Point-Map Probabilities of a Point Process

F. Baccelli, UT AUSTIN

Joint work with M.O. Haji-Mirsadeghi

TexAMP, Austin, November 23, 2014

Structure of the talk

- **Point processes**
- **Palm probabilities**
- **Mecke's invariant measure equation**
- **Point maps**
- **Dynamical systems**
- **Point map probabilities**

COUNTING MEASURES

- **Let ϕ be a finite or countably-infinite collection of points of $I\!R^d$, without accumulation**

- **One can think of ϕ**

  – either as a set of points: $\phi = \{t_n\} \subset I\!R^d$

  – or as a counting measure: $\phi = \sum_n \varepsilon_{t_n}$

POINT PROCESS

- N: **space of all counting measures** ϕ
- $\mathcal{N}$: σ**-field of** $\mathbb{M}$ **generated by** $\phi \mapsto \phi(B) \in \overline{I\!\!N}$, B **Borel sets of** $I\!\!R^d$
- $(\Omega, \mathcal{F}, I\!\!P)$: **probability space**
- **A point process** Φ **is a** $(N, \mathcal{N})$**-valued random variable on** $(\Omega, \mathcal{F}, I\!\!P)$

STATIONARY POINT PROCESS

- **Let $\{\theta_t\}_{t\in \mathbb{R}^d}$ be a measure preserving flow on $(\Omega, \mathcal{F}, \mathbb{P})$**
- **A point process Φ is stationary if**

the translations of Φ are a factor of the flow θ_t:

$$\Phi \circ \theta_t(B) = \Phi(B+t) \quad \forall t, \forall B$$

- **Implies the existence of an intensity λ assumed finite below**

Palm Probability of a Stationary Point Process

- **Probability $\mathbb{P}_0 = \mathbb{P}_0^{\Phi}$ on $(N, \mathcal{N})$ such that**
  - **Mecke:**

$$\int_{\mathbb{R}^d \times \Omega} f(t, \Phi(\theta_t(\omega)))\Phi(\omega, dt)\mathbb{P}(d\omega) = \int_{\mathbb{R}^d \times N} f(t, z)\lambda dt \mathbb{P}_0(dz), \quad \forall f \geq 0$$

  - **Matthes:**

$$\mathbb{P}_0(A) = \frac{\mathbb{E} \sum 1_{t_n \in B} 1_{\Phi \circ \theta_{t_n} \in A}}{\mathbb{E} \sum 1_{t_n \in B}}, \quad \forall A \in \mathcal{N}, \forall B \text{ Borel}$$

- **The support of $\mathbb{P}_0$ is contained in N^0: space of counting measures with a point at the origin.**

- **Interpretation**
  - **Conditional: distribution of the point process given that the origin is included in the point process**
  - **Ergodic: empirical distribution of the sequence $\{\Phi \circ \theta_{t_n}\}$ for all points t_n of Φ in a large ball**

