COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 2

Due Friday, January 31, 2014, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

(a) For which $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{C})$ are the Möbius transformations

$$T_A : z \mapsto \frac{az+b}{cz+d}$$

automorphisms of $\mathbb{C}$, respectively automorphisms of $\mathbb{C}_\infty$?
(b) Consider the inversion map $J : z \mapsto \frac{1}{z}$ on $\mathbb{C}_\infty$. Determine $\Phi^{-1} \circ J \circ \Phi : S^2 \to S^2$, where $\Phi : S^2 \to \mathbb{C}_\infty$ denotes the stereographic projection.

2. Problem

(a) Consider the matrices corresponding to the Cayley map and its inverse,

$$C := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \ , \ C' := \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} .$$

Prove that for any $A \in SL(2,\mathbb{R})$, one has $\frac{1}{2i} CAC' = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 - |\beta|^2 = 1$.
(b) Prove that for arbitrary $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$, there exists $s \in \mathbb{C}^*$ such that

$$s \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} = \begin{bmatrix} \eta & -\eta\omega \\ \overline{\omega} & -1 \end{bmatrix}$$

for some $\eta, \omega \in \mathbb{C}$ with $|\eta| = 1$, $|\omega| < 1$.
(c) Prove that for any $\eta, \omega \in \mathbb{C}$ with $|\eta| = 1$, $|\omega| < 1$, the Möbius transform

$$\phi_\omega : z \mapsto \eta \frac{z - \omega}{z\overline{\omega} - 1}$$

is an automorphism of $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$.

3. Problem

Does there exist a holomorphic function $f(z) = f(x+iy)$ with real part given by

$$u(x,y) = x + x^2 - y^2 \ ?$$

If yes, find it. If no, explain why not.