PDE I – HOMEWORK ASSIGNMENT 5

Due Monday, October 4, 2010. **Please write clearly, and staple your work !**

1. Problem

Show that

$$u(x,t) \;=\; \begin{cases} -\frac{2}{3}(t+\sqrt{3x+t^2}) & \text{if } 4x+t^2>0 \\ 0 & \text{if } 4x+t^2<0 \end{cases}$$

is an entropy solution of

$$\begin{aligned} u_t + (\frac{u^2}{2})_x \;&=\; 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}_+ \\ u \;&=\; g \qquad \text{in } \mathbb{R}\times\{t=0\}\,, \end{aligned}$$

where $g(x)=0$ if $x\le 0$, and $g(x)=-\frac{2}{3}\sqrt{3x}$ if $x>0$.

2. Problem

Assume that $u(x+z)-u(x)\le Ez$ for all $z>0$. Let $u^\epsilon := \eta_\epsilon * u$, where η_ϵ is a standard mollifier (satisfying $\int_{\mathbb{R}}\eta_\epsilon(y)dy=1$), and show that

$$u^\epsilon_x \le E\,.$$

3. Problem

Compute explicitly the unique entropy solution of

$$\begin{aligned} u_t + (\frac{u^2}{2})_x \;&=\; 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}_+ \\ u \;&=\; g \qquad \text{in } \mathbb{R}\times\{t=0\}\,, \end{aligned}$$

with

$$g(x) \;=\; \begin{cases} 1 & \text{if } x<-1 \\ 0 & \text{if } -1<x<0 \\ 2 & \text{if } 0<x<1 \\ 0 & \text{if } x>1\,. \end{cases}$$

Draw a picture to illustrate the solution at $t>0$.