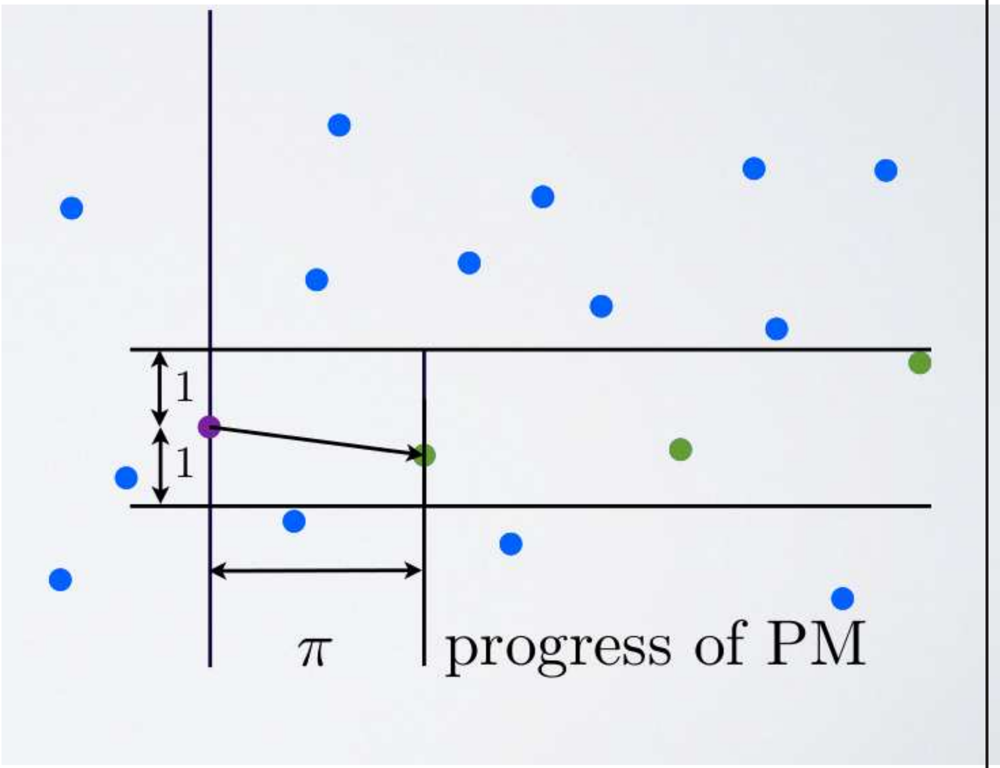
Point-Shift on Point Processes

- **Maps each point of Φ to some point of Φ**
- **Point-Shifts in the literature:**
  - **Point-Shift H. Thorisson 00**
  - **Allocation rule e.g. by A. Holroyd & Y. Peres 05**
- **Initial motivations:**
  - **Palm calculus**
  - **Navigation on the points of Φ**
  - **Cracks in materials**

Example 1 of Point-Shifts on Point Processes

Strip Routing PS

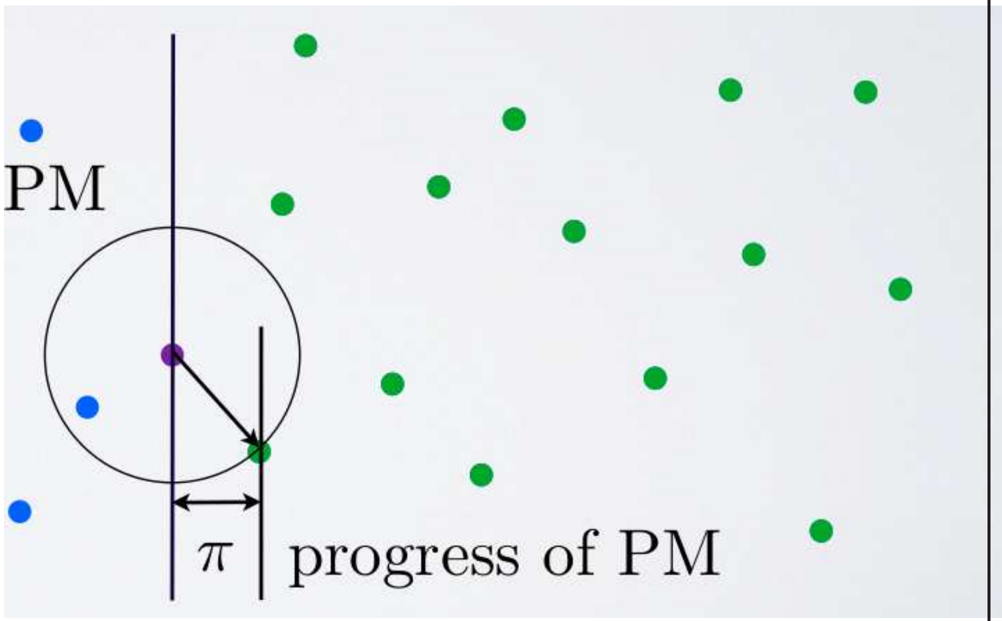
P. A. Ferrari,
C. Landim,
H. Thorisson
05



