COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 11

Due Friday, April 26, 2013, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Prove that an elliptic function has as many poles as zeros.

2. Problem

Let Λ be the lattice generated by the linearly independent vectors (ω_1, ω_2), and $\mathcal{P}$ the corresponding Weierstrass function. Prove that every meromorphic function on the torus $\mathbb{C}/\Lambda$, $f \in \mathcal{M}(\mathbb{C}/\Lambda)$, can be written in the form

$$f(z) = R(\mathcal{P}(z)) + Q(\mathcal{P}(z))\mathcal{P}'(z)\,,$$

where R, Q are rational functions, and $\mathcal{P}'$ is the complex derivative of $\mathcal{P}$.
Hints: See next page.

3. Problem

(i) Verify that $SL(2, \mathbb{Z})$ is generated by the elements

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \;\; , \;\; T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(give a simple interpretation of the Möbius transformations corresponding to J and T). That is, every $A \in SL(2, \mathbb{Z})$ can be written as a finite word

$$A = J^{\epsilon_1} T^{m_1} J T^{m_2} J \cdots T^{m_\ell} J^{\epsilon_2}$$

with $m_j \in \mathbb{Z}$, and $\epsilon_1, \epsilon_2 \in \{0, 1\}$.
Hints: See next page.

(ii) The *order* m of an element $g \in SL(2, \mathbb{Z})$ (or any group) is the smallest positive integer such that $g^m = \mathbf{1}$. Determine the orders of J, T, and $U := J^2TJ = -TJ$.

(iii) Consider the fundamental domain of $\mathbb{H}/SL(2, \mathbb{Z})$ (here including all boundaries),

$$\mathcal{F} := \Big\{ z \in \mathbb{C} \,\Big|\, Im(z) > 0\,,\; |z| \geq 1\,,\; Re(z) \in [-\frac{1}{2}, \frac{1}{2}] \Big\} \;\subset \mathbb{H}\,.$$

For $A \in SL(2, \mathbb{Z})$, let $A\mathcal{F} := \{T_A(z) \,|\, z \in \mathcal{F}\}$, where T_A is the Möbius transformation corresponding to A. Determine the regions $J\mathcal{F}$, $T\mathcal{F}$, $U\mathcal{F}$, $UJ\mathcal{F}$, $UT\mathcal{F}$, $U^2\mathcal{F}$.

Hints for Problem 2

First consider $f \in \mathcal{M}(\mathbb{C}/\Lambda)$ even, of degree $m \in 2\mathbb{N}$ (why is the degree even ?). Let $m = 2k$. Let $\mathcal{B} := \{z \in \mathbb{C}/\Lambda \,|\, f'(z) = 0\}$ denote the set of branch points. Assume that $w \notin f(\mathcal{B})$. Verify that $f(z) = w$ has $2k$ distinct solutions $\{c_1, \cdots, c_k, c_1', \cdots, c_k'\} \subset \mathbb{C}/\Lambda$ which appear in pairs satisfying $c_j + c_j' \in \Lambda$, where in particular c_j and c_j' are different.

Moreover, let $u \neq w$ with $u \notin f(\mathcal{B})$, and let $\{d_j, d_j'\}_{j=1}^k \subset \mathbb{C}/\Lambda$ be the solutions of $f(z) = u$.

Then, compare the functions

$$g(z) := \frac{f(z) - w}{f(z) - u} \quad \text{and} \quad h(z) := \prod_{j=1}^{k} \frac{\mathcal{P}(z) - \mathcal{P}(c_j)}{\mathcal{P}(z) - \mathcal{P}(d_j)}\,.$$

Next, for f odd, note that f can be written as $f = f_{even}\mathcal{P}'(z)$, where $f_{even} = \frac{f}{\mathcal{P}'}$ is even.

Hints for Problem 3

Problem 3(i): Let $H \subseteq SL(2, \mathbb{Z})$ denote the subgroup generated by J and T. Let $A = J = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$. Prove by induction in $|c|$ that $A \in H$.