General soliton solution to the vector nonlinear Schrödinger equations

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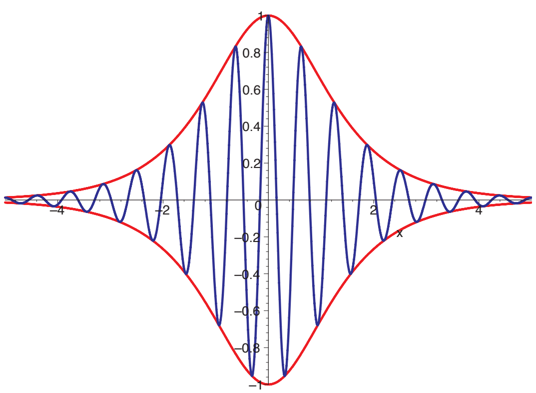
Outline

- Brief review on the vNLS equations
- Two-bright-one-dark and one-bright-two-dark soliton solutions to 3-coupled NLS equation
- General bright-dark soliton solution to the vNLS equation
- General soliton solution to the vNLS equation
- Based on B.F, J. Phys. A: Math. Theor 47 (2014) 355203 (22pp)

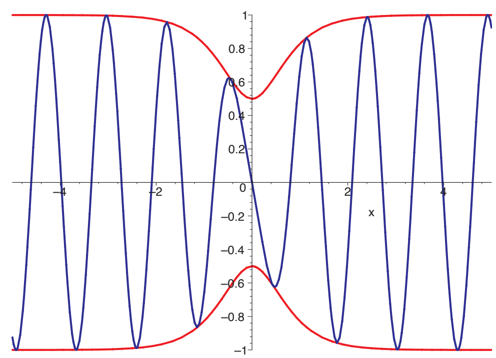
Nonlinear Schrödinger Equation

$$\mathrm{i}u_t + u_{xx} + \tfrac{1}{2}\sigma|u|^2u = 0, \qquad \sigma = \pm 1$$

- Arises in numerous physical applications including:
  - ▶ water waves (**Benney & Roukes [1969]; Zakharov [1968]**);
  - ▶ optical fibres (**Hasegawa & Tappert [1973]**);
  - ▶ plasmas (**Zakharov [1972]**);
  - ▶ magnetostatic spin waves (**Kalinikos *et al*. [1997]; Xia *et al*. [1997]**).
- A **soliton equation** solvable by inverse scattering (**Zakharov & Shabat [1972]**).
- **Bright solitons**, which decay as $|x| \to \infty$, arise when $\sigma = 1$ ("**focusing**")
- **Dark solitons**, which don't decay as $|x| \to \infty$, arise when $\sigma = -1$ ("**defocusing**")



Bright soliton



Dark soliton

Two-coupled NLS equation (Manakov system)

$$\mathrm{i}q_{1,t} + q_{1,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2\right) q_1 = 0\,,$$
$$\mathrm{i}q_{2,t} + q_{2,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2\right) q_2 = 0\,,$$

- Bright-bright solitons to focusing-focusing case ($\sigma_1 = \sigma_2 = 1$)
  Manakov, Sov. Phys. JETP 38, 248 (1974)
  Radhakrishnan and Lakshmanan, J. Phys. A 28, 2683 (1995).

Brief review on the coupled NLS equation

Two-coupled NLS equation (Manakov system)

$$\mathrm{i}q_{1,t} + q_{1,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2\right) q_1 = 0\,,$$

$$\mathrm{i}q_{2,t} + q_{2,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2\right) q_2 = 0\,,$$

- Bright-bright solitons to focusing-focusing case ($\sigma_1 = \sigma_2 = 1$)
  Manakov, Sov. Phys. JETP 38, 248 (1974)
  Radhakrishnan and Lakshmanan, J. Phys. A 28, 2683 (1995).
- Dark-dark and bright-dark solitons to defocusing-defocusing case
  Kivshar, Phys. Rev. E 55, 4773 (1997);
  B. Prinari, M. J. Ablowitz etc. J. Math. Phys. 47, 063508 (2006).