Example 2 of Point-Shifts on Point Processes

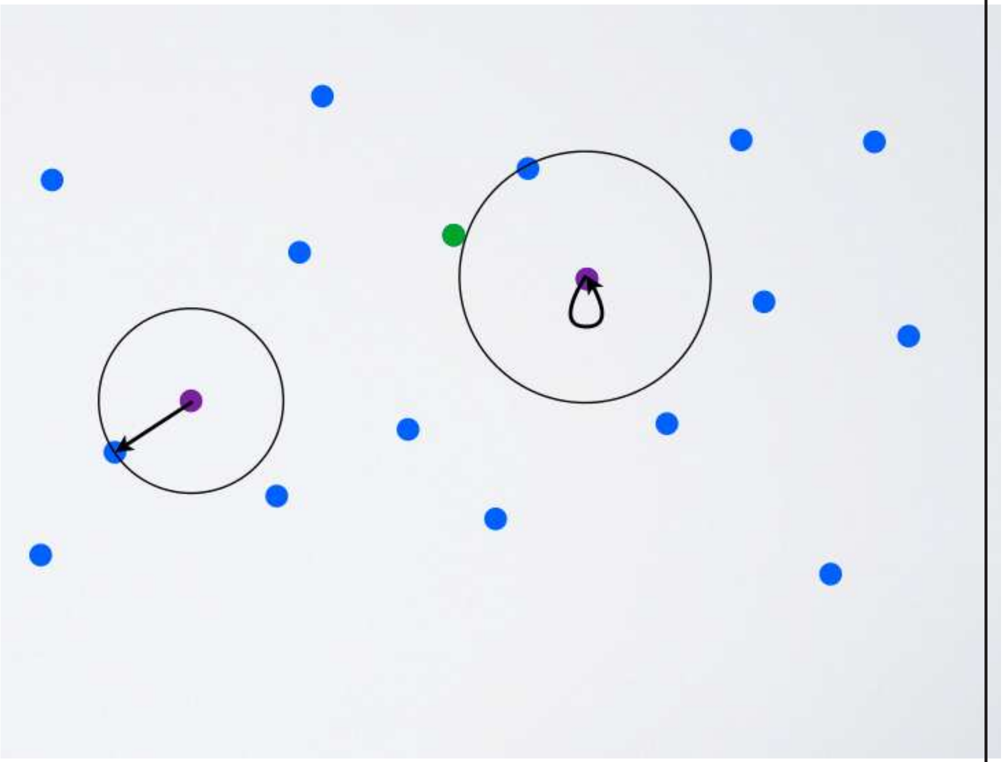
Directional PS

F.B. & C. BORDENAVE 07



Example 3 of Point-Shifts on Point Processes

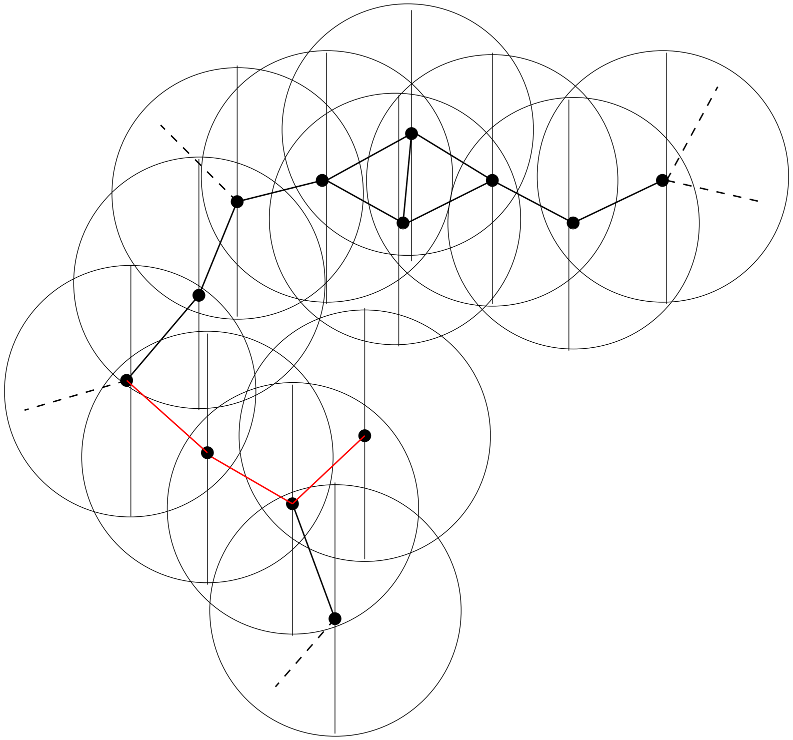
Closest-Closest PS



Example 4 of Point Shifts on Point Processes

Directional PS on the supercritical random geometric graph

The PS a.s. leads to a trap even when departing from points in the infinite connected component.



