COMPLEX ANALYSIS – PRACTICE PROBLEMS - MARCH 7, 2014

1. Problem

Let Ω_1 and Ω_2 be two disjoint open subsets of the complex plane. Let $n \mapsto f_n$ be a sequence of analytic functions on Ω_1 , with values in Ω_2 . If this sequence converges pointwise to a function $f : \Omega_1 \to \mathbb{C}$, show that f is analytic and $f(\Omega_1) \subset \Omega_2$.

2. Problem

Assume that f is meromorphic on $\mathbb{C}$ satisfying $f(z) = f(-\frac{1}{\bar{z}})$ for all $z \in \mathbb{C}$, and $\lim_{z\to\infty} f(z) = 1$. Let $\gamma_\theta : \mathbb{R} \to \mathbb{C}$, $t \mapsto e^{i\theta}t$ be a contour that traces out a straight line at inclination angle θ containing the origin, and which contains no zeros or poles of f. Determine

$$\int_{\gamma_\theta} \frac{f'(z)}{f(z)} dz \,.$$

3. Problem

Assume that $f : \mathbb{D}^* \to \Omega$ is a biholomorphic map where $\mathbb{D}^* := \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$ is the punctured disc, and $\Omega \subset \mathbb{C}$ is a bounded region.

(1) Prove that f has a removable singularity at 0.

(2) Let $g \in \mathcal{H}(\mathbb{D})$ denote the holomorphic extension of $f \in \mathcal{H}(\mathbb{D}^*)$. Prove that $g'(0) \neq 0$.

4. Problem

Assume that $f \in \mathcal{H}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ (that is, holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$), and satisfies $|f(z)| = 1$ for $|z| = 1$. Moreover, assume that f vanishes nowhere inside $\mathbb{D}$.

(1) Prove that the function $g(z)$, defined by

$$g(z) := f(z) \quad \text{if } z \in \overline{\mathbb{D}} \,,$$

$$g(z) := \frac{1}{\overline{f(\frac{1}{\bar{z}})}} \quad \text{if } z \in \mathbb{C} \setminus \overline{\mathbb{D}} \,,$$

is holomorphic on $\mathbb{C}$.

(2) Prove that f must be a constant.

5. Problem

If $\Omega \subset \mathbb{C}$ is the complement of a compact connected set containing $\pm 1 \pm i$, show that there exists an analytic function $g : \Omega \to \mathbb{C}$ such that $g(z)^4 = z^4 + 4$.