Two-coupled NLS equation (Manakov system)

$$\mathrm{i}q_{1,t} + q_{1,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2\right) q_1 = 0\,,$$
$$\mathrm{i}q_{2,t} + q_{2,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2\right) q_2 = 0\,,$$

- Bright-bright solitons to focusing-focusing case ($\sigma_1 = \sigma_2 = 1$)
  Manakov, Sov. Phys. JETP 38, 248 (1974)
  Radhakrishnan and Lakshmanan, J. Phys. A 28, 2683 (1995).
- Dark-dark and bright-dark solitons to defocusing-defocusing case
  Kivshar, Phys. Rev. E 55, 4773 (1997);
  B. Prinari, M. J. Ablowitz etc. J. Math. Phys. 47, 063508 (2006).
- Bright-dark soliton solution to focusing-defocusing case
  Kanna, Lakshmanan etc., Phys. Rev. E 73, 026604 (2006).
  Y. Ohta, D.-S. Wang and J. Yang, Stud. Appl. Math. 127, 345 (2011).
  Vijayajayanthi, Kanna, Lakshmanan, Phys. Rev. A 77, 013820 (2008).

- Construct a unified formula for the general soliton solution for the vector NLS equation of all types

$$\mathrm{i}q_{j,t} + q_{j,xx} + 2\left(\sum_j \sigma_j |q_j|^2\right) q_j = 0\,, \quad j = 1, 2, \cdots, M$$

based on the KP theory.

Two-bright-one-dark soliton solution to 3-coupled NLS equation (I)

Assuming q_1 and q_2 are of bright type, q_3 is of dark type

$$q_j = \frac{g_j}{f} e^{2\mathrm{i}\sigma_3|\rho_1|^2 t}, j = 1, 2, \quad q_3 = \rho_1 \frac{h_1}{f} e^{\mathrm{i}(\beta_1 x + (2\sigma_3|\rho_1|^2 - \beta_1^2)t)}, \tag{1}$$

transform three-coupled NLS equation

$$\mathrm{i}q_{j,t} + q_{j,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2 + \sigma_3|q_3|^2\right) q_j = 0, \quad j = 1, 2, 3 \tag{2}$$

into the following bilinear equations

$$\begin{cases} (\mathrm{i}D_t + D_x^2) g_j \cdot f = 0, \quad j = 1, 2 \\ (\mathrm{i}D_t + D_x^2 + 2\mathrm{i}\beta_1 D_x) h_1 \cdot f = 0, \\ \left(\frac{1}{2} D_x^2 + \sigma_3|\rho_1|^2\right) f \cdot f = \sum_{j=1}^{2} \sigma_j |g_j|^2 + \sigma_3 |\rho_1|^2 |h_1|^2, \end{cases} \tag{3}$$

Two-bright-one-dark soliton solution to 3-coupled NLS equation (II)

Theorem

The 3-coupled NLS equation admits two-bright-one-dark soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad h_1 = \begin{vmatrix} A^{(1)} & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_j & 0 \end{vmatrix}, \tag{4}$$

where the elements defined respectively by

$$a_{ij} = \frac{1}{p_i + \bar{p}_j} e^{\xi_i + \bar{\xi}_j}, \quad a_{ij}^{(1)} = \frac{1}{p_i + \bar{p}_j} \left(-\frac{p_i - \mathrm{i}\beta_1}{\bar{p}_j + \mathrm{i}\beta_1} \right) e^{\xi_i + \bar{\xi}_j}, \tag{5}$$

Two-bright-one-dark soliton solution to 3-coupled NLS equation (III)

$$b_{ij} = \frac{\sum_{k=1}^{2} \bar{\alpha}_i^{(k)} \sigma_k \alpha_j^{(k)}}{(\bar{p}_i + p_j)\left(1 + \frac{\sigma_3 |\rho_1|^2}{(\bar{p}_i + \mathrm{i}\beta_1)(p_j - \mathrm{i}\beta_1)}\right)}, \tag{6}$$

$$\Phi = (e^{\xi_1}, e^{\xi_2}, \cdots, e^{\xi_N}), \quad C_j = -(\alpha_1^{(j)}, \alpha_2^{(j)}, \cdots, \alpha_N^{(j)}), \tag{7}$$

with $\xi_i = p_i x + \mathrm{i} p_i^2 t + \xi_{i0}$, p_i, α_i and ξ_{i0} $(i = 1, 2, \cdots, N)$ are complex constants.

