k}')P_0(i\phi)\psi \parallel \, . \end{aligned}$$

This expression gives the estimate

$$\begin{aligned} &\parallel a(J)P_0(i\phi)\zeta \parallel \parallel a(J)P_0(i\phi)\psi \parallel \leq \\ &\qquad \parallel \omega^{-\frac{1}{2}}\chi_{\omega\leq\rho_0} J \parallel_{L^2}^2 \parallel \mathbf{1}_{el} \otimes H_f^{\frac{1}{2}} P_0(i\phi)\zeta \parallel \parallel \mathbf{1}_{el} \otimes H_f^{\frac{1}{2}} P_0(i\phi)\psi \parallel \, , \end{aligned}$$

due to lemma 4.16. By definition, $\Gamma(\beta) = \parallel [1+\omega^{-1-\beta}]^{\frac{1}{2}} J \parallel_{L^2}$. Hence, we find

$$\parallel \omega^{-\frac{1}{2}}\chi_{\omega\leq\rho_0} J \parallel_{L^2} \leq \sup_{\omega\leq\rho_0} \omega^{-\frac{1}{2}}[1+\omega^{-1-\beta}]^{-\frac{1}{2}}\Gamma(\beta) \leq \rho_0^{\frac{\beta}{2}}\Gamma(\beta) \, .$$

Furthermore, $\parallel \mathbf{1}_{el} \otimes H_f^{\frac{1}{2}} P_0(i\phi)\psi \parallel \leq \rho^{\frac{1}{2}} \parallel \psi \parallel$, and the same holds for ζ. This concludes the proof. $\bullet$

Lemma 4.18 *Let β and $\Gamma(\beta)$ be the constants specified in hypothesis 1. Then,*

$$\parallel gP_0(i\phi)W_1(i\phi)P_0(i\phi) \parallel \leq 2g\rho_0^{\frac{1+\beta}{2}}\Gamma(\beta)$$

Proof The interaction term W_1 consists of two parts which can be estimated in the same way. We only demonstrate the proof for $\int d^3\vec{k} G_{01}^{(i\phi)}(\vec{k}) \otimes a(\vec{k})$. For the expression

$$\| gP_0(i\phi)W_1(i\phi)P_0(i\phi)\psi \|^2 = \langle \int d^3\vec{k}\, G_{01}^{(i\phi)}(\vec{k}) \otimes a(\vec{k})P_0(i\phi)\psi, \int d^3\vec{k}'\, G_{01}^{(i\phi)}(\vec{k}') \otimes a(\vec{k})P_0(i\phi)\psi\rangle ,$$

we find the upper bound $\rho_0^{1+\beta}\Gamma(\beta)^2 \| \psi \|^2$ from lemma 4.17, which concludes the proof. •

We see that the term $P_0(i\phi)W_g(i\phi)P_0(i\phi)$ can be made small by choosing a small cutoff ρ_0. This cutoff-dependency originates from the presence of creation and annihilation operators in the complex dilated interaction Hamiltonian $W_g(i\phi)$. There remains a last contribution $\Sigma_{11}(z)$ to the Feshbach map which we have not estimated yet. By use of the pull-through formula, it obtains the form

$$\Sigma_{11}(z) = \Sigma_{111}(z) + \Sigma_{112}(z) ,$$

where

$$\Sigma_{111}(z) = \int d^3\vec{k}\, d^3\vec{k}'\, P_0(i\phi)[G_{10}^{(i\phi)} \otimes \mathbf{1}_f]R_{\bar{P}_0,H_0}(\omega(\vec{k}))[G_{10}^{(i\phi)} \otimes \mathbf{1}_f]P_0(i\phi) ,$$

and where $\Sigma_{112}(z)$ is given by

$$\begin{aligned} &\int d^3\vec{k}\, d^3\vec{k}'\, P_0(i\phi)[G_{10}^{(i\phi)} \otimes a^\dagger(\vec{k})a^\dagger(\vec{k}')]R_{\bar{P}_0,H_0}(\omega(\vec{k}'))[G_{10}^{(i\phi)} \otimes \mathbf{1}_f]P_0(i\phi) \\ + &\int d^3\vec{k}\, d^3\vec{k}'\, P_0(i\phi)[G_{01}^{(i\phi)} \otimes \mathbf{1}_f]R_{\bar{P}_0,H_0}(\omega(\vec{k}))[G_{01}^{(i\phi)} \otimes a(\vec{k})a(\vec{k}')]P_0(i\phi) \\ + &\int d^3\vec{k}\, d^3\vec{k}'\, P_0(i\phi)[G_{10}^{(i\phi)} \otimes a^\dagger(\vec{k})]R_{\bar{P}_0,H_0}(0)[G_{01}^{(i\phi)} \otimes a(\vec{k})]P_0(i\phi) \\ + &\int d^3\vec{k}\, d^3\vec{k}'\, P_0(i\phi)[G_{01}^{(i\phi)} \otimes a^\dagger(\vec{k}')]R_{\bar{P}_0,H_0}(\omega(\vec{k}) + \omega(\vec{k}'))[G_{10}^{(i\phi)} \otimes a(\vec{k})]P_0(i\phi) . \end{aligned}$$

For notational convenience, we have introduced the operator

$$R_{\bar{P}_0,H_0}(\omega) := \{\bar{P}_0^\omega(i\phi)[H_0(i\phi) + e^{-i\phi}\omega - z]\}^{-1} ,$$

where $\bar{P}_0^\omega(i\phi) := \mathbf{1} - P_0^\omega(i\phi)$ with $P_0^\omega(i\phi) := P_j(i\phi) \otimes \chi_{H_f+\omega\leq\rho_0}$. One observes that all terms in $\Sigma_{112}(z)$ include creation and annihilation operators. Consequently, one can make them small by a suitable choice of ρ_0. We note that the cutoff ρ_0 has been introduced as an auxiliary parameter for the proof. For our intention to study the spectrum of the Feshbach map in a $\frac{\rho_0}{2}$-ball around E_j, we are free to minimize ρ_0. However, if ρ_0 is too small, we will only find an estimate on the spread of the resolvent set of $H_g(i\phi)$ in the vicinity of E_j, but not on the location of the spectrum.

