On the genericity of non-degenerate spectral edges

Ngoc Do, Peter Kuchment, Frank Sottile
Texas A & M University

Texas Analysis and Mathematical Physics Symposium
November 22nd, 2014

A bit about Floquet-Bloch theory

A bit about Floquet-Bloch theory

- Floquet-Bloch theory: tool to study periodic differential operators

A bit about Floquet-Bloch theory

- Floquet-Bloch theory: tool to study periodic differential operators
- Hamiltonian $H = -\Delta + V(x)$
$V(x) = V(x + e_i)$ for a basis $e = \{e_i\}_{i=1}^{n} \in \mathbb{R}^n$

A bit about Floquet-Bloch theory

- Floquet-Bloch theory: tool to study periodic differential operators

- Hamiltonian $H = -\Delta + V(x)$
$V(x) = V(x + e_i)$ for a basis $e = \{e_i\}_{i=1}^n \in \mathbb{R}^n$

- Bloch-Hamiltonian: $H^\theta = -\Delta + V(x)$
Floquet condition: $f(x + pe) = f(x)e^{ip\theta}$
$p = \{p_i\}_{i=1}^n \in \mathbb{Z}^n$, $\theta = \{\theta_i\}_{i=1}^n \in B$, $B = [-\pi, \pi]^n$ - Brillouin zone

A bit about Floquet-Bloch theory

- Floquet-Bloch theory: tool to study periodic differential operators
- Hamiltonian $H = -\Delta + V(x)$
$V(x) = V(x + e_i)$ for a basis $e = \{e_i\}_{i=1}^n \in \mathbb{R}^n$
- Bloch-Hamiltonian: $H^\theta = -\Delta + V(x)$
Floquet condition: $f(x + pe) = f(x)e^{ip\theta}$
$p = \{p_i\}_{i=1}^n \in \mathbb{Z}^n$, $\theta = \{\theta_i\}_{i=1}^n \in B$, $B = [-\pi, \pi]^n$ - Brillouin zone
  - $\sigma(H^\theta)$ - discrete spectrum

A bit about Floquet-Bloch theory

- Floquet-Bloch theory: tool to study periodic differential operators
- Hamiltonian $H = -\Delta + V(x)$
$V(x) = V(x + e_i)$ for a basis $e = \{e_i\}_{i=1}^n \in \mathbb{R}^n$
- Bloch-Hamiltonian: $H^\theta = -\Delta + V(x)$
Floquet condition: $f(x + pe) = f(x)e^{ip\theta}$
$p = \{p_i\}_{i=1}^n \in \mathbb{Z}^n$, $\theta = \{\theta_i\}_{i=1}^n \in B$, $B = [-\pi, \pi]^n$ - Brillouin zone
  - $\sigma(H^\theta)$ - discrete spectrum
  - Direct integral decomposition $H = \bigoplus_{\theta \in B} H^\theta$

$$\sigma(H) = \bigcup_{\theta \in B} \sigma(H^\theta)$$

A bit about Floquet-Bloch theory

- Floquet-Bloch theory: tool to study periodic differential operators
- Hamiltonian $H = -\Delta + V(x)$
$V(x) = V(x + e_i)$ for a basis $e = \{e_i\}_{i=1}^{n} \in \mathbb{R}^n$
- Bloch-Hamiltonian: $H^\theta = -\Delta + V(x)$
Floquet condition: $f(x + pe) = f(x)e^{ip\theta}$
$p = \{p_i\}_{i=1}^{n} \in \mathbb{Z}^n$, $\theta = \{\theta_i\}_{i=1}^{n} \in B$, $B = [-\pi, \pi]^n$ - Brillouin zone
  - $\sigma(H^\theta)$ - discrete spectrum
  - Direct integral decomposition $H = \bigoplus_{\theta \in B} H^\theta$

$$\sigma(H) = \bigcup_{\theta \in B} \sigma(H^\theta)$$

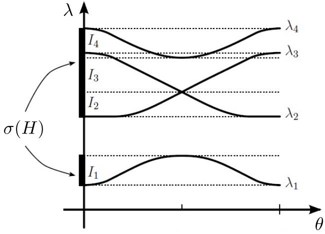
Periodic spectral prob. $\rightarrow$ spectral prob. on fundamental domain

A bit about Floquet-Bloch theory (cont.)

- Dispersion relation - multi-valued function: $\theta \to \{\lambda_j(\theta)\}$

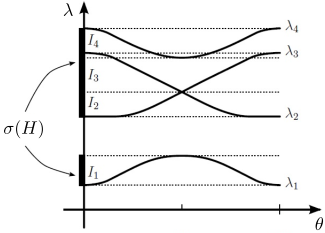
A bit about Floquet-Bloch theory (cont.)

- Dispersion relation - multi-valued function: $\theta \to \{\lambda_j(\theta)\}$



A bit about Floquet-Bloch theory (cont.)

- Dispersion relation - multi-valued function: $\theta \to \{\lambda_j(\theta)\}$



Definition

An extremum of dispersion relation is degenerate if its Hessian is zero.