Three-component KP hierarchy and its Gram-type solution

Define a tau-function for three-component KP hierarchy,

$$\tau_{0,0}^{k_1} = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}_{2N\times 2N}, \tag{8}$$

where A and B are $N \times N$ matrices whose elements are

$$a_{ij}^{k_1} = \frac{1}{p_i + \bar{p}_j}\left(-\frac{p_i - c}{\bar{p}_j + c}\right)^{k_1} e^{\xi_i + \bar{\xi}_j},$$

$$b_{ij} = \frac{1}{q_i + \bar{q}_j} e^{\eta_i + \bar{\eta}_j} + \frac{1}{r_i + \bar{r}_j} e^{\chi_i + \bar{\chi}_j},$$

with

$$\xi_i = \frac{x_{-1}^{(1)}}{p_i - c} + p_i x_1 + p_i^2 x_2 + \xi_{i0},\ \bar{\xi}_j = \frac{x_{-1}^{(1)}}{\bar{p}_i + c} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}_{j0},$$

$$\eta_i = q_i y_1^{(1)} + \eta_{i0}, \quad \bar{\eta}_j = \bar{q}_j y_1^{(1)} + \bar{\eta}_{j0},$$

$$\chi_i = r_i y_1^{(2)} + \chi_{j0},\ \bar{\chi}_j = \bar{r}_j y_1^{(2)} + \bar{\chi}_{j0}.$$

Furthermore, we define the following tau-functions

$$\tau_{1,0}^{k_1} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & -\bar{\Psi} & 0 \end{vmatrix}, \quad \tau_{-1,0}^{k_1} = \begin{vmatrix} A & I & 0^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & 0 & 0 \end{vmatrix}, \tag{9}$$

$$\tau_{0,1}^{k_1} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & -\bar{\Upsilon} & 0 \end{vmatrix}, \quad \tau_{0,-1}^{k_1} = \begin{vmatrix} A & I & 0^T \\ -I & B & \Upsilon^T \\ -\bar{\Phi} & 0 & 0 \end{vmatrix}, \tag{10}$$

Φ, Ψ, Υ, $\bar{\Phi}$, $\bar{\Psi}$ and $\bar{\Upsilon}$ are N-component row vectors whose elements are defined respectively as

$$\Phi = \left(e^{\xi_1}, \cdots, e^{\xi_N}\right), \ \Psi = \left(e^{\eta_1}, \cdots, e^{\eta_N}\right), \ \Upsilon = \left(e^{\chi_1}, \cdots, e^{\chi_N}\right),$$

$$\bar{\Phi} = \left(e^{\bar{\xi}_1}, \cdots, e^{\bar{\xi}_N}\right), \ \bar{\Psi} = \left(e^{\bar{\eta}_1}, \cdots, e^{\bar{\eta}_N}\right), \ \bar{\Upsilon} = \left(e^{\bar{\chi}_1}, \cdots, e^{\bar{\chi}_N}\right),$$

Three-component KP hierarchy and its Gram-type solution

Based on the KP theory, the above tau functions satisfies the following bilinear equations

$$\left\{\begin{array}{l}(D_{x_2}-D_{x_1}^2)\tau_{1,0}^{k_1}\cdot\tau_{0,0}^{k_1}=0\,,\\ (D_{x_2}-D_{x_1}^2)\tau_{0,1}^{k_1}\cdot\tau_{0,0}^{k_1}=0\,,\\ (D_{x_2}-D_{x_1}^2-2cD_{x_1})\tau_{0,0}^{k_1+1}\cdot\tau_{0,0}^{k_1}=0\,,\\ D_{x_1}D_{y_1^{(1)}}\tau_{0,0}^{k_1}\cdot\tau_{0,0}^{k_1}=-2\tau_{1,0}^{k_1}\tau_{-1,0}^{k_1}\,,\\ D_{x_1}D_{y_1^{(2)}}\tau_{0,0}^{k_1}\cdot\tau_{0,0}^{k_1}=-2\tau_{0,1}^{k_1}\tau_{0,-1}^{k_1}\,,\\ \left(D_{x_1}D_{x_{-1}^{(1)}}-2\right)\tau_{0,0}^{k_1}\cdot\tau_{0,0}^{k_1}=-2\tau_{0,0}^{k_1+1}\tau_{0,0}^{k_1-1}\,.\end{array}\right. \tag{11}$$

The proof of above equations can be done by using the Grammian technique.

