Approximate Analysis to the KdV-Burgers Equation

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Outline

1. Introduction
   - Generalized KdV-Burgers Equation
   - KdV-Burgers Equation
   - Planar Polynomial Systems and Abel Equation

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   - Planar Polynomial Systems and Abel Equation
2. Qualitative Analysis
   - Generalized Abel Equation
   - Property of Our Operator
   - Two Theorems

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- Generalized Korteweg-de Vries-Burgers equation [1, 2]

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where u is a function of x and t, α, β and $p > 0$ are real constants, μ and δ are coefficients of dissipation and dispersion, respectively.

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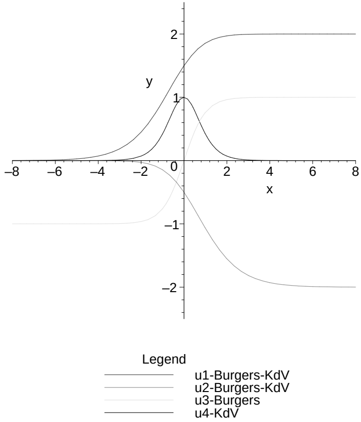
Figures of Wave Solutions



Planar Polynomial Systems and Abel Equation

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- For the Poincaré center problem, setting $x = r\cos\theta,\ y = r\sin\theta$ gives

$$\frac{dr}{d\theta} = \frac{r^k \xi(\theta)}{1 + r^{k-1}\eta(\theta)}, \tag{7}$$

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$$\rho = \frac{r^{k-1}}{1 + r^{k-1}\eta(\theta)},$$
we get an Abel equation
$$\frac{d\rho}{d\theta} = a(\theta)\rho^2 + b(\theta)\rho^3,$$
where $a = (k-1)\xi + \eta'$ and $b = (1-k)\xi\eta$.

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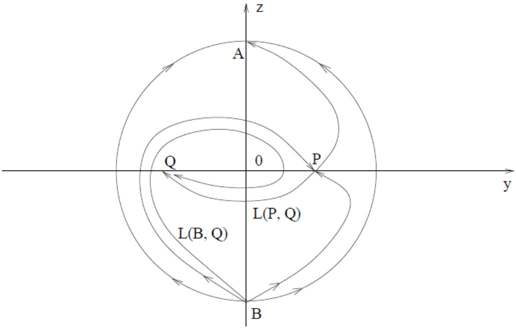
- Assume that $y = u$ and $u' = z$, then equation (9) is equivalent to

$$\begin{cases} y' = z, \\ z' = ey + gz + fy^p + d. \end{cases} \tag{10}$$

Global Structure of $p = 2$



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- Open Problem: There have been two longstanding problems, called the Poincaré center-focus problem and the local Hilbert 16th problem. Both are closely related to the Bautin quantities and the Bautin ideal of the Abel equation.

Integral Form

- Consider the generalized Abel equation

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- Rewrite equation (15) as

$$r(t) = c\left(1 + A(t) + r(t)\int_{t_0}^{t} b(\tau)r^{n-2}d\tau\right). \tag{16}$$

A Nonlinear Operator

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$$T_c : \mathcal{C}[0,1] \to \mathcal{C}[0,1],$$

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[10] Z. Feng, *Z. angew. Math. Phys.* under review.

Property of Our Operator

Lemma (1)

For $f \in \mathcal{C}[0,1]$ *and* $c \in \mathbb{R}$ *with* $\|f\| \leq M$ *and* $|c| < c_0 \stackrel{def}{=} (\|a\| + \|b\| M^{n-2})^{-1}$, $T_c(f)$ *is well defined and differentiable, and satisfies*

$$\frac{d}{dt} T_c(f)(t) = a(t)[T_c(f)(t)]^2 + b(t)[T_c(f)(t)]^2 f(t)^{n-2}.$$

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Furthermore, we have an identity

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for arbitrary $f,\ g \in \mathcal{C}[0,1]$ *and* $c \in \mathcal{R}$ *with* $\|f\|,\ \|g\| \leq M$ *and* $|c| < c_0$.

