COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 11

Due Friday, May 2, 2014, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

Let $L := \{m + in \,|\, m, n \in \mathbb{Z}\}$, and $L^* := L \setminus \{(0,0)\}$.

(i) Prove that
$$f(z) := \frac{1}{z^2} + \sum_{\omega \in L^*} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big)$$
defines a meromorphic function in $\mathbb{C}$ satisfying $f(z+\omega) = f(z)$ for all $\omega \in L$.

(ii) Verify that f defines an analytic map $\mathbb{T} \to \mathbb{C}_\infty$, from the torus $\mathbb{T}$ to the Riemann sphere. Determine the degree of f, and find branch points and their valencies, if there are any. Compare your results with the Riemann-Hurwitz formula.

2. Problem

Prove that an elliptic function has as many poles as zeros.

3. Problem

Prove that an elliptic function f with degree $deg(f) = 2$ has exactly 4 branch points, each with valency 2.

4. Problem

Let Λ be the lattice generated by the linearly independent vectors (ω_1, ω_2), and $\mathcal{P}$ the corresponding Weierstrass function. Prove that every meromorphic function on the torus $\mathbb{C}/\Lambda$, $f \in \mathcal{M}(\mathbb{C}/\Lambda)$, can be written in the form
$$f(z) = R(\mathcal{P}(z)) + Q(\mathcal{P}(z))\mathcal{P}'(z)\,,$$
where R, Q are rational functions, and $\mathcal{P}'$ is the complex derivative of $\mathcal{P}$.
Hints: See next page.

Hints for Problem 4

First consider $f \in \mathcal{M}(\mathbb{C}/\Lambda)$ even, of degree $m \in 2\mathbb{N}$ (why is the degree even ?). Let $m = 2k$. Let $\mathcal{B} := \{z \in \mathbb{C}/\Lambda \,|\, f'(z) = 0\}$ denote the set of branch points. Assume that $w \notin f(\mathcal{B})$. Verify that $f(z) = w$ has $2k$ distinct solutions $\{c_1, \cdots, c_k, c_1', \cdots, c_k'\} \subset \mathbb{C}/\Lambda$ which appear in pairs satisfying $c_j + c_j' \in \Lambda$, where in particular c_j and c_j' are different.

Moreover, let $u \neq w$ with $u \notin f(\mathcal{B})$, and let $\{d_j, d_j'\}_{j=1}^k \subset \mathbb{C}/\Lambda$ be the solutions of $f(z) = u$.

Then, compare the functions

$$g(z) := \frac{f(z) - w}{f(z) - u} \quad \text{and} \quad h(z) := \prod_{j=1}^{k} \frac{\mathcal{P}(z) - \mathcal{P}(c_j)}{\mathcal{P}(z) - \mathcal{P}(d_j)}\,.$$

Next, for f odd, note that f can be written as $f = f_{even}\mathcal{P}'(z)$, where $f_{even} = \frac{f}{\mathcal{P}'}$ is even.