Reduction 1: Complex conjugate reduction
Assume $\bar{p}_j$ means the complex conjugate of p_j, $x_1, x_{-1}^{(1)}, y_1^{(1)}, y_1^{(2)}$ are real, x_2, $c = \mathrm{i}\beta_1$ are pure imaginary
For simplicity, we define

$$f = \tau_{0,0}^{0}, \quad g_1 = \tau_{1,0}^{0}, \quad g_2 = \tau_{0,1}^{0}, \quad h_1 = \tau_{0,0}^{1}.$$

Then we can show that f is real and

$$\bar{g}_1 = -\tau_{-1,0}^{0}, \quad \bar{g}_2 = -\tau_{0,-1}^{0}, \quad \bar{h}_1 = \tau_{0,0}^{-1},$$

Reductions to two-bright-one-dark soliton

In this way, the bilinear equations (11) read

$$\begin{cases} (D_{x_2} - D_{x_1}^2) g_j \cdot f = 0\,, \quad j = 1, 2 \\ (D_{x_2} - D_{x_1}^2 - 2\mathrm{i}\beta_1 D_{x_1}) h_1 \cdot f = 0\,, \\ D_{x_1} D_{y_1^{(j)}} f \cdot f = 2 g_j \bar{g}_j\,, \quad j = 1, 2 \\ \left(D_{x_1} D_{x_{-1}^{(1)}} - 2 \right) f \cdot f = -2 h_1 \bar{h}_1. \end{cases} \tag{12}$$

$$\begin{cases} (\mathrm{i}D_t + D_x^2) g_j \cdot f = 0\,, \quad j = 1, 2 \\ (\mathrm{i}D_t + D_x^2 + 2\mathrm{i}\beta_1 D_x) h_1 \cdot f = 0\,, \\ \left(\frac{1}{2} D_x^2 + \sigma_3 |\rho_1|^2 \right) f \cdot f = \sum_{j=1}^{2} \sigma_j |g_j|^2 + \sigma_3 |\rho_1|^2 |h_1|^2\,, \end{cases}$$

Reduction 2: Dimension reduction

If the following dimension reduction holds

$$f_{x_1} = \sigma_1 f_{y_1^{(1)}} + \sigma_2 f_{y_1^{(2)}} - \sigma_3 |\rho_1|^2 f_{x_{-1}^{(1)}}\,, \tag{13}$$

thus,

$$f_{x_1 x_1} = \sigma_1 f_{x_1 y_1^{(1)}} + \sigma_2 f_{x_1 y_1^{(2)}} - \sigma_3 |\rho_1|^2 f_{x_1 x_{-1}^{(1)}}\,. \tag{14}$$

Then from the last two bilinear equations in (12)

$$f_{x_1 y_j^{(1)}} f - f_{x_1} f_{y_j^{(1)}} = |g_j|^2\,,\ f_{x_1 x_{-1}^{(1)}} f - f_{x_1} f_{x_{-1}^{(1)}} - f^2 = -|h_1|^2\,, \tag{15}$$

we have

$$f_{x_1 x_1} - f_{x_1}^2 + \sigma_3 |\rho_1|^2 f^2 = \sigma_1 |g_1|^2 + \sigma_2 |g_2|^2 + \sigma_3 |\rho_1|^2 |h_1|^2\,, \tag{16}$$

which is exactly the last bilinear equation in (3)

How to realize the dimension reduction?

Note that, by row operations, f can be rewritten as

$$f = \begin{vmatrix} A' & I \\ -I & B' \end{vmatrix}, \quad a'_{ij} = \frac{1}{p_i + \bar{p}_j},$$

$$b'_{ij} = \frac{1}{q_i + \bar{q}_j} e^{(\eta_i + \bar{\xi}_i) + (\bar{\eta}_j + \xi_j)} + \frac{1}{r_i + \bar{r}_j} e^{(\chi_i + \bar{\xi}_i) + (\bar{\chi}_j + \xi_j)},$$

where each exponent in b'_{ij} can be divided into two parts

$$\eta_i + \bar{\xi}_i = q_i y_1^{(1)} + \frac{1}{\bar{p}_i + c} x_{-1}^{(1)} + \bar{p}_i x_1 + \cdots,$$

$$\bar{\eta}_j + \xi_j = \bar{q}_j y_1^{(1)} + \frac{1}{p_j - c} x_{-1}^{(1)} + p_j x_1 + \cdots,$$

$$\chi_i + \bar{\xi}_i = r_i y_1^{(2)} + \frac{1}{\bar{p}_i + c} x_{-1}^{(1)} + \bar{p}_i x_1 + \cdots,$$

$$\bar{\chi}_j + \xi_j = \bar{r}_j y_1^{(2)} + \frac{1}{p_j - c} x_{-1}^{(1)} + p_j x_1 + \cdots.$$

How to realize the dimension reduction?

