Finding the Stokes Wave: Low Steepness to Highest Wave

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Outline

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Figure : Nearly plane waves in the wake of a boat in Maas-Waal Canal.

Equations of free surface hydrodynamics

Consider a potential flow of ideal fluid in a seminfinite strip $(x, y) \in [-\pi, \pi]x[-\infty, \eta(x, t)]$ periodic in x-variable.

$$\nabla^2\Phi = 0$$

with BC and domain defined by:

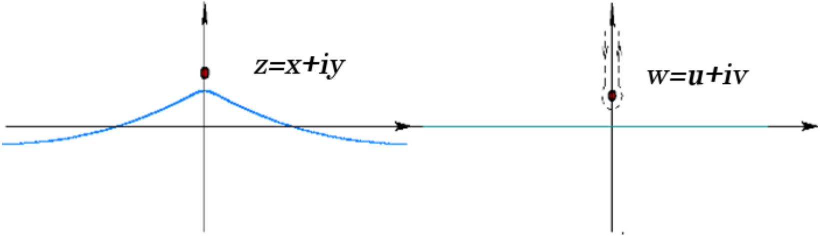
$$\frac{\partial \eta}{\partial t} + \left(-\frac{\partial \eta}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \right)\Bigg|_{y=\eta(x,t)} = 0$$

$$\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\nabla\Phi\right)^2 \right)\Bigg|_{y=\eta(x,t)} + g\eta = 0$$

$$\frac{\partial \Phi}{\partial y}\Bigg|_{y\to-\infty} = 0$$

Conformal Map

We introduce the conformal map $z(w,t) = w + \tilde{z}(w,t)$, so that in new variable $w = u + iv$ fluid occupies region $[-\pi, \pi]x[-\infty, 0]$.



Analyticity of a function $\tilde{z}(w)$ in $\mathbb{C}^-$ imposes a relation between real and imaginary parts of $\tilde{z}$ on the free surface $v = 0$:

$$\tilde{z}(u) = \tilde{x}(u) + iy(u) = \tilde{x}(u) + i\hat{H}\tilde{x}(u) = 2\hat{P}\tilde{x}$$

where $\hat{H}$ is Hilbert operator, and $\hat{P} = \frac{1}{2}\left(1 + i\hat{H}\right)$ is projector.

Hamiltonian

In the absence of viscosity, free surface hydrodynamics constitute a Hamiltonian system with:

$$\mathcal{H} = \frac{1}{2}\int\int_{-\infty}^{\eta(x)} \nabla\Phi^2 \, dy \, dx + \frac{g}{2}\int \eta^2 \, dx$$

$$\mathcal{H} = \frac{1}{2}\int \Phi\Phi_v|_{v=0} \, du + \frac{g}{2}\int y^2 x_u \, du$$

Introduce complex potential $\Pi(w) = \Phi(w) + i\Theta(w)$ analytic in $\mathbb{C}^-$. Define $\psi(u) = \Phi(u, v = 0)$ and we have:

$$\Theta(u) = \hat{H}\psi(u)$$

$$\Phi_v|_{v=0} = -\,\Theta_u|_{v=0} = -\hat{H}\psi_u$$

at $v = 0$, where $-\hat{H}\partial_u$ is Dirichlet-Neumann operator.

Equations of motion

Equations of motion are found from extremizing action $\mathcal{S}$:

$$\mathcal{L} = \int \psi \eta_t \, du - \mathcal{H}, \qquad \mathcal{S} = \int \mathcal{L} \, dt$$

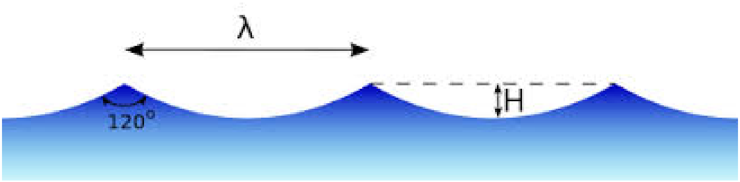
And are:

$$\psi_t + gy = -\frac{\psi_u^2 - \left(\hat{H}\psi_u\right)^2}{2|z_u|^2} + \psi_u \hat{H}\left(\frac{\hat{H}\psi_u}{|z_u|^2}\right)$$

$$z_t = z_u(\hat{H} - i)\frac{\hat{H}\psi_u}{|z_u|^2}$$

The results of simulation of a flavour of these equations is described in further sections.