Outline of the Proof

Step 1: well-defined

$$1 - cA(t) - c\int_0^t b(\tau)f(\tau)^{n-2}d\tau = 0 \Rightarrow$$

$$|c| \geq \frac{1}{|A(t)| + \int_0^t |b(\tau)f(\tau)^{n-2}|\, d\tau} \geq \frac{1}{\|a\| + \|b\|M^{n-2}}.$$

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Step 2: A direct calculation gives

$$\frac{d}{dt}T_c(f)(t) = \frac{-c[-ca(t) - cb(t)f(t)^{n-2}]}{(1 - cA(t) - c\int_0^t b(\tau)f(\tau)^{n-2}d\tau)^2}$$

$$= \frac{c^2a(t)}{(1 - cA(t) - c\int_0^t b(\tau)f(\tau)^{n-2}d\tau)^2} + \frac{c^2b(t)f(t)^{n-2}}{(1 - cA(t) - c\int_0^t b(\tau)f(\tau)^{n-2}d\tau)^2}$$

$$T_c(f)(t) - T_c(g)(t) = \frac{c}{H(f)} \cdot \frac{c}{H(g)} \cdot \int_0^t b(\tau)(f(\tau)^{n-2} - g(\tau)^{n-2})d\tau$$

Lemma 2

Lemma (2)

Let $c_1 = (\|a\| + \|b\| + 1)^{-1}$. *Then we have*

$$\|T_c f\| \leq 1 \quad \textit{if} \;\; \|f\| \leq 1 \;\; \textit{and} \;\; |c| \leq c_1.$$

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$$\|T_c f\| \leq 1 \quad \textit{if} \quad \|f\| \leq 1 \;\; \textit{and} \;\; |c| \leq c_1.$$

Outline of the Proof.

If $\|f\| \leq 1$ and $|c| \leq c_1$, then we have

$$\begin{aligned} \|T_c f\| &\leq \frac{|c|}{1 - |c| \left(\|a\| + \|b\| \|f\|^{n-2}\right)} \\ &\leq \frac{|c|}{1 - |c| \left(\|a\| + \|b\|\right)} \\ &\leq 1. \end{aligned}$$

The conclusion follows. $\square$

Lemma 3

Lemma (3)

Let $c_2 = (\sqrt{(n-2)\|b\|} + \|a\| + \|b\| + 1)^{-1}$. *If* $|c| \le c_2$, *then* T_c *is a contraction mapping on the close unit ball* $\mathcal{B}_1 = \{f \in \mathcal{C}[0,1] | \; \|f\| \le 1\}$ *of* $\mathcal{C}[0,1]$.

Outline of the Proof.

It follows from Lemmas 1 and 2 that

$$\begin{aligned} \|T_c(f)(t) - T_c(g)(t)\| &\le \|T_c(f)\|\|T_c(g)\|\|b\|\|f^{n-2} - g^{n-2}\| \\ &= C\|(f-g)(f^{n-3} + f^{n-4}g + \cdots + fg^{n-4} + g^{n-3})\| \\ &\quad \le (n-2)c\|f-g\|, \end{aligned}$$

where

$$c \stackrel{def}{=} \left(\frac{|c|}{1 - |c|\left(\|a\| + \|b\|\right)}\right)^2 \|b\|.$$

Theorem 1

Theorem (1)

For given $a,\, b \in \mathcal{C}[0,1]$ *and* $c \in \mathbb{R}$ *with* $|c| \leq (\sqrt{(n-2)\|b\|} + \|a\| + \|b\| + 1)^{-1}$*, the solution* $r(t,\, c)$ *of equation* (1) *with* $r(0,\, c) = c$ *can be uniformly approximated by an iterated sequence* $\{T_c^n(f)(t)\}$*:*

$$r(t,\, c) = \lim_{n\to\infty} T_c^n(f)(t), \quad 0 \leq t \leq 1, \tag{17}$$

that is,

$$r(t,\, c) = \cfrac{c}{1 - cA(t) - c^{n-1}\int_0^t \cfrac{b(t_1)dt_1}{1 - cA(t_1) - c^{n-1}\int_0^{t_1} \cfrac{b(t_2)dt_2}{1 - cA(t_2) - c^{n-1}\int_0^{t_2}\cdots}}} \tag{18}$$

for arbitrary $f \in \mathcal{C}[0,1]$ *with* $\|f\| \leq 1$*. Furthermore, the following error estimate holds*

$$r(t,\, c) - T_c^n(f)(t) = \mathcal{O}(c^{2n}).$$

Theorem 2: Case of $n = 3$

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Theorem (2)

Suppose a, $b \in \mathcal{C}[0,1]$ and $c \in \mathbb{R}$ with

$$|c| \leq \max\{(\sqrt{\|b\|} + \|a\| + \|b\| + 1)^{-1},\ (2M)^{-1}\}.$$

Then, in formula (18), the following part is bounded

$$\begin{aligned} & \frac{b(t_1)}{1 - cA(t_1) - c^2 \int_0^{t_1} \frac{b(t_2)dt_2}{1-cA(t_2)-c^2\int_0^{t_2}\cdots}} \\ = \ & \frac{1}{c}\cdot b(t_1)\cdot \frac{c}{1 - cA(t_1) - c^2\int_0^{t_1} b(t_2)\cdot \frac{c}{1-cA(t_2)-c^2\int_0^{t_2}\cdots} dt_2}. \end{aligned}$$

