Periodic block Jacobi matrices and the XY spin chain

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This talk was presented on the blackboard. The following extended abstract is provided in lieu of slides.

Our work concerns the anisotropic XY spin chain, which is an infinite sequence of spin-1/2 particles interacting through a specific nearest neighbor interaction. The state of each particle is described by an element of $\mathbb{C}^2$, so the state of a finite "box" $\Lambda = [m, n] \cap \mathbb{Z}$ is described by an element of $\otimes_{j=m}^{n} \mathbb{C}^2$. The Hamiltonian on Λ is

$$H^{(\Lambda)} = \sum_{j=m}^{n-1} \mu_j [(1 + \gamma_j)\sigma_j^{(x)}\sigma_{j+1}^{(x)} + (1 - \gamma_j)\sigma_j^{(y)}\sigma_{j+1}^{(y)}] + \sum_{j=m}^{n} \nu_j \sigma_j^{(z)},$$

where $\sigma_j^{(x)}, \sigma_j^{(y)}, \sigma_j^{(z)}$ are the usual Pauli matrices acting on spin j. We also assume that μ, γ, ν are bounded sequences and that all $\mu_j \neq 0$.

The support of an operator A, denoted $\operatorname{supp} A$, is the smallest set S such that A can be represented as a tensor product of an operator on S and the identity on $\Lambda \setminus S$. If two operators, A and B, have disjoint supports, then $[A, B] = 0$, so time-propagation of A can be quantified by bounds on $\|[e^{itH^{(\Lambda)}} A e^{-itH^{(\Lambda)}}, B]\|$.

Theorem 1 (Lieb–Robinson, Nachtergaele–Sims)**.** *For any $\Lambda = [m, n] \cap \mathbb{Z}$, any operators A, B with disjoint supports and all $t \in \mathbb{R}$,*

$$\left\| [\tau_t^{(\Lambda)}(A), B] \right\| \leq C \|A\| \|B\| e^{-\eta(d(S_1,S_2) - v|t|)} \tag{1}$$

with uniform constants $\eta, v > 0$ and a constant C which can depend solely on the size of the interaction boundaries of S_1, S_2. If we restrict to operators with non-interlacing supports, $\max \operatorname{supp} A < \min \operatorname{supp} B$*, then C is uniform as well.*

The quantity v above is thought of as the velocity, as the Lieb–Robinson bound is a statement about exponential decay beyond the distance $v|t|$. In the case of $\gamma_j = 0$, constant μ_j, and suitable random i.i.d. ν_j, Hamza–Sims–Stolz proved a zero-velocity Lieb–Robinson bound, which should be viewed as a localization result. We were interested in capturing the opposite effect of ballistic transport by proving that, in some spin chains, there is a non-zero lower bound on the velocity v.

Theorem 2 (DLY). *For the anisotropic XY chain on $\mathbb{Z}$ described above, with periodic sequences of $\mu_j \in \mathbb{R} \setminus \{0\}$, $\gamma_j \in \mathbb{R} \setminus \{\pm 1\}$, $\nu_j \in \mathbb{R}$, there exists a $v_0 > 0$ such that if the Lieb-Robinson bound* (1) *holds for some C, η, v in the sense of Theorem 1, then $v \geq v_0$.*

Our proof is based on an observation of Lieb–Schultz–Mattis that the dynamics of the XY spin chain is closely related to the dynamics of a block Jacobi matrix. Hamza–Sims–Stolz used this to prove an upper bound on propagation in the random case, and we use it to prove a lower bound on propagation in the periodic case, reducing the problem to ballistic transport for periodic block Jacobi matrices.

Block Jacobi matrices are operators $J : \ell^2(\mathbb{Z})^m \to \ell^2(\mathbb{Z})^m$ of the form

$$(Ju)_n = a_{n-1}^* u_{n-1} + b_n u_n + a_n u_{n+1}$$

where a_n and b_n are $m \times m$ complex matrices with $\det a_n \neq 0$ and $b_n^* = b_n$.

While ballistic transport for Jacobi matrices is considered folklore, the results available in the literature concern only scalar Jacobi matrices and do not guarantee a non-zero velocity. We proved a result which does, using an approach of Asch–Knauf for periodic (continuum) Schrödinger operators.

Theorem 3 (DLY). *Let J be a periodic block Jacobi matrix as described above. Let X be the position operator, $(Xu)_n = nu_n$. There is a bounded self-adjoint operator Q with* $\operatorname{Ker} Q = \{0\}$ *such that for any ψ in the domain of X,*

$$\lim_{t\to\infty} \frac{1}{t} e^{itJ} X e^{-itJ} \psi = Q\psi.$$

To illustrate how this is a statement about ballistic motion, notice that it implies, in particular, that

$$\lim_{t\to\infty} \frac{1}{t} \| X e^{-itJ} \psi \| = \| Q\psi \| \neq 0,$$

i.e., the expectation value of position at time t behaves as $\|Q\psi\|\, t$.

If J is the block Jacobi matrix corresponding to the spin chain of Theorem 2, then the velocity v_0 of Theorem 2 is simply $v_0 = \|Q\|$.