What is a Stokes wave?



- A **Stokes wave** is a fully nonlinear wave propagating over one dimensional free surface of an ideal fluid.
- The defining parameters of a Stokes wave is **steepness**.
- The steepness is measured as a ratio of crest to trough height H over a wavelength λ

Equation on the Stokes wave

A solution propagating with fixed velocity c:

$$z(u,t) = u + z(u-ct)$$
$$\psi(u,t) = \psi(u-ct)$$

satisfies:

$$\left(\frac{c^2}{c_0^2}\hat{k} - 1\right) y - \left(\frac{1}{2}\hat{k}y^2 + y\hat{k}y\right) = 0$$

here $c_0 = \sqrt{g/k_0}$ - phase velocity of linear gravity waves, and $\hat{k} = |k| = \sqrt{-\Delta}$

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Numerical method

We solve equation:

$$\left(\frac{c^2}{c_0^2}\hat{k} - 1\right) y - \left(\frac{1}{2}\hat{k}y^2 + y\hat{k}y\right) = 0$$

with Newton Conjugate-Gradient iterations in Fourier space:

- Write exact solution $y_k^* = y_k^{(n)} + \delta y_k^{(n)}$ and linearize at $y_k^{(n)}$:

$$\hat{L}_0\left(y_k^{(n)} + \delta y_k^{(n)}\right) = \hat{L}_0 y_k^{(n)} + \hat{L}_1 \delta y_k^{(n)} = 0$$

- Solve linearized system:

$$\hat{L}_1 \delta y_k^{(n)} = -\hat{L}_0 y_k^{(n)}$$
$$y_k^{(n+1)} = y_k^{(n)} + \delta y_k^{(n)}$$

These simulations were performed in quadruple precision.

Stokes Waves

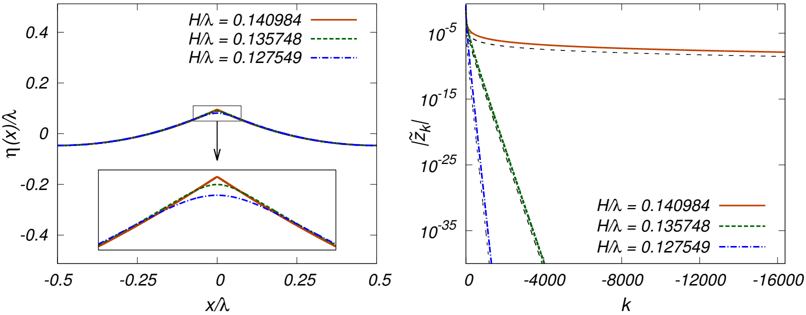
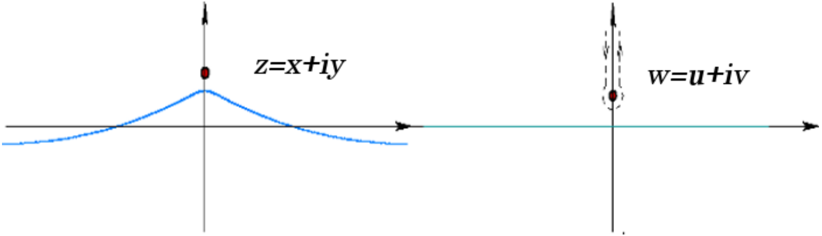


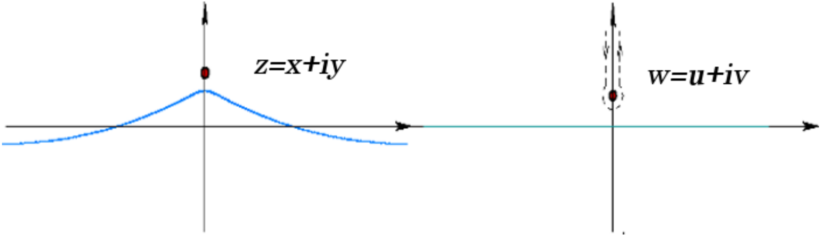
Figure : Stokes waves computed with Newton-CG method (left) and their spectra (right). Simulations with $N = 2048$ (blue), $N = 4096$ (green) and $N = 4194304$ (orange) Fourier modes. Black dashed lines are asymptotic decay predicted by theory.