Lemma 4.19 *Let β and $\Gamma(\beta)$ be the constants specified in hypothesis 1, and let $c \in \mathbf{R}$ satisfy $c < E_0$. Then, there is a constant γ_c, such that*

$$\| \Sigma_{112}(z) \| \leq 8g^2\gamma_c\Gamma_c\Gamma(\beta)^2[E_0 - c]^{-\frac{1}{2}}\rho_0^{1+\beta} .$$

Proof $\Sigma_{112}(z)$ consists of four terms which have to be estimated separately, but in the same way. We will only demonstrate the proof for the term which contains the operator $a(\vec{k})a(\vec{k}')$ on Fock space. Let $c \in \mathbf{R}$ satisfy $c < E_0$. Then, there is a constant γ_c, such that

$$\| [-\Delta + 1]^{\frac{1}{4}} [H_0 - c]^{-\frac{1}{4}} \| \leq \gamma_c^{\frac{1}{2}} .$$

This can easily be shown with the spectral theorem, since both operators in question are self-adjoint. From hypothesis 1, we therefore find that

$$\| [H_0 - c]^{-\frac{1}{4}} G_{m,n}^{(i\phi)}(\vec{k}) [H_0 - c]^{-\frac{1}{4}} \| \leq \gamma_c J(\vec{k})$$

for $m + n = 1$. We denote the term in $\Sigma_{112}(z)$ proportional to $a(\vec{k})a(\vec{k}')$ by T_{aa}. Then, we have that

$$\| T_{aa}\psi \| \leq \gamma_c \int d^3\vec{k}\, d^3\vec{k}'\, J(\vec{k}) J(\vec{k}') \| [H_0 - c]^{\frac{1}{4}} R_{\bar{P}_0, H_0}(\omega(\vec{k})) [H_0 - c]^{\frac{1}{4}} \|$$
$$\| \mathbf{1}_{el} \otimes a(\vec{k}) a(\vec{k}') \chi_{H_f \leq \rho_0} \psi \| \; .$$

We introduce the operator $R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda)$ on $\mathcal{H}_{el}$ by the spectral decomposition $R_{\bar{P}_0, H_0}(\omega(\vec{k})) =: \int R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda) \otimes dQ_\lambda$, where dQ_λ is the spectral measure associated to H_f. The term $\| [H_0 - c]^{\frac{1}{4}} R_{\bar{P}_0, H_0}(\omega(\vec{k})) [H_0 - c]^{\frac{1}{4}} \|$ is then identical to

$$\sup_{\lambda > 0} \| \int_{t' \geq t \geq E_0} \frac{[t + \lambda - c]^{\frac{1}{4}}}{[t' + \lambda - c]^{\frac{1}{4}}} dP_t R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda) [H_0 + \lambda - c] dP_{t'}$$
$$+ \int_{t \geq t' \geq E_0} \frac{[t' + \lambda - c]^{\frac{1}{4}}}{[t + \lambda - c]^{\frac{1}{4}}} dP_t [H_0 + \lambda - c] R_{\bar{P}_0, H_0}(\omega(\vec{k}) \lambda) dP_{t'} \| ,$$

which is bounded by

$$2 \sup_{\lambda > 0} [E_0 + \lambda - c]^{-\frac{1}{2}} \| [H_{el} + \lambda - c] R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda) \|$$
$$\leq 2 [E_0 - c]^{-\frac{1}{2}} \sup_{\lambda > 0} \| [H_{el} + \lambda + \omega(\vec{k}) - c] R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda) \| \cdot$$
$$\| [H_{el} + \lambda + \omega(\vec{k}) - c]^{-1} [H_{el} + \lambda - c] \|$$

By definition, we have that $R_{\bar{P}_0(i\phi)}(H_0(i\phi)) = R_{\bar{P}_0, H_0}(0)$. $R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda)$ differs from $R_{\bar{P}_0, H_0}(0, \lambda)$ only by the substitution $\lambda \to \lambda + \omega(\vec{k})$ of the photon variable. Hence we obtain that

$$\sup_{\lambda > 0} \| [H_{el} + \lambda + \omega(\vec{k}) - c] R_{\bar{P}_0, H_0}(\omega(\vec{k}), \lambda) \|$$
$$= \sup_{\lambda > \omega(\vec{k})} \| [H_{el} + \lambda - c] R_{\bar{P}_0, H_0}(0, \lambda) \|$$
$$\leq \sup_{\lambda > 0} \| [H_{el} + \lambda - c] R_{\bar{P}_0, H_0}(0, \lambda) \| \leq \Gamma_c$$

Furthermore, $\sup_{\lambda > 0} \| [H_{el} + \lambda + \omega(\vec{k}) - c]^{-1} [H_{el} + \lambda - c] \|$ is bounded by 1. We summarize that

$$\begin{aligned}
\| T_{aa}\psi \| &\leq 2\gamma_c \Gamma_c [E_0 - c]^{-\frac{1}{2}} \| \mathbf{1}_{el} \otimes a(J) a(J) \chi_{H_f \leq \rho_0} \psi \| \\
&\leq 2\gamma_c \Gamma_c [E_0 - c]^{-\frac{1}{2}} \| \omega^{-\frac{1}{2}} \chi_{\omega \leq \rho_0} J \|_{L^2}^2 \| \mathbf{1}_{el} \otimes H_f \chi_{H_f \leq \rho_0} \psi \| \\
&\leq 2\gamma_c \Gamma_c \Gamma(\beta)^2 [E_0 - c]^{-\frac{1}{2}} \rho_0^{1+\beta} \| \psi \| \; . \bullet
\end{aligned}$$

We will now give the proof of lemma 4.9, which is much in the same line as the proofs of the last three lemmata.