Therefore, under the reduction conditions

$$\sigma_1 q_i = \bar{p}_i + \frac{\sigma_3 |\rho_1|^2}{\bar{p}_i + c}\,, \quad \sigma_1 \bar{q}_i = p_i + \frac{\sigma_3 |\rho_1|^2}{p_i - c}\,, \tag{17}$$

$$\sigma_2 r_i = \bar{p}_i + \frac{\sigma_3 |\rho_1|^2}{\bar{p}_i + c}\,, \quad \sigma_2 \bar{r}_i = p_i + \frac{\sigma_3 |\rho_1|^2}{p_i - c}\,, \tag{18}$$

the following relation holds

$$\partial_{x_1} b'_{ij} = \left(\sigma_1 \partial_{y_1^{(1)}} + \sigma_2 \partial_{y_1^{(2)}} - \sigma_3 |\rho_1|^2 \partial_{x_{-1}^{(1)}} \right) b'_{ij}\,,$$

which realizes the dimension reduction condition

$$f_{x_1} = \sigma_1 f_{y_1^{(1)}} + \sigma_2 f_{y_1^{(2)}} - \sigma_3 |\rho_1|^2 f_{x_{-1}^{(1)}}\,.$$

Applying variable transformations

$$x_1 = x, \quad x_2 = \mathrm{i}t\,, \tag{19}$$

i.e.,

$$\partial_{x_1} = \partial_x, \quad \partial_{x_2} = -\mathrm{i}\partial_t\,, \tag{20}$$

Under above variable transformations, the variables $y_1^{(1)}$, $y_1^{(2)}$, $x_{-1}^{(1)}$ become dummy variables, which basically can be treated as constants. Consequently, we could let $e^{\eta_i} = \bar{\alpha}_i^{(1)}$, $e^{\bar{\eta}_i} = \alpha_i^{(1)}$,$e^{\chi_i} = \bar{\alpha}_i^{(2)}$, $e^{\bar{\chi}_i} = \alpha_i^{(2)}(i = 1, 2, \cdots, N)$, further, we let

$$-\bar{\Psi} = C_1\,, \quad -\bar{\Upsilon} = C_2.$$

Obviously (8) - (10) recover the two-bright-one-dark soliton solution (4).

One-bright-two-dark soliton solution to 3-coupled NLS equation (I)

Assuming q_1 is of bright type, q_2 and q_3 is of dark type

$$q_1 = \frac{g_1}{f} e^{2\mathrm{i}\left(\sigma_2|\rho_1|^2+\sigma_3|\rho_2|^2\right)t},$$

$$q_{l+1} = \rho_l \frac{h_l}{f} e^{\mathrm{i}(\beta_l x+(\sigma_2|\rho_1|^2+\sigma_3|\rho_2|^2-\beta_l^2)t)}, l = 1, 2$$

transform three-coupled NLS equation

$$\mathrm{i}q_{j,t} + q_{j,xx} + 2\left(\sigma_1|q_1|^2 + \sigma_2|q_2|^2 + \sigma_3|q_3|^2\right) q_j = 0, \quad j = 1, 2, 3$$

into the following bilinear equations

$$\begin{cases} (\mathrm{i}D_t + D_x^2)g_1 \cdot f = 0, \\ (\mathrm{i}D_t + D_x^2 + 2\mathrm{i}\beta_l D_x)h_l \cdot f = 0, \quad l = 1, 2 \\ \left(\frac{1}{2}D_x^2 + \sum_{l=1}^{2} \sigma_{l+1}|\rho_l|^2\right) f \cdot f = \sigma_1|g_1|^2 + \sum_{l=1}^{2} \sigma_{l+1}|\rho_l|^2|h_l|^2, \end{cases} \tag{21}$$

