COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 1

Due Friday, January 24, 2014, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Assume that the power series $\sum_{j=0}^{\infty} a_j z^j$ has convergence radius $0 < R < \infty$.
(a) Prove that the series converges uniformly to a continuous function $f(z)$ on the disc $D(0, r)$, for any $0 < r < R$.
(b) Taking termwise derivatives, prove that the series $\sum_{j=1}^{\infty} j a_j z^{j-1}$ has the same convergence radius R, and that it converges uniformly on $D(0, r)$ to $f'(z)$.

2. Problem

(a) Prove that the exponential function e^z is entire (holomorphic on $\mathbb{C}$), with $(e^z)' = e^z$.
(b) Let $\log(z)$ denote the branch of logarithm with branch cut at $L_\theta = \{re^{i\theta} \,|\, r \geq 0\}$, for some angle $\theta \in [0, 2\pi)$. Using (a), prove that log is holomorphic on $\mathbb{C} \setminus L_\theta$, with $(\log z)' = \frac{1}{z}$.

3. Problem

Consider the stereographic projection,

$$\Phi : (x, y, t) \mapsto \frac{x + iy}{1 - t} \, ,$$

for $x^2 + y^2 + t^2 = 1$, and $-1 \leq t < 1$.
(a) Prove that Φ defines a conformal map $S^2 \subset \mathbb{R}^3 \to \mathbb{C}_\infty$ (where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$).
(b) Define a metric on $\mathbb{C}_\infty$ by the Euclidean distance on $S^2 \subset \mathbb{R}^3$ as follows: For $z, w \in \mathbb{C}$, let $d(z, w) := |\Phi^{-1}(z) - \Phi^{-1}(w)|$. Show that

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} \, .$$

(c) Prove that Φ maps circles on S^2 to circles or straight lines in $\mathbb{C}$, using (b).