Proof of lemma 4.9 We will only demonstrate the proof of the boundedness of the expression $\| [H_0 - c]^{-\frac{1}{2}} W_g(i\phi)[H_0 - c]^{-\frac{1}{2}} \|$. The other proofs are almost identical. $c \in \mathbf{R}$ satisfies $c < E_0$. We will consider W_1 and W_2 separately. For W_1, we only demonstrate the proof for the term $\int d^3\vec{k}\, G_{01}^{(i\phi)}(\vec{k}) \otimes a(\vec{k})$. For $\psi \in \mathcal{H}$, and by use of the pull-through formula, we estimate

$$\| \int d^3\vec{k}\, [H_0 - c]^{-\frac{1}{2}} G_{01}^{(i\phi)}(\vec{k})[H_0 + \omega(\vec{k}) - c]^{-\frac{1}{2}} \otimes a(\vec{k})\psi \| \ .$$

Using hypothesis 1 and $\| [-\Delta + 1]^{\frac{1}{4}}[H_0 - c]^{-\frac{1}{4}} \| \leq \gamma_c^{\frac{1}{2}}$, we obtain the bound

$$\begin{aligned} \gamma_c \| \int d^3\vec{k}\ \| [H_0 + \omega(\vec{k}) - c]^{-\frac{1}{2}} \|\ J(\vec{k}) \otimes a(\vec{k})\psi \| \\ \leq \gamma_c \| \int d^3\vec{k}\, [E_0 + \omega(\vec{k}) - c]^{-\frac{1}{2}}\, J(\vec{k}) \otimes a(\vec{k})\psi \| \ . \end{aligned}$$

This is the same as $\gamma_c \| \mathbf{1}_{el} \otimes a([E_0 + \omega - c]^{-\frac{1}{2}} J)\psi \|$, which is smaller than

$$\gamma_c \sup_{\omega>0} [E_0 + \omega(\vec{k}) - c]^{-\frac{1}{2}} [1 + \omega^{-1-\beta}]^{-\frac{1}{2}} \omega^{\frac{1}{2}}\, \Gamma(\beta) \| \psi \| \ .$$

This expression is bounded, as can be easily checked, and the first part of the proof is finished. To prove that the term proportional to W_2 is bounded, we only demonstrate the argumentation for $\int d^3\vec{k}\, d^3\vec{k}'\, G_{02}^{(i\phi)}(\vec{k}, \vec{k}') \otimes a(\vec{k})a(\vec{k}')$. By use of the pull-through formula, we find the expression

$$\| \int d^3\vec{k}\, d^3\vec{k}'\, [H_0 - c]^{-\frac{1}{2}} G_{02}^{(i\phi)}(\vec{k}, \vec{k}')[H_0 + \omega(\vec{k}) + \omega(\vec{k}') - c]^{-\frac{1}{2}} \otimes a(\vec{k})a(\vec{k}')\psi \| \,,$$

which is bounded by

$$\gamma_c \| \int d^3\vec{k}\, d^3\vec{k}'\, [E_0 + \omega(\vec{k}) + \omega(\vec{k}') - c]^{-\frac{1}{2}}\, J(\vec{k})J(\vec{k}') \otimes a(\vec{k})a(\vec{k})\psi \| \ .$$

It is now straightforward to show that this expression is finite. •

Now we can extract the transition matrix T_j from the term $\Sigma_{111}(z)$. First, we partition $\Sigma_{111}(z)$ into $\Sigma_{111}^{od}(z) + \Sigma_{111}^{d}(z)$, where the superscripts od and d stand for "off-diagonal" and "diagonal". The contribution $\Sigma_{111}^{od}(z)$, given by

$$\begin{aligned} \Sigma_{111}^{od}(z) = \int d^3\vec{k}\, P_j(i\phi)[G_{01}^{(i\phi)}(\vec{k}) \otimes \mathbf{1}_f][\bar{P}_j(i\phi)[H_{el}(i\phi) \otimes \mathbf{1}_f] \\ + e^{-i\phi}\mathbf{1}_{el} \otimes [H_f + \omega(\vec{k})]]^{-1} \cdot [G_{10}^{(i\phi)}(\vec{k}) \otimes \mathbf{1}_f] P_j(i\phi) \otimes \chi_{H_f \leq \rho_0} \,, \end{aligned}$$

describes self-energy processes which involve intermediate states outside of the eigenspace spanned by $P_j(i\phi)$, whereas the contribution

$$\Sigma_{111}^{d}(z) := \int d^3\vec{k}\, [P_j(i\phi) G_{01}^{(i\phi)}(\vec{k}) P_j(i\phi) G_{10}^{(i\phi)}(\vec{k}) P_j(i\phi) \otimes \mathbf{1}_f] \cdot$$

$$[e^{-i\phi}\mathbf{1}_{el} \otimes [H_f + \omega(\vec{k})] + E_j - z]^{-1}\chi_{H_f \leq \rho_0}\chi_{H_f+\omega(\vec{k}) \geq \rho_0}$$

describes self-energy processes that remain within this given eigenspace of $H_{el}(i\phi)$. We recall that the transition matrix has the form