One-bright-two-dark soliton solution

Theorem

The 3-coupled NLS equation admits one-bright-two-dark soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \; h_l = \begin{vmatrix} A^{(l)} & I \\ -I & B \end{vmatrix}, \; g_1 = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_1 & 0 \end{vmatrix}, \qquad (22)$$

where the elements defined respectively by

$$a_{ij} = \frac{1}{p_i + \bar{p}_j} e^{\xi_i + \bar{\xi}_j}, \quad a_{ij}^{(l)} = \frac{1}{p_i + \bar{p}_j} \left(-\frac{p_i - \mathrm{i}\beta_l}{\bar{p}_j + \mathrm{i}\beta_l} \right) e^{\xi_i + \bar{\xi}_j}, \qquad (23)$$

$$b_{ij} = \frac{\bar{\alpha}_i^{(1)} \sigma_1 \alpha_j^{(1)}}{(\bar{p}_i + p_j)\left(1 + \sum_{l=1}^{2} \frac{\sigma_{l+1}|\rho_l|^2}{(\bar{p}_i + \mathrm{i}\beta_l)(p_j - \mathrm{i}\beta_l)}\right)}, \qquad (24)$$

Two-component KP hierarchy and its Gram-type solution

Define a tau-function for two-component KP hierarchy,

$$\tau_0^{k_1,k_2} = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}_{2N\times 2N}, \qquad (25)$$

where A and B are $N \times N$ matrices whose elements are

$$a_{ij}^{k_1,k_2} = \frac{1}{p_i + \bar{p}_j}\left(-\frac{p_i - c}{\bar{p}_j + c}\right)^{k_1}\left(-\frac{p_i - d}{\bar{p}_j + d}\right)^{k_2} e^{\xi_i + \bar{\xi}_j},$$

$$b_{ij} = \frac{1}{q_i + \bar{q}_j} e^{\eta_i + \bar{\eta}_j},$$

with

$$\xi_i = \frac{1}{p_i - c} x_{-1}^{(1)} + \frac{1}{p_i - d} x_{-1}^{(2)} + p_i x_1 + p_i^2 x_2 + \xi_{i0},$$

$$\bar{\xi}_j = \frac{1}{\bar{p}_i + c} x_{-1}^{(1)} + \frac{1}{\bar{p}_i + d} x_{-1}^{(2)} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}_{j0},$$

$$\eta_i = q_i y_1^{(1)} + \eta_{i0}, \quad \bar{\eta}_j = \bar{q}_j y_1 + \bar{\eta}_{j0}.$$

Furthermore, we define the following tau-functions

$$\tau_1^{k_1,k_2} = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & -\bar{\Psi} & 0 \end{vmatrix}, \quad \tau_{-1}^{k_1,k_2} = \begin{vmatrix} A & I & 0^T \\ -I & B & \Psi^T \\ -\bar{\Phi} & 0 & 0 \end{vmatrix},$$

Then the following bilinear equations hold

$$\begin{cases} (D_{x_2} - D_{x_1}^2)\tau_1^{k_1,k_2} \cdot \tau_0^{k_1,k_2} = 0\,, \\ (D_{x_2} - D_{x_1}^2 - 2cD_{x_1})\tau_0^{k_1+1,k_2} \cdot \tau_0^{k_1,k_2} = 0\,, \\ (D_{x_2} - D_{x_1}^2 - 2dD_{x_1})\tau_0^{k_1,k_2+1} \cdot \tau_0^{k_1,k_2} = 0\,, \\ D_{x_1}D_{y_1^{(1)}}\tau_0^{k_1,k_2} \cdot \tau_0^{k_1,k_2} = -2\tau_1^{k_1,k_2}\tau_{-1}^{k_1,k_2}\,, \\ \left(D_{x_1}D_{x_{-1}^{(1)}} - 2\right)\tau_0^{k_1,k_2} \cdot \tau_0^{k_1,k_2} = -2\tau_0^{k_1+1,k_2} \cdot \tau_0^{k_1-1,k_2}\,, \\ \left(D_{x_1}D_{x_{-1}^{(2)}} - 2\right)\tau_0^{k_1,k_2} \cdot \tau_0^{k_1,k_2} = -2\tau_0^{k_1,k_2+1} \cdot \tau_0^{k_1,k_2-1}\,. \end{cases} \tag{26}$$

Reductions to one-bright-two-dark soliton

Reduction 1: Complex conjugate reduction
Assume $\bar{p}_j$ means the complex conjugate of p_j, $x_1, x_{-1}^{(1)}, y_1^{(1)}$ are real, x_2, $c = \mathrm{i}\beta_1$, $d = \mathrm{i}\beta_2$ are pure imaginary
For simplicity, we define

$$f = \tau_{0,0}^{0}, \quad g_1 = \tau_1^{0,0}, \quad h_1 = \tau_0^{1,0}, \quad h_2 = \tau_0^{0,1},$$