Factor Point-Shifts

- **Point Shift**

$$f : \textbf{supp}(\Phi) \to \textbf{supp}(\Phi) : \quad f(\Phi, t_n) = t_m$$

- **Factor Point-Shift:**
there exists a function g called the point map, defined on N^0 which associates to each $\phi \in N^0$ a point of its support and s.t.

$$f(t_n) = t_n + g \circ \theta_{t_n}, \quad \forall n$$

- **Notation: f_g or g_f**

Point Stationarity

- **Theorem** **Mecke (1975)**
  If f is an a.s. bijective factor point shift, then its associated point map g preserves the Palm probability of all stationary point processes. i.e.

$$(\theta_g)_* I\!P_0^\Phi = I\!P_0^\Phi, \quad \forall \Phi$$

Palm probabilities are the only probability measures on N^0 preserved by all a.s. bijective factor point shifts.

- **Example: Closest-closest PS**
- **Example of a.s. bijective factor point shifts visiting all points of a Poisson point process in $I\!R^2$ in Ferrari, Landim, Thorisson 04**

Mecke's Invariant Measure Equation

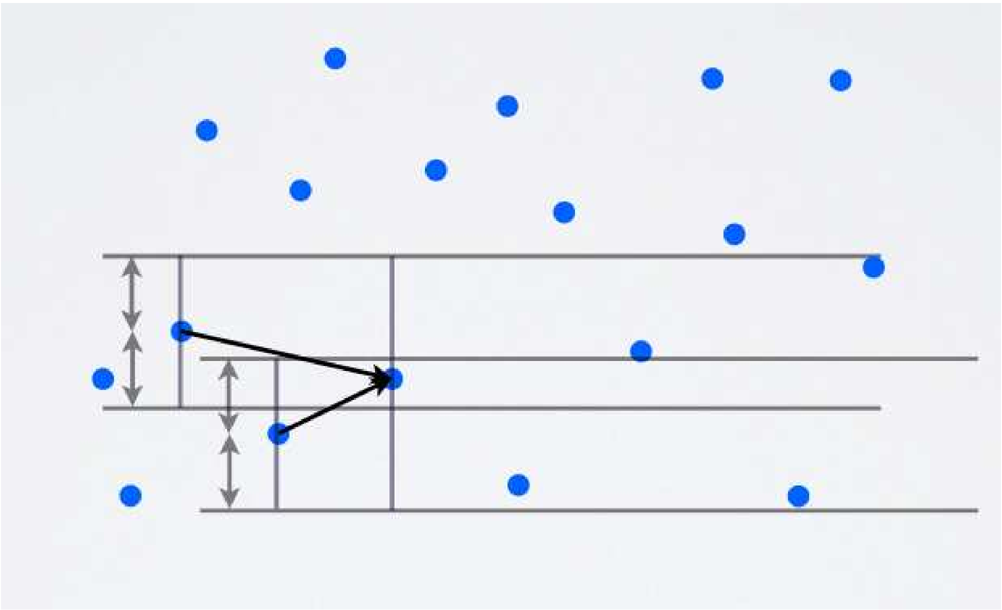
- **Question**
  Let f be a factor point shift. Let g denote its point map.
  What is the set of all probability measures on N^0 satisfying

$$(\theta_g)_* \mathcal{Q} = \mathcal{Q} \quad ? \tag{1}$$

- **The point stationarity theorem says that if f is bijective, the Palm measure of any stationary point process solves (1).**

Invariant Measure Equation

- **Question** If f is not bijective, can we construct a solution to (1) from the Palm probability of a stationary point process?
- Neither strip PM nor directional PM are bijective. Can we find solutions to (1) for these PMs?