2D Korteweg-de Vries-Burgers Equation

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$$(U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx})_x + \gamma U_{yy} = 0, \tag{19}$$

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$$U(x, y, t) = U(\xi), \quad \xi = hx + ly - wt. \tag{20}$$

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- Substitution of (20) into equation (19) and performing integration twice yields

$$U''(\xi) + \lambda U'(\xi) + aU^2(\xi) + bU(\xi) + d = 0, \tag{21}$$

where $v = U(\xi) \in [v_0,\ v_1]$, $\lambda = \frac{\beta}{sh}$, $a = \frac{\alpha}{2sh^2}$, $b = \frac{\gamma l^2 - wh}{sh^4}$ and $d = -\frac{C}{sh^4}$.

Resultant Abel Equation

- Let $v = U(\xi)$ and $y = U'(\xi)$. Equation (21) becomes

$$\frac{dy}{dv}y + \lambda y + av^2 + bv + d = 0. \tag{22}$$

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where $h(\eta),\ k(\eta) \in \mathcal{C}[0,\ 1]$, and

$$h(\eta) = (v_1 - v_0)\lambda,$$

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Solution to Equation (24)

- By virtue of Theorem 1, if $|c| \leq (\sqrt{\|k\|} + \|h\| + \|k\| + 1)^{-1}$, the solution to equation (24) is

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$$\begin{aligned} H(\eta) &= \int_0^\eta h(x)\,dx = \int_0^\eta (v_1 - v_0)\lambda\,dx = (v_1 - v_0)\lambda\eta, \\ k(x) &= (v_1 - v_0)\left(a(v_0 + (v_1 - v_0)x)^2 + b(v_0 + (v_1 - v_0)x) + d\right). \end{aligned}$$

Approximate Solution to 2D-KdV-Burgers Equation

- Recall that $r = \frac{1}{y}$, $y = U'(\xi)$, $\eta = \frac{v - v_0}{v_1 - v_0}$ and $v = U(\xi)$. When conditions of Theorem 1 are fulfilled, we have

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- When c is small, according to Theorem 2, the coefficient of c^2 is bounded. So we can drop the term containing c^2 and get

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That is,

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- Solving equation (27) gives

$$U(x, y, t) = \frac{\frac{1}{c} + \lambda v_0}{\lambda} + ce^{-\lambda\xi}, \quad \xi = hx + ly - wt$$

where $\lambda = \frac{\beta}{sh}$.

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  where $\lambda = \frac{\beta}{sh}$.
- If we take $v_0 = \frac{b}{2a}$ and choose $c = \frac{-2a}{\lambda\sqrt{b^2 - 4ad}}$ sufficiently small, when $\lambda\xi \to +\infty$, we obtain

$$U(x,\, y,\, t) \sim \frac{b^2 - 4ad}{-2a} + \frac{b}{2a}. \tag{28}$$

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[8] Z. Feng, *Nonlinearity*, **20** (2007), 343–356.

Boundedness of Solutions

- Note that equation (26) can be rewritten as

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- When Φ is a quadratic or cubic function or special function of $U(\xi)$, we can analyze equation (29) qualitatively and numerically with classifications. For instance, if Φ is quadratic, we take $v_0 = \frac{b}{2a}$ and choose $c = \frac{-2a}{\lambda\sqrt{b^2-4ad}}$ sufficiently small, we can obtain the solution of the type

$$u(x, y, t) = \frac{3\beta^2 + \gamma + c}{25\alpha s}\operatorname{sech}^2\xi - \frac{6\beta^2 + \gamma + c}{25\alpha s}\tanh\xi \pm \frac{6\beta^2}{25\alpha s} + C_0.$$

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- When Φ is a function with the lower and upper bounds, we can also find bounds of solutions of equation (29) by the comparison principle, which match well with the phase analysis described in [7].

Summary

- In this talk, we provided a connection between the Abel equation of the first kind, an ordinary differential equation that is cubic in the unknown function, and the Korteweg-de Vries-Burgers equation, a partial differential equation that describes the propagation of waves on liquid-filled elastic tubes. We presented an integral form of the Abel equation with the initial condition.

Summary

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- By virtue of the integral form and the Banach Contraction Mapping Principle we derived the asymptotic expansion of bounded solutions in the Banach space, and used the asymptotic formula to construct approximate solutions to the Korteweg-de Vries-Burgers equation.

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- As an example, we presented the asymptotic behavior of traveling wave solution for a 2D KdV-Burgers equation which agrees well with existing results in the literature.
- Under certain conditions, we can also study bounds of traveling wave solutions of KdV-Burgers type equations by the comparison principle.

Acknowledgement

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- Thank you.