Then we can show that f is real and

$$\bar{g}_1 = -\tau_{-1}^{0,0}, \quad \bar{h}_1 = \tau_0^{-1,0}, \quad \bar{h}_2 = \tau_0^{0,-1}$$

Reductions to one-bright-two-dark soliton

In this way, the bilinear equations (26) read

$$\begin{cases} (D_{x_2} - D_{x_1}^2)g_1 \cdot f = 0\,, \\ (D_{x_2} - D_{x_1}^2 - 2\beta_l D_{x_1})h_l \cdot f = 0\,, \quad l = 1, 2\,, \\ D_{x_1} D_{y_1^{(1)}} f \cdot f = 2|g_1|^2\,, \\ \left(D_{x_1} D_{x_{-1}^{(l)}} - 2\right) f \cdot f = -2|h_l|^2 \quad l = 1, 2\,. \end{cases} \tag{27}$$

$$\begin{cases} (\mathrm{i}D_t + D_x^2)g_1 \cdot f = 0\,, \\ (\mathrm{i}D_t + D_x^2 + 2\mathrm{i}\beta_l D_x)h_l \cdot f = 0\,, \quad l = 1, 2 \\ \left(\frac{1}{2}D_x^2 + \sum_{l=1}^{2} \sigma_{l+1}|\rho_l|^2\right) f \cdot f = \sigma_1|g_1|^2 + \sum_{l=1}^{2} \sigma_{l+1}|\rho_l|^2|h_l|^2, \end{cases}$$

Reduction 2: Dimension reduction, if

$$\frac{1}{q_i+\bar{q}_j}=\frac{\sigma_1}{(\bar{p}_i+p_j)\left(1+\frac{\sigma_2|\rho_1|^2}{(\bar{p}_i+c)(p_j-c)}+\frac{\sigma_3|\rho_2|^2}{(\bar{p}_i+d)(p_j-d)}\right)},$$

then

$$f_{x_1}=\sigma_1 f_{y_1^{(1)}}-\sigma_2|\rho_1|^2 f_{x_{-1}^{(1)}}-\sigma_3|\rho_2|^2 f_{x_{-1}^{(2)}}, \qquad (28)$$

which also implies

$$f_{x_1x_1}=\sigma_1 f_{x_1y_1^{(1)}}-\sigma_2|\rho_1|^2 f_{x_1x_{-1}^{(1)}}-\sigma_3|\rho_2|^2 f_{x_1x_{-1}^{(2)}}. \qquad (29)$$

From the last two bilinear equations in (27) are reduced to

$$f_{x_1x_1}-f_{x_1}^2+\sum_{l=1}^{2}\sigma_{l+1}|\rho_l|^2 f^2=\sigma_1|g_1|^2+\sum_{l=1}^{2}\sigma_{l+1}|\rho_l|^2|h_l|^2,$$

which is exactly the last bilinear equation in (21)

General bright-dark soliton solution to M-coupled NLS equation (I)

A general soliton solution with m-bright solitons and $(M-m)$-dark solitons

$$q_j = \frac{g_j}{f} e^{2\mathrm{i}\sum_{k=1}^{M-m} \sigma_{k+m}|\rho_k|^2 t}, j = 1, 2 \cdots, m$$

$$q_{m+l} = \rho_l \frac{h_l}{f} e^{\mathrm{i}(\beta_l x + (2\sum_{k=1}^{M-m} \sigma_{k+m}|\rho_k|^2 - \beta_l^2)t)}, l = 1, 2, \cdots, M-m$$

transform M-coupled NLS equation

$$\mathrm{i}q_{j,t} + q_{j,xx} + 2\left(\sum_j \sigma_j |q_j|^2\right) q_j = 0, \quad j = 1, 2, \cdots, M$$

into the following bilinear equations

$$\begin{cases} (\mathrm{i}D_t + D_x^2) g_j \cdot f = 0, \quad j = 1, 2, \cdots, m, \\ (\mathrm{i}D_t + D_x^2 + 2i\beta_l D_x) h_l \cdot f = 0, \quad l = 1, 2, \cdots, M-m, \\ \left(\frac{1}{2}D_x^2 + \sum_{l=1}^{M-m} \sigma_{l+m}|\rho_l|^2\right) f \cdot f = \sum_{j=1}^{m} \sigma_j |g_j|^2 + \sum_{l=1}^{M-m} \sigma_{l+m}|\rho_l|^2 |h_l|^2. \end{cases}$$