Notation for Point Shifts

- **Image of a point**

$$x \in \Phi,\ f_\Phi(x) := f(\Phi, x).$$

- **Image of the point process**

$$f_\Phi(\Phi) := \{f_\Phi(x), x \in \Phi\}.$$

- **Notation:**
  - $M^1(N)$**: set of probability measures on** N
  - $M^1(N^0)$**: set of probability measures on** N^0

Four Actions of $I\!N$

- **Actions $\pi = \{\pi_n\}$ of $(\mathbb{N}, +)$ on the topological space X:**

1 $X = N$**, equipped with the vague topology, and for all** $n \in \mathbb{N}$**,**

$$\pi_n(\phi) = f_\phi^n \phi$$

1^* $X = M^1(N)$**, equipped with the weak convergence of probability measures on** N**, and for all** $\mathcal{Q} \in X$**,** $n \in I\!N$**,**

$$\pi_n(\mathcal{Q}) = (f_\phi \phi)_*^n \mathcal{Q} \in M^1(N)$$

2 $X = N^0$, **equipped with vague topology, and** $\forall \phi \in N^0$ **and** $n \in \mathbb{N}$**,**

$$\pi_n(\phi) = \theta_{g^n(\phi)}(\phi) \in N^0$$

2^* $X = M^1(N^0)$, **equipped with the same topology as** $M^1(N)$**, and for all** $\mathcal{Q} \in M^1(N^0)$ **and** $n \in \mathbb{N}$**,**

$$\pi_n(\mathcal{Q}) = (\theta_{g^n(\phi)})_* \mathcal{Q} \in M^1(N^0)$$

Action 1

- **Possible behaviors:**
  - p**-periodicity**
  - **Evaporation: Action 1 converges a.s. to the null measure on** Φ
- **Let**

$$I := \{\phi \in N^0; \forall n \in I\!N, \exists\, y \in \phi \textbf{ s.t. } f_\phi^n(y) = 0\}$$

- **Lemma 1 For all factor point shifts f and all stationary point processes Φ, there is evaporation of Φ under f if and only if $I\!P_0^\Phi[I] = 0$**

- **Example of p-periodicity**
  - **Directional PS on the super-critical RGG:** $p = 1$
  - **Closest-Closest PS:** $p = 2$
- **Examples of Evaporation**
  - **Ferrari, Landim, Thorisson 04: If Φ is a stationary Poisson point process in $\mathbb{R}^2$, then the strip PS evaporates Φ**
  - **The same holds true for the directional PS on a stationary Poisson point process in $\mathbb{R}^2$**

Actions 1^* and 2

- **Actions 1^* is only of interest in the non evaporation case**
- **Action 2 is not a topological dynamical system in dim. > 1:**

- **Lemma 2 For $d \geq 2$, there is no continuous point map on the whole N^0 other than the identity.**

Action 2^*

- **First g-Palm probability of Φ:**

$$\mathbb{P}_0^{g,1} = (\theta_g)_* \mathbb{P}_0$$

- **Interpretation: distribution of Φ given that the origin is in $f(\Phi)$, considering multiplicities of the points of the image process**

- **n-th g-Palm probability of Φ:**
  distribution of Φ given that 0 is in $f^n(\Phi)$

$$\mathbb{P}_0^{g,n} = (\theta_g)_* \mathbb{P}_0^{g,n-1}$$

  taking multiplicities into account.

- **Interpretation:**
  distribution of Φ seen from a typical point of $f^n(\Phi)$.

Definition of Point Map Probabilities

- **Let**
  - g **be a point map**
  - Φ **be a stationary point process with Palm distribution** $\mathbb{P}_0$
- **Definition**
  - **Every element of the** ω**-limit set of** $\mathbb{P}_0$ **under the action of** $\{(\theta_{g^n})_*\}_{n\in\mathbb{N}}$ **will be called a** g **probability of** $\mathbb{P}_0$
  - **If the limit of the sequence**

$$\{(\theta_{g^n})_*\mathbb{P}_0\}_{n=1}^{\infty} = \{\mathbb{P}_0^{g,n}\}_{n=1}^{\infty}$$

exists, it is called the g **probability of** $\mathbb{P}_0$ **and denoted by** $\mathbb{P}_0^g$