Conjecture

Conjecture

The dispersion relation of a generic periodic Schrödinger operator has only non-degenerate extrema

Conjecture

Conjecture

The dispersion relation of a generic periodic Schrödinger operator has only non-degenerate extrema

Motivation: Anderson localization, effective masses, etc

Conjecture

Conjecture

The dispersion relation of a generic periodic Schrödinger operator has only non-degenerate extrema

Motivation: Anderson localization, effective masses, etc

• The bottom of the spectrum of periodic Schrödinger operator is non-degenerate, W. Kirsch and B. Simon, J. Funct. Anal. '87

Conjecture

Conjecture

The dispersion relation of a generic periodic Schrödinger operator has only non-degenerate extrema

Motivation: Anderson localization, effective masses, etc

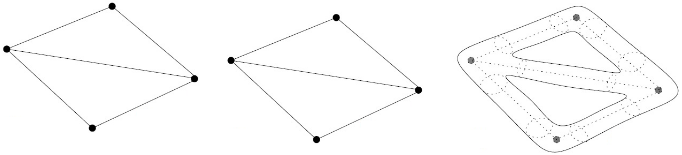
• The bottom of the spectrum of periodic Schrödinger operator is non-degenerate, W. Kirsch and B. Simon, J. Funct. Anal. '87

• Spectral edges of generic Schrödinger operator are extrema of single band function, F. Klopp and J. Ralston, Meth. Appl. Anal. '00

Graph approach and some justification

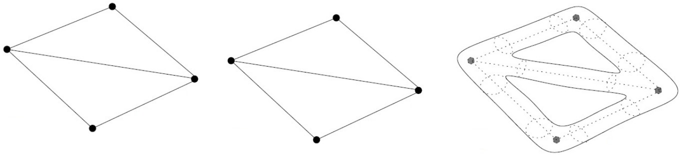
Graph approach and some justification

Discrete graph → Quantum graph → Graph-like thin domain



Graph approach and some justification

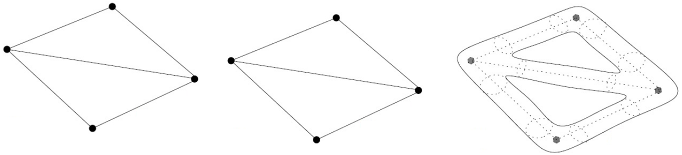
Discrete graph → Quantum graph → Graph-like thin domain



Why graph approach???

Graph approach and some justification

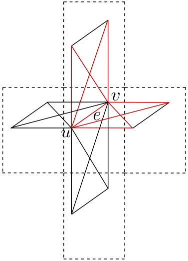
Discrete graph → Quantum graph → Graph-like thin domain



Why graph approach???

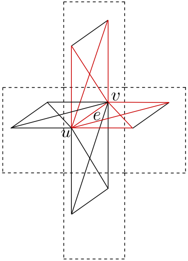
- Simple
- Highly effective
  (P. Exner, P. Kuchment, O. Post, J. Rubinstein, M. Schatzman, H. Zeng, etc)

Structures and Laplacians



A simple class of discrete graphs

Structures and Laplacians



A simple class of discrete graphs

$$L_\mu f(u) = \sum_{e=(u,v)\in E_u} \mu_e(f(u) - f(v)), \mu \in \mathbb{R}^9$$

Results

Theorem

The dispersion relation of a generic operator $L_\mu, \mu \in \mathbb{R}^9$, *has only non-degenerate extrema.*

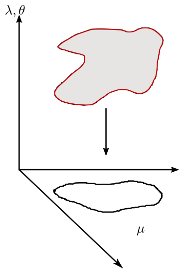
Results

Theorem

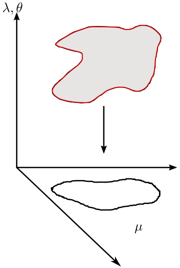
The dispersion relation of a generic operator $L_\mu, \mu \in \mathbb{R}^9$, has only non-degenerate extrema.

The set of $\mu, \mu \in \mathbb{R}^9$, such that L_μ has degenerate spectral edges is of codimension 1.

Proof outline

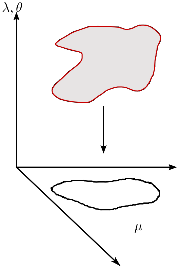


Proof outline



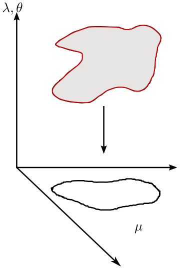
- Obtain set A that "describes" all degenerate spectral edges (Floquet-Bloch theory)

Proof outline



- Obtain set A that "describes" all degenerate spectral edges (Floquet-Bloch theory)
- Prove that $codim(A) = 4$
  (numerical algebraic geometry: *Bertini^{TM}software*)

Proof outline



- Obtain set A that "describes" all degenerate spectral edges (Floquet-Bloch theory)
- Prove that $codim(A) = 4$
  (numerical algebraic geometry: *Bertini*^{TM} *software*)
- Project A to onto the space of weights
  (Saidenberg-Tarski theorem)

Future plans

Discrete case $\rightarrow$ Quantum graph case $\rightarrow$ Graph-like thin domain

Future plans

Discrete case $\rightarrow$ Quantum graph case $\rightarrow$ Graph-like thin domain

- Eliminate numerical proof part

Future plans

Discrete case $\rightarrow$ Quantum graph case $\rightarrow$ Graph-like thin domain

- Eliminate numerical proof part
- Increase number of vertices inside the fundamental domain to arbitrarily finitely many vertices

Future plans

Discrete case $\rightarrow$ Quantum graph case $\rightarrow$ Graph-like thin domain

- Eliminate numerical proof part
- Increase number of vertices inside the fundamental domain to arbitrarily finitely many vertices
- Bootstrap to quantum graph

Future plans

Discrete case $\rightarrow$ Quantum graph case $\rightarrow$ Graph-like thin domain

- Eliminate numerical proof part
- Increase number of vertices inside the fundamental domain to arbitrarily finitely many vertices
- Bootstrap to quantum graph
- Bootstrap to graph-like thin domain

Future plans

Discrete case $\rightarrow$ Quantum graph case $\rightarrow$ Graph-like thin domain

- Eliminate numerical proof part
- Increase number of vertices inside the fundamental domain to arbitrarily finitely many vertices
- Bootstrap to quantum graph
- Bootstrap to graph-like thin domain
- Continuous (???)

Thank you!