$$[T_j]_{ll'} = \int_{-\infty}^{E_j - 0} dE \int d^3\vec{k}\, \langle G_{10}(\vec{k})\psi_{j,l}, d\chi_{H_f \leq E} G_{10}(\vec{k})\psi_{j,l'}\rangle\, \delta[\omega(\vec{k}) - E_j + E] ,$$

and define the $N_j \times N_j$ matrices

$$Z_j^{(i\phi)od} := \int d^3\vec{k}\, P_j(i\phi) G_{01}^{(i\phi)}(\vec{k})[\bar{P}_j(i\phi) - E_j + e^{-i\phi}\omega(\vec{k})]^{-1} G_{10}^{(i\phi)}(\vec{k}) P_j(i\phi)\chi_{H_f \leq \rho_0}$$

and

$$Z_j^{(i\phi)d} := \int d^3\vec{k}\, [e^{-i\phi}\omega(\vec{k})]^{-1} P_j(i\phi) G_{01}^{(i\phi)}(\vec{k}) P_j(i\phi) G_{10}^{(i\phi)}(\vec{k}) P_j(i\phi) .$$

Lemma 4.20 *The following estimates hold.*

$$\begin{aligned} \| \Sigma_{111}^{od}(z) - Z_j^{(i\phi)od} \| &= O(\rho_0 g^2) \\ \| \Sigma_{111}^{d}(z) - Z_j^{(i\phi)d} \| &= O(\rho_0^{\beta} g^2) \ \bullet \end{aligned}$$

The proof is very similar to the proofs given in the last four lemmata and can be found in [5]. We reverse the complex dilation in $Z_j^{(\phi)d}$ and $Z_{(\phi)j}^{od}$ to carry out the $\vec{k}$ integration, which yields

$$Z_j^{(0)od} = \int d^3\vec{k}\, P_j(i\phi) G_{01}^{(0}_{01})(\vec{k})[\bar{P}_j(i\phi) - E_j + \omega(\vec{k}) - i0]^{-1} G_{10}^{(0)}(\vec{k}) P_j(i\phi)\chi_{H_f \leq \rho_0}$$

and

$$Z_j^{(0)d} = \int d^3\vec{k}\, \omega(\vec{k})^{-1}\, P_j(i\phi) G_{01}^{(0)}(\vec{k}) P_j(i\phi) G_{10}^{(0)}(\vec{k}) P_j(i\phi) .$$

We observe that $Z_j^{(0)d} = Re[Z_j^{(0)d}]$, and that $Im[Z_j^{(0)od}] = T_j$, where we have used the notation $Re[Z] := \frac{1}{2}[Z + Z^*]$ and $Im[Z] := \frac{1}{2i}[Z - Z^*]$. Hence we conclude that the Lamb shift and the inverse life-times of resonances are given by

$$\begin{aligned} \Delta E_j &:= g^2 Z_j^{(0)d} \\ \Gamma_j &:= g^2 Im[Z_j^{(0)od}] = g^2 T_j . \end{aligned}$$

By hypothesis 2, Γ_j is self-adjoint and positive definite. Hence, there is a constant γ_j, such that $\Gamma_j \geq \gamma_j \mathbf{1}_{el} > 0$. We apply complex dilation on these matrices, and obtain $P_j(i\phi)\Delta E_j P_j(i\phi)$ and $P_j(i\phi)\Gamma_j P_j(i\phi)$, respectively. We conclude the results of this section in the following theorem.

Thm. 2 *Let $z \in D(E_j, \frac{\rho_0}{2})$, and let β be the constant specified in hypothesis 1. Assume that hypotheses 1 and 2 hold, and set $\rho_0 = g^{2-\frac{2\beta}{2+\beta}}$. Consequently, the Feshbach map is given by*

$$\begin{aligned} f_{P_0(i\phi)}(H_g(i\phi) - z) = P_0(i\phi)[E_j - z + [\Delta E_j - i\Gamma_j] + e^{-i\phi}\mathbf{1}_{el} \otimes H_f] P_0(i\phi) \\ + O(g^{2+\frac{\beta}{2+\beta}}) , \end{aligned}$$

which is invertible on $\mathcal{O}' := D(E_j, \frac{\rho_0}{2}) \cap \{Im(z) > -g^2\gamma_j + O(g^{2+\frac{\beta}{2+\beta}})\}$.

Proof We have to prove the error estimate of order $O(g^{2+\frac{\beta}{2+\beta}})$. Hence we have to consider all error estimates from lemma 4.17 to lemma 4.20. The two largest terms stem from lemma 4.18, which is of order $O(g\rho_0^{\frac{1+\beta}{2}})$, and from lemma 4.20, which is of order $O(g^2\rho_0^\beta)$. For the choice $\rho_0 = g^{2-\frac{2\beta}{2+\beta}}$, one can easily verify that the largest error among all small terms is of order $O(g^{2+\frac{\beta}{2+\beta}})$. This concludes the proof. •

As a corollary, we have the following theorem.

Thm. 3 *Let $z \in D(E_j, \frac{\rho_0}{2})$, and let β be the constant specified in hypothesis 1. Assume that hypotheses 1 and 2 hold, and define the region*

$$\mathcal{O} := [E_j - D, E_j + D] - i\gamma_j + i\mathbf{R}^+ + O(g^{2+\frac{\beta}{2+\beta}}),$$

where D is smaller than $[1 - O(g^2)]$ times the distance between E_j and the closest neighboring eigenvalue. Then, there exists an angle ϕ such that $H_g(i\phi)$ is invertible on $\mathcal{O}$.