Analytical Continuation



Recall that $\tilde{z}(u)$ is analytic in $\mathbb{C}^-$, but does it have an analytic continuation into the upper half plane?

Analytical Continuation



Recall that $\tilde{z}(u)$ is analytic in $\mathbb{C}^-$, but does it have an analytic continuation into the upper half plane?

Yes! Stokes wave can be written as a Cauchy integral over branch cut extending from iv_c to $+i\infty$

Asymptotics of Fourier Coefficients

$$z_k = \int_{-\pi}^{\pi} \tilde{z}(u) \exp(-iku)\, du$$

We deform the integration contour into $\mathbb{C}^+$ and assume the local behaviour about iv_c to be:

$$z(w) \sim (w - iv_c)^\beta$$

It is easy to show that asymptotically

$$z_k \to |k|^{-\beta-1} \exp(-|k|v_c)$$
$$k \to -\infty$$

Result from 1973 by M. Grant shows that $\beta = \frac{1}{2}$ and it is supported by analysis of the spectra we have found.

Simulations in high steepness regimes

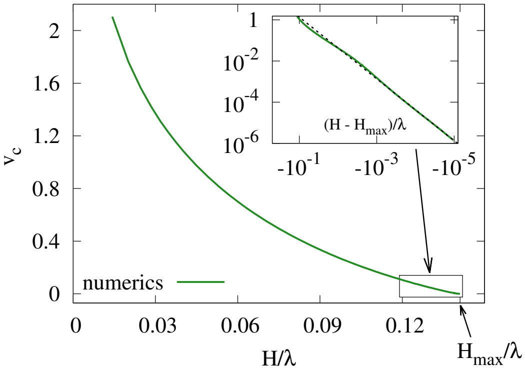


Figure : Position of the singularity v_c as a function of wave steepness.

Our estimate for steepness of highest $\dfrac{H_{max}}{\lambda} = 0.1410633 \pm 4 \cdot 10^{-7}$

Limiting Stokes Wave

The limiting Stokes wave a.k.a wave of greatest height has a jump in derivative η_x and forms a $\frac{2\pi}{3}$ angle on the surface.

The singularity of limiting Stokes appears as a result of coalescence of more than one singularity, e.g.:

$$z(w) \sim f(w)\,(w - iv_c)^{1/2} + h.o.t. \to w^{2/3}$$
$$\text{as} \quad v_c \to 0$$

where $f(w)$ is some regular function. Finding the limiting Stokes wave from presented equations faces several major difficulties.

Everwidening Spectrum

The Asymptotics of Fourier coefficients as $k \to \infty$:

$$y_k \sim \frac{1}{|k|^{3/2}} \exp\left(-|k| v_c\right)$$

$$(\hat{k}y)_k \sim \frac{1}{|k|^{1/2}} \exp\left(-|k| v_c\right)$$

Take a limit $v_c \to 0$ and:

$$(\hat{k}y)_k \sim \frac{1}{|k|^{1/2}}$$

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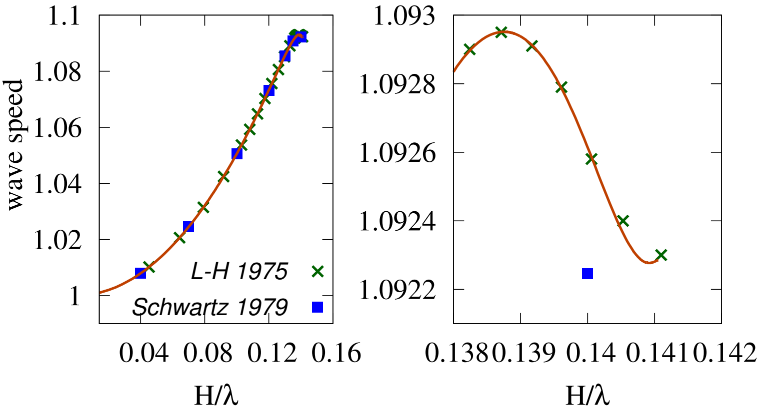
Divergent!

