PDE I – HOMEWORK ASSIGNMENT 4

Due Monday, September 27, 2010. **Please write clearly, and staple your work !**

1. Problem

Let $U \subset \mathbb{R}^n$ be an open subset. Find the characteristic ODE of the Hamilton-Jacobi equation,

$$u_t + H(D_x u, x, t) = 0$$

for $u \in C^2(U \times \mathbb{R}_+)$ as a function of $(x,t) \in U \times \mathbb{R}_+$, with H being C^2 in all its arguments. Show that it has the form

$$x_t(t) = D_p H(p,x,t) \quad , \quad u_t(t) = (p \cdot D_p H - H)(p,x,t) \quad , \quad p_t(t) = -(D_x H)(p,x,t)\,,$$

where $u(t) = u(x(t),t)$, and $p(t) = D_x u(x(t),t)$.

Hint: You might want to consider $y := (x,t)$ and $D_y u$, to bring the above equation into the form $F(D_y u, y) = 0$ which has been discussed in class. Show that the characteristic curve $y(s)$ is such that in fact, $t(s) = s$ (which is why the above characteristic ODE is parametrized by t, not s).

2. Problem

Consider the Hamilton-Jacobi equation

$$u_t + \frac{1}{2}(D_x u)^2 - x = 0$$

for $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$, with

$$u(x,0) \,=\, x\,.$$

Determine the initial conditions for the characteristic ODE, and solve the characteristic ODE. Subsequently, find the solution $u(x,t)$.

Hint: Again, consider $y := (x,t)$, $D_y u$, and $F(D_y u, y) = 0$. Then, the initial condition at $t = 0$ becomes a boundary condition for $y \in \mathbb{R} \times \mathbb{R}_+$. Determine the initial conditions for the characteristic ODE as discussed in class.