Proof By suitable choice of ϕ, the spectrum of $H_0(i\phi)$ does not intersect $\mathcal{O} \setminus \mathcal{O}'$. Hence, $[z - H_0(i\phi)]^{-1}$ is bounded for all $z \in \mathcal{O} \setminus \mathcal{O}'$. By an argument similar to the proof of lemma 4.5, one can use the information on $Num(H_0(i\phi))$ from lemma 4.3 to show that this norm is bounded by a constant which is independent of z. Therefore, it is possible to prove the boundedness of $[z - H_g(i\phi)]^{-1}$ on $\mathcal{O} \setminus \mathcal{O}'$ by use of the second resolvent formula, as in the proof of lemma 4.11. From theorem 1 and 2, we conclude that $H_g(i\phi)$ is invertible on $\mathcal{O}'$. Hence, the assertion of the theorem follows. •

5 Decay of resonances

In this section, we will prove the exponential decay of eigenstates of the unperturbed Hamiltonian H_{el} if the interaction with the radiation field is turned on. We consider the time evolution of the expectation value

$$\langle \psi, e^{-itH_g}\psi \rangle \,. \tag{66}$$

We have $\psi = \psi_j \otimes \Omega_f \in \mathcal{H}$, where Ω_f is the Fock vacuum. ψ_j is an eigenvector of H_{el} which corresponds to the eigenvalue E_j. We denote the orthogonal projector on the eigenspace associated to E_j by P_j, and the projector $P_j \otimes P_{\Omega_f}$ by P_0. P_{Ω_f} is the projector on the linear subspace spanned by the Fock vacuum. Due to the self-adjointness of the Hamiltonian H_g, we can employ the spectral theorem to analyze (66). The spectral measure dP_E associated to H_g can be represented by the resolvent in the following way:

$$\begin{aligned}\langle \psi, e^{-itH_g}\psi \rangle &= \int e^{-itE}\langle \psi, dP_E\psi \rangle \\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int dE e^{-itE}[\langle \psi, [E - i\epsilon - H_g]^{-1}\psi \rangle - \langle \psi, [E + i\epsilon - H_g]^{-1}\psi \rangle]\,.\end{aligned} \tag{67}$$

The integration contour encloses a thin strip $I_\epsilon := \{z | Im(z) \le \epsilon\}$ which contains the real axis. The resolvent $R_{H_g}(z) := [z - H_g]^{-1}$ is evaluated on the first Riemann sheet for both terms in the bracket. Since we know the approximate locations of the resonances next to E_j from section 4, we will deform the integration contour into their vicinity in the negative half-plane on the second sheet of the Riemann surface. The factor e^{-itE} will then approximately generate the $e^{-t\Gamma}$ decay rate, where $-\Gamma$ is the imaginary part of the resonance next to E_j, which lies closest to the real axis. The first term in the brackets is evaluated below the real axis, hence it will cause no difficulty if the integration path is deformed into $\mathbf{C}^-$. However, the second term is evaluated in $\mathbf{C}^+$, and the deformed integration path crosses the real line, where $R_{H_g}(z)$ is singular.

This problem can be solved by using dilation analyticity of H_g. According to Balslev-Combes theory, we start with

$$\langle \psi, R_{H_g}(z)\psi\rangle = \langle U(\theta)\psi, R_{H_g(\theta)}(z)U(\theta)\psi\rangle \,, \tag{68}$$

where $U(\theta)$ is the unitary dilation operator for $\theta \in \mathbf{R}$, and where $H_g(\theta) := U(\theta)H_gU(\theta)^{-1}$. The equality of the lhs and the rhs of (68) is due to the unitarity of $U(\theta)$. We assume that the vector $\psi \in \mathcal{H}$ is dilation analytic, and that z lies in the upper half-plane, above the continuous spectrum of H_g. Since H_g is dilation analytic, the rhs of (68) can be analytically continuated to $\theta \in \mathcal{V}^+$, where $\mathcal{V}^+$ is a complex neighborhood of $\{0\}$ which lies entirely in the upper half-plane. Because the identity (68) holds for all real θ, it must also hold for all complex $\theta \in \mathcal{V}^+$. Consequently,

$$\langle U(i\phi)\psi, R_{H_g(i\phi)}(z)U(i\phi)\psi\rangle \,, \tag{69}$$

with $\phi > 0$, is an analytic continuation of the lhs of (68). From section 4, we know that (69) can be continuated from $z \in \mathbf{C}^+$ to the part of the resolvent set of $H_g(i\phi)$ which lies in $\mathbf{C}^-$. We define the deformed integration contour $J \cup \bar{J}$ by way of

$$\begin{aligned}
\bar{J} &= \mathbf{R} \setminus \{E_j + [-D, D]\} \\
J &= J_< \cup J_\parallel \cup J_> \\
J_\parallel &= E_j - i(\Gamma - \delta) + [-D, D] \\
J_< &= E_j + D - i[0, \Gamma - \delta] \\
J_> &= E_j - D - i[0, \Gamma - \delta] \,,
\end{aligned}$$

where δ is a small number which ensures that the integration path does not intersect the spectrum of $H_g(i\phi)$. The number D is chosen such that $2D$ is smaller than the spacing between E_j and the eigenvalue which lies closest to it. Note that the region enclosed by J and the real axis is a subset of the region $\mathcal{O}$ defined in theorem 3. (67) is then equal to

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{J\cup\bar{J}} dz e^{-itz}[\langle\psi, R_{H_g}(z - i\epsilon)\psi\rangle - \langle\psi(i\phi), R_{H_g(i\phi)}(z + i\epsilon)\psi(i\phi)\rangle] \,. \tag{70}$$

We will first consider the contribution to (70) which stems from $J_\parallel$, given by