General bright-dark soliton solution

Theorem

The M-coupled NLS equation admits m-bright-$M-m$-dark soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad h_l = \begin{vmatrix} A^{(l)} & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_j & 0 \end{vmatrix}, \qquad (31)$$

where the elements defined respectively by

$$a_{ij} = \frac{1}{p_i + \bar{p}_j} e^{\xi_i + \bar{\xi}_j}, \quad a_{ij}^{(l)} = \frac{1}{p_i + \bar{p}_j} \left(-\frac{p_i - \mathrm{i}\beta_l}{\bar{p}_j + \mathrm{i}\beta_l} \right) e^{\xi_i + \bar{\xi}_j}, \qquad (32)$$

$$b_{ij} = \frac{\sum_{k=1}^{m} \bar{\alpha}_i^{(k)} \sigma_k \alpha_j^{(k)}}{(\bar{p}_i + p_j)\left(1 + \sum_{l=1}^{M-m} \frac{\sigma_{l+m}|\rho_l|^2}{(\bar{p}_i + \mathrm{i}\beta_l)(p_j - \mathrm{i}\beta_l)}\right)}, \qquad (33)$$

We consider a $(m+1)$-component KP hierarchy with $(M-m)$-copies of shifted singular points in first component. Based on the KP theory, the following bilinear equations hold

$$\begin{cases} (D_{x_2} - D_{x_1}^2) g_j \cdot f = 0\,, \\ (D_{x_2} - D_{x_1}^2 - 2\mathrm{i}\beta_l D_{x_1}) h_l \cdot f = 0\,, \\ D_{x_1} D_{y_1^{(j)}} f \cdot f = 2 g_j \bar{g}_j\,, \\ D_{x_1} D_{x_{-1}^{(l)}} f \cdot f - 2f^2 = -2 h_l \bar{h}_l\,, \end{cases} \tag{34}$$

where $j = 1, 2, \cdots, m$, $l = 1, 2, \cdots, M-m$

Dimension reduction

$$\sum_{j=1}^{m} \sigma_j f_{y_1^{(j)}} - \sum_{l=1}^{M-m} \sigma_{l+m} |\rho_l|^2 f_{x_{-1}^{(l)}} = f_x\,, \tag{35}$$

Unified formula for the general soliton solution to the vector CNLS equation(I)

Theorem

The M-coupled NLS equation admits general soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad h_l = \begin{vmatrix} A^{(l)} & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_j & 0 \end{vmatrix},$$

- *The general bright-dark soliton solution has been given previously. The condition for the regularity of the soluton is given by*

$$\left(\sum_{k=1}^{m} \bar{\alpha}_1^{(k)} \sigma_k \alpha_1^{(k)}\right)\left(1 + \sum_{l=1}^{M-m} \frac{\sigma_{l+m}|\rho_l|^2}{|p_i - \mathrm{i}\beta_l|^2}\right) > 0\,, \quad i = 1, 2, \cdots, N\,.$$

Unified formula for the general soliton solution to the vector CNLS equation(II)

Theorem

The M-coupled NLS equation admits general soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C_j & 0 \end{vmatrix},$$

- *The general all bright soliton solution can be given if the element in matrix B can be adjusted into*

$$b_{ij} = \frac{\sum_{k=1}^{M} \bar{\alpha}_i^{(k)} \sigma_k \alpha_j^{(k)}}{(\bar{p}_i + p_j)}$$

Unified formula for the general soliton solution to the vector CNLS equation(II)

Theorem

The M-coupled NLS equation admits general soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \; h_l = \begin{vmatrix} A^{(l)} & I \\ -I & B \end{vmatrix},$$

- *The general all dark soliton solution can be given if $b_{ij} = \delta_{ij}$. The reason lies in a simple fact that*

$$\begin{vmatrix} A & I \\ -I & I \end{vmatrix} = |I + A| \; .$$

$$\sum_{l=1}^{M} \frac{\sigma_l |\rho_l|^2}{(\bar{p}_i + \mathrm{i}\beta_l)(p_i - \mathrm{i}\beta_l)} = -1 \, ,$$

Thank you!