Velocity Oscillation



How can we study the analytic properties?

1. Study of Fourier spectrum:
   - Accurate, but only allows to study the singularities closest to the real axis
2. Construction of Padé interpolant and analysis of its poles:
   - Straight-forward construction of Padé interpolant suffers from catastrophic loss of precision in finite digit arithmetic.

Solution: Alpert-Greengard-Hagström algorithm(AGH) can construct Padé approximation using many points on the grid.

Branch Cut

We expand periodic interval $u \in [-\pi, \pi]$ to infinite interval $\zeta \in (-\infty, \infty)$ with auxiliary transform:

$$\zeta = \tan\left(\frac{w}{2}\right)$$

Applying Padé approximation we observe that Stokes wave is a branch cut:

$$\tilde{z}(u) = z(u) - u \approx \sum_{k=1}^{d} \frac{\alpha_k}{\tan\left(\frac{u}{2}\right) - i\chi_k} \approx \int_{\chi_c}^{1} \frac{\rho(\chi)d\chi}{\tan\left(\frac{u}{2}\right) - i\chi}$$

The branch cut spans along the positive imaginary axis from point $i\chi_c$ to $+i\infty$

Padé Approximation of Stokes Wave

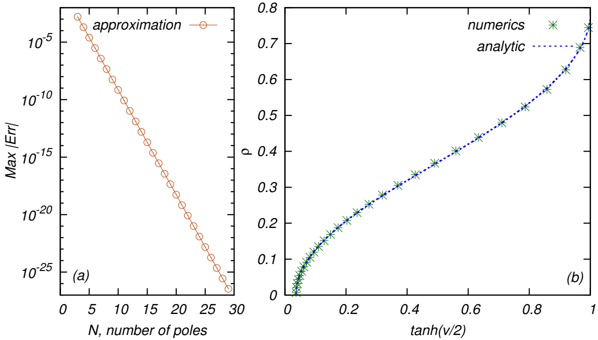
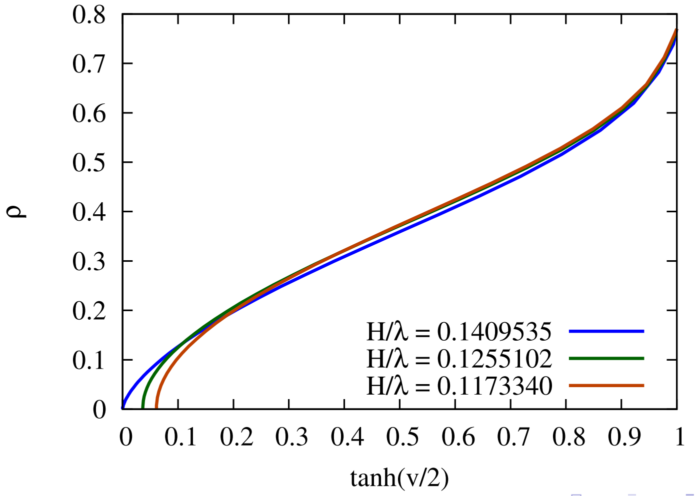


Figure : (a) Maximum of absolute error between the Stokes wave solution and its approximation by poles as a function of the number of poles; (b) Reconstructed jump on the branch cut using residues and position of poles from AGH algorithm