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{J_\parallel} dz e^{-itz}[\langle\psi, R_{H_g}(z - i\epsilon)\psi\rangle - \langle\psi(i\phi), R_{H_g(i\phi)}(z + i\epsilon)\psi(i\phi)\rangle] \,, \tag{71}$$

where we have defined $\psi(i\phi) := U(i\phi)\psi$. The norm of this expression is bounded by

$$e^{-[\Gamma-\delta]t}\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{J_\parallel} dz|[\langle\psi, R_{H_g}(z-i\epsilon)\psi\rangle - \langle\psi(i\phi), R_{H_g(i\phi)}(z+i\epsilon)\psi(i\phi)\rangle]|$$
$$\le e^{-[\Gamma-\delta]t}\frac{D}{2\pi}\sup_{z\in J_\parallel}[|\langle\psi, R_{H_g}(z-i\epsilon)\psi\rangle| + |\langle\psi(i\phi), R_{H_g(i\phi)}(z+i\epsilon)\psi(i\phi)\rangle|] . \tag{72}$$

The norm of the resolvent in the first, undilated term can be estimated by the inverse imaginary part of the variable z, which is simply $|Im(z)|^{-1} = [\Gamma-\delta]^{-1}$. From theorem 3, it is known that $H_g(i\phi)$ has no singularities on J, hence there exists a constant $\gamma_\parallel(\delta)$, such that for all $z \in J_\parallel$,

$$\| R_{H_g(i\phi)}(z) \| \le \gamma_\parallel(\delta) , \tag{73}$$

and a constant $\gamma_\perp$, such that for all $z \in J_<$ or $J_>$,

$$\| R_{H_g(i\phi)}(z) \| \le \gamma_\perp . \tag{74}$$

We conclude that (72) is bounded by

$$e^{-[\Gamma-\delta]t}\frac{D}{2\pi}[[\Gamma-\delta]^{-1} + \gamma_\parallel(\delta)] .$$

In order to study the contributions from the rest of J and from $\bar{J}$, we prove two easy lemmata.

Lemma 5.1 *There is a constant γ_c, such that*

$$\| W_g\psi \| , \| W_g(i\phi)\psi(i\phi) \| \le g\gamma_c . \tag{75}$$

Proof This follows from

$$\| W_g(i\phi)\psi(i\phi) \| \le \| W_g(i\phi)[H_0(i\phi)-c]^{-1} \| \cdot \| [H_0(i\phi)-c]\psi(i\phi) \|$$
$$\le g\Gamma_c^{(W)}[E_j - c] \| [H_0 - c][H_0(i\phi)-c]^{-1} \| ,$$

where lemma 4.9 has been used, and where c is a real number which satisfies $c < E_0$. One can show that the norm on the second line is bounded with the argumentation used to prove lemma 4.7. Obviously, this is also valid for $\phi = 0$. •

Lemma 5.2 *For all $|\phi| \le \phi_0$, and all $z \in \rho(H_0(i\phi))$, the following identity holds:*

$$\langle\psi(i\phi), R_{H_g(i\phi)}(z)\psi(i\phi)\rangle = [z-E_j]^{-1}$$
$$+[z-E_j]^{-2}\langle\psi(i\phi), W_g(i\phi)R_{H_g(i\phi)}(z)W_g(i\phi)\psi(i\phi)\rangle .$$

Proof By use of the second resolvent identity, the lhs of this expression is given by

$$\langle\psi(i\phi), R_{H_g(i\phi)}(z)\psi(i\phi)\rangle = [z-E_j]^{-1} + [z-E_j]^{-2}\langle\psi(i\phi), W_g(i\phi)\psi(i\phi)\rangle$$
$$+[z-E_j]^{-2}\langle\psi(i\phi), W_g(i\phi)R_{H_g(i\phi)}(z)W_g(i\phi)\psi(i\phi)\rangle ,$$

The term $\langle\psi(i\phi), W_g(i\phi)\psi(i\phi)\rangle$ vanishes, due to the following reason: $W_1(i\phi)$ acts on $\mathcal{H}_f$ by way of a single creation or annihilation operator, which results in zero expectation value for any state with a definite number of photons. $W_2(i\phi)$ is normal ordered, hence it has no expectation value with respect to $\psi = \psi_j \otimes \Omega_f$, because the photon part of ψ is the Fock vacuum. Note that these arguments do not touch the analyticity properties of the given operators. Hence, the assertion holds independently of the sign of ϕ. •

We will now estimate the contributions from the parts $J_<$ and $J_>$ of the integration contour. Both are equally analyzed, and we will restrict our considerations only on $J_<$. Hence, we will study

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{J_<} dze^{-itz}[\langle\psi, R_{H_g}(z-i\epsilon)\psi\rangle - \langle\psi(i\phi), R_{H_g(i\phi)}(z+i\epsilon)\psi(i\phi)\rangle] \,. \tag{76}$$

We apply lemma 5.2 to both terms in the integrand, and obtain

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{J_<} dze^{-itz}\{[z-i\epsilon-E_j]^{-1} - [z+i\epsilon-E_j]^{-1}\}\\ &+\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{J_<} dze^{-itz}\{[z-i\epsilon-E_j]^{-2}\langle\psi, W_gR_{H_g}(z-i\epsilon)W_g\psi\rangle\\ &\qquad -[z+i\epsilon-E_j]^{-2}\langle\psi(i\phi), W_g(i\phi)R_{H_g(i\phi)}(z+i\epsilon)W_g(i\phi)\psi(i\phi)\rangle\}\,.\end{aligned}$$

The first integral vanishes, since there are no singularities of the function $z-E_j$ on $J_<$. The norm of the complex dilated term in the second integral is bounded by $g^2D^{-2}\gamma_c^2\gamma_\perp$, which follows from lemma 5.1 and (74).

