COMPLEX ANALYSIS – HOMEWORK ASSIGNMENT 9

Due Friday, April 14, 2014, at the beginning of class.

Please write clearly, and staple your work !

1. Problem

(i) Assume that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Prove that

$$|f(z_0)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(z_0 + re^{i\theta})|^2 r dr d\theta$$

holds for any $z_0 \in \mathbb{D}$ and for $R < \text{dist}(z_0, \partial\mathbb{D})$.

(ii) Consider the set A of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ that satisfy

$$\int_{\mathbb{D}} |f(x + iy)|^2 dx dy \leq 1 \,.$$

Show that A is a normal family.

2. Problem

Let f be a holomorphic function on a simply connected open subset $\Omega \subset \mathbb{C}$, taking values in a compact subset of Ω. Show that f has a unique fixed point.

3. Problem

Assume that $f \in \mathcal{H}(\mathbb{C})$ is an entire function with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \,.$$

Prove that the order λ of f is finite if and only if

$$\mu := \limsup_{n \to \infty} \frac{n \ln n}{\ln(1/|a_n|)}$$

is finite, and that then, $\lambda = \mu$.
Hint: Proceed by proving that $\mu \leq \lambda$ and $\mu \geq \lambda$. Also, recall that

$$\lambda = \limsup_{n \to \infty} \frac{\ln \ln M(r)}{\ln r}$$

where $M(r) := \sup_{|z|=r} |f(z)|$ (see Ahlfors). Moreover, it is useful to express a_n via Cauchy's integral formula.