Jump on the Branch Cut



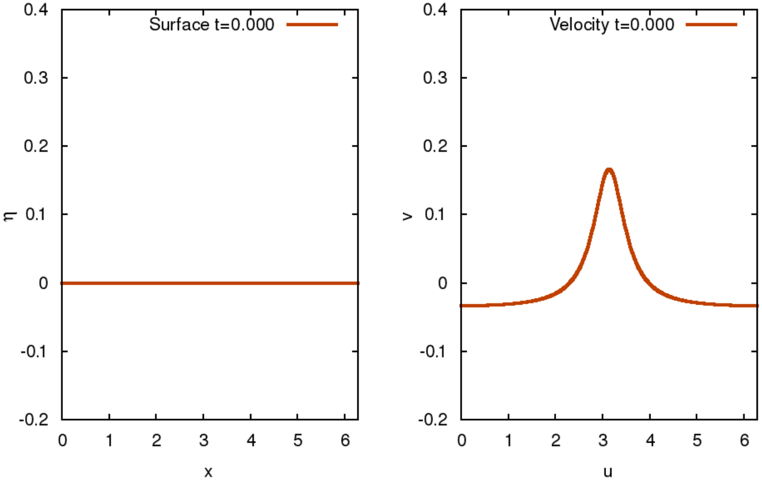
Numerical simulation of time-dependent problem

The equations on $\psi(u,t)$ and $z(u,t)$ are not optimal for numerical simulations, instead equations are formulated in terms of $R(u,t) = \frac{1}{z_u}$ and $V(u,t) = \frac{i\psi_u}{z_u}$:

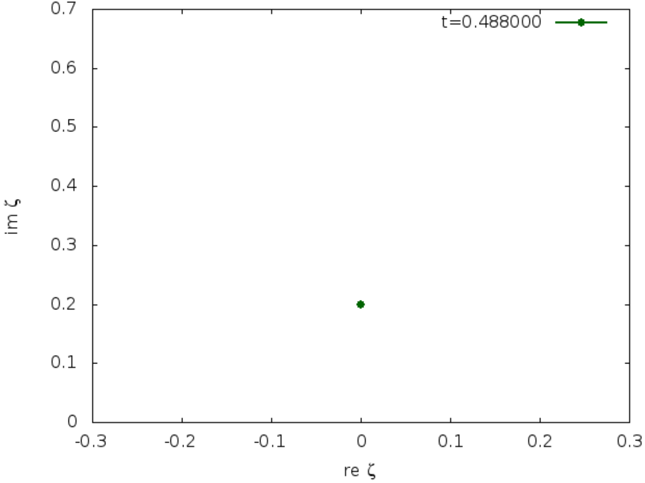
$$R_t = i\left(UR' - U'R\right)$$
$$V_t = i\left(UV' - B'R\right) + g\left(R - 1\right)$$

it is convenient to introduce projection operator $\hat{P} = \frac{1}{2}(1 + i\hat{H})$, then $U = 2\hat{P}\left(\frac{-\hat{H}\psi_u}{|z_u|^2}\right)$ and $B = \hat{P}\left(\frac{|\Phi_u|^2}{|z_u|^2}\right)$

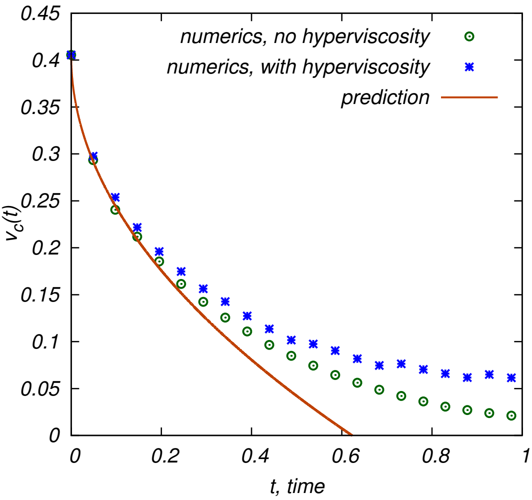
Simulation with Simple Pole in V as Initial Data



Evolution of branch cut corresponding to velocity



Position of branch cut in the absence of gravity



Conclusions

- Analytical properties of Stokes wave are fully determined by a single branch cut.
- High steepness waves have been constructed numerically.
- The prediction of oscillatory approach to limiting wave was confirmed, with several of the oscillations being well-resolved.
- Closed integral equations on $\rho(\chi)$ were found.