The norm of $\langle\psi, W_gR_{H_g}(z-i\epsilon)W_g\psi\rangle$ remains to be estimated. Due to the self-adjointness of W_g, it equals

$$\langle W_g\psi, R_{H_g}(z-i\epsilon)W_g\psi\rangle \,. \tag{77}$$

Again, we use dilatation analyticity of H_g, but this time, we consider the analytic continuation of $H_g(-i\phi)$ into the *upper* half-plane, for *negative* angles $-\phi$ ($\phi > 0$). (77) is analytically continuated to

$$\langle W_g(-i\phi)\psi(-i\phi), R_{H_g(-i\phi)}(z-i\epsilon)W_g(-i\phi)\psi(-i\phi)\rangle \,.$$

From [5] and section 4, we know that the spectrum of H_g has no embedded eigenvalues on the interval $[E_0,\Sigma)$, where Σ is the smallest threshold. The spectrum of $H_g(-i\phi)$ therefore consists of a branch of continuous spectrum which emanates from E_0, and which is rotated by ϕ into the upper half-plane. Other branches of continuous spectrum emanate at Σ and at points with larger real parts than Σ. Thus, $J_<$ and $J_>$ do not intersect the spectrum of $H_g(-i\phi)$, and consequently, there is a constant $\eta_\perp$, such that

$$\| \; [z-H_g(-i\phi)]^{-1} \; \| \leq \eta_\perp$$

for all z in some neighborhood of $J_<$ and $J_>$. We find the estimate

$$\begin{aligned}&|\langle W_g(-i\phi)\psi(-i\phi), R_{H_g(-i\phi)}(z-i\epsilon)W_g(-i\phi)\psi(-i\phi)\rangle| \leq\\ &\qquad\qquad \eta_\perp\,\langle W_g(-i\phi)\psi(-i\phi), W_g(-i\phi)\psi(-i\phi)\rangle\end{aligned}$$

for all $z \in J_<$ and $J_>$. $\langle W_g(-i\phi)\psi(-i\phi), W_g(-i\phi)\psi(-i\phi)\rangle$ is the (constant) analytic continuation of $\langle W_g\psi, W_g\psi\rangle$, which is bounded by $g^2\gamma_c^2$, according to lemma 5.1. We conclude that

$$\sup_{z\in J_<, J_>} |[z-E_j]^{-2}\langle W_g\psi, R_{H_g}(z-i\epsilon)W_g\psi\rangle| \le g^2\gamma_c^2\eta_\perp D^{-2}\,.$$

Therefore, the norm of the integrand of (76) is of order $O(g^2)$. The length of the paths $J_<$ and $J_>$ is given by $\Gamma - \delta$, which is also of order $O(g^2)$, as known from theorems 2 and 3. The norm of (76) is thus of order $O(g^4)$.

We will now estimate the contribution to (70), which stems from the integration on $\bar{J}$, given by

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\bar{J}} dz e^{-itz}[\langle\psi, R_{H_g}(E-i\epsilon)\psi\rangle - \langle\psi, R_{H_g}(z+i\epsilon)\psi\rangle]\,. \tag{78}$$

Using lemma 5.2, we obtain

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\bar{J}} dE e^{-itE}\{[E-i\epsilon-E_j]^{-1} - [E+i\epsilon-E_j]^{-1}\}\\ &+\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\bar{J}} dE e^{-itE}\{[E-i\epsilon-E_j]^{-2}\langle\psi, W_g R_{H_g}(E-i\epsilon)W_g\psi\rangle\\ &\qquad -[E+i\epsilon-E_j]^{-2}\langle\psi, W_g R_{H_g}(E+i\epsilon)W_g\psi\rangle\}\,.\end{aligned}$$

The first integral vanishes, because the function $[E-E_j]^{-1}$ has no singularity on $\bar{J}$. Due to the same reason, the second integral is identical to

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\bar{J}} dE e^{-itE}[E-E_j]^{-2}\{&\langle\psi, W_g R_{H_g}(E-i\epsilon)W_g\psi\rangle\\ &-\langle\psi, W_g R_{H_g}(E+i\epsilon)W_g\psi\rangle\}\,.\end{aligned}$$

We use the self-adjointness of the interaction Hamiltonian W_g, and arrive at

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\bar{J}} dE e^{-itE}[E-E_j]^{-2}\{&\langle W_g\psi, R_{H_g}(E-i\epsilon)W_g\psi\rangle\\ &-\langle W_g\psi, R_{H_g}(E+i\epsilon)W_g\psi\rangle\}\,,\end{aligned}$$

which is just an alternative representation of

$$\int_{\bar{J}} e^{-itE}[E-E_j]^{-2}\langle W_g\psi, dP_E W_g\psi\rangle\,,$$

where dP_E is the spectral measure associated to H_g. The norm of this expression can be estimated by

$$\sup_{E\in\bar{J}}|e^{-itE}[E-E_j]^{-2}|\ \|\, W_g\psi\,\|^2\,,$$

which is smaller than $g^2\gamma_c^2 D^{-2}$.

We summarize the results of this section in the following theorem.

Thm. 4 *Let* $\psi = \psi_j \otimes \Omega_f \in \mathcal{H}$, *where* Ω_f *is the Fock vacuum, and where* ψ_j *is an eigenvector of* H_{el} *which corresponds to the eigenvalue* E_j. *Let* $-\Gamma$ *be the imaginary part of the resonance in the vicinity of* E_j, *which lies closest to the real axis. For every* $\delta > 0$, *there exists a constant* $C(\delta)$, *such that*

$$|\langle \psi, e^{-itH_g} \psi \rangle| \leq C(\delta)\, e^{-[\Gamma - \delta]} + O(g^2)\,.$$

References

[1] C. Itzykson, J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill International Editions, 1985.

[2] P. Ramond, *Field Theory: A Modern Primer*, Addison-Wesley

[3] E. Balslev, J. M. Combes, *Spectral Properties of Many-body Schrödinger Operators with Dilatation-analytic Interactions*, Commun. math. Phys. 22, 280 - 294 (1971)

[4] M. Reed, B. Simon, *Methods of modern mathematical Physics, IV: Analysis of Operators*, Academic press (1978)

[5] V. Bach, J. Fröhlich, I. M. Sigal, *Quantum Electrodynamics of Confined Non-Relativistic Particles*, Preprint

[6] C. Fefferman, J. Fröhlich, G. M. Graf, *Stability of Ultraviolet-Cutoff Quantum Electrodynamics with Non-Relativistic Matter*, Preprint

[7] A. Soffer, M. I. Weinstein, *Time Dependent Resonance Theory*, Preprint

[8] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger operators*, Berlin, Heidelberg, New York: Springer 1987

[9] T. Kato, *Perturbation Theory of Linear Operators*

[10] W. Hunziker, *Resonances, Metastable States and Exponential Decay Laws in Perturbation Theory*, Commun. math. Phys. 132, 177 - 188 (1990)

[11] H. Spohn, *Asymptotic Completeness for Rayleigh Scattering*, Preprint

Spectral analysis of N-body Schrödinger operators by use of Balslev-Combes theory has been subject to intensive study ever since the original paper [3] appeared.

Dilation of the radiation field Hamiltonian H_f requires the choice of a one-photon basis $\{f_i\}$ on $\mathbf{C}_0(\mathbf{R}^3)$ to represent H_f in terms of $a^\dagger(f_i)$, $a(f_j)$

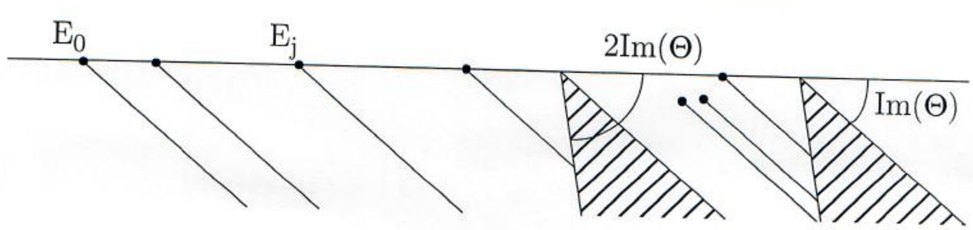
$$H_f = \sum_{i,j} \langle f_i, \omega f_j \rangle \, a^\dagger(f_i) a(f_j) \ .$$

Using $\langle f_i, \omega f_j \rangle = \int d^3\vec{k} \bar{f}_i(\vec{k}) \omega(\vec{k}) f_j(\vec{k})$ and the usual definition of $a^\sharp(f)$, it follows that

$$H_f^{(\theta)} = \sum_{i,j} \langle f_i, \omega_\theta f_j \rangle \, a^\dagger(f_i) a(f_j) \ ,$$

where $\omega_\theta(\vec{k}) := \omega(e^{-\theta}\vec{k})$. For the dispersion relation $\omega(\vec{k}) = |\vec{k}|$, one finds that $H_f^{(\theta)} = e^{-\theta} H_f$. Note that since the modulus function $|\cdot| : \mathbf{R}^3 \to \mathbf{R}^+$ is not analytic, the factor $e^{-\theta}$ must be extracted from $|e^{-\theta}\vec{k}|$ before analytic continuation. The spectrum of H_f consists of a single eigenvalue $\{0\}$ at the bottom of its continuous spectrum, corresponding to the vacuum state Ω_f. There is no separation between the point spectrum and the continuous spectrum of H_f, since photons are massless particles. For $\theta = i\phi \in \mathbf{C}^+$, the continuous spectrum of $H_f^{(i\phi)}$ that emanates from $\{0\}$ is rotated by ϕ into $\mathbf{C}^-$.

The spectrum of the complex dilated, free Hamiltonian $H_{g=0}^{(i\phi)}$ thus consists of branches of continuous spectrum of $H_f^{(i\phi)}$, which emanate from each element of the spectrum of $H_{el}^{(i\phi)}$, from each threshold, and from each resonance, cf. the following figure.



We have introduced the interaction Hamiltonian W_g at the end of section 1.2. The unitarily dilated $W_g^{(\theta)} = gW_1^{(\theta)} + g^2 W_2^{(\theta)}$ is characterized by the coupling functions $G_{m,n}^{(\theta)}$, $1 \le m+n \le 2$, which are defined by

$$\begin{aligned} W_1^{(\theta)} &= \int d^3\vec{k} [G_{10}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) + G_{01}^{(\theta)}(\vec{k}) \otimes a(\vec{k})] \\ W_2^{(\theta)} &= \int d^3\vec{k} d^3\vec{k}' [G_{20}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) a^\dagger(\vec{k}') + G_{02}^{(\theta)}(\vec{k}) \otimes a(\vec{k}) a(\vec{k}') \\ &\quad + G_{11}^{(\theta)}(\vec{k}) \otimes a^\dagger(\vec{k}) a(\vec{k}')] \,. \end{aligned}$$

We have $G_{m,n}^{(\theta)}(\vec{k}) = e^{3\theta/2} G_{m,n}(e^{-\theta}\vec{k})$ for $m+n = 1$, and $G_{m,n}^{(\theta)}(\vec{k}, \vec{k}') = e^{3\theta} G_{m,n}(e^{-\theta}\vec{k}, e^{-\theta}\vec{k}')$ for $m + n = 2$. Thus, the coupling functions are dilation analytic, and we choose $\theta = i\phi \in \mathbf{C}^+$. The properties of the coupling functions are specified in the following hypothesis.