Neither Existence nor Uniqueness are granted

- **Example with no converging subsequence:**

$B_r(x)$**: Ball of radius r and center** x

$$m(x) := \Phi(B_1(x))$$

$$f(x) := \mathbf{argmin}|y - x|; m(y) > 2m(x)$$

$$I\!P_0^{g,n}[\Phi(B_1(0)) > 2n] = 1$$

- **Examples with convergent subsequences with different limits**

Periodic Case

- **Theorem 1** **If $f := f_g$ is 1-periodic on Φ, then the g probability $\mathbb{P}_0^g$ of $\mathbb{P}_0$ exists and is given by**

$$\mathbb{P}_0^g = \Phi_* \, \mathbb{P}_0^{f_\Phi^\infty \Phi}$$

Furthermore $\mathbb{P}_0^g$ is absolutely continuous with respect to $\mathbb{P}_0$, with

$$\frac{d\mathbb{P}_0^g}{d\mathbb{P}_0} = \frac{\left(f_\phi^\infty \phi\right)(\{0\})}{\phi(\{0\})}.$$

In addition, $\mathbb{P}_0^g = (\theta_g)_* \mathbb{P}_0^g$

- **Similar statements hold in the p-periodic case**

Evaporation Case

- **Theorem 2**
  Assume

  – Φ **evaporates under the action of** f_g**,**

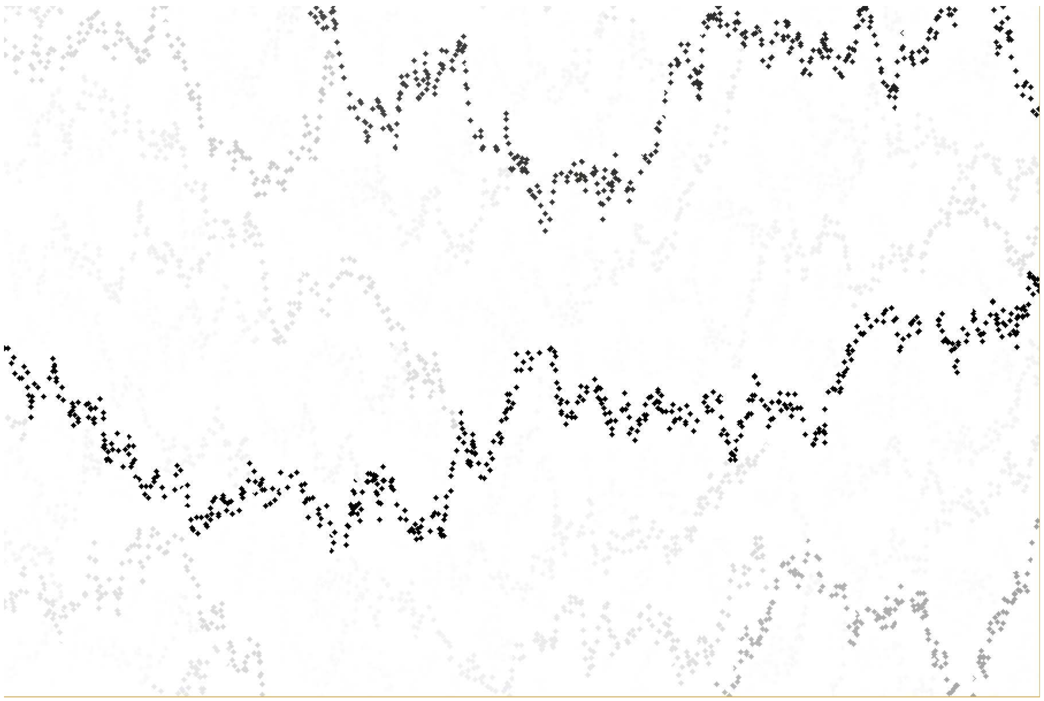
  – the g **probability** $\mathbb{P}_0^g$ **of** $\mathbb{P}_0$ **exists**

  – it satisfies $\mathbb{P}_0^g = (\theta_g)_* \mathbb{P}_0^g$**,**

then $\mathbb{P}_0^g$ **is singular with respect to** $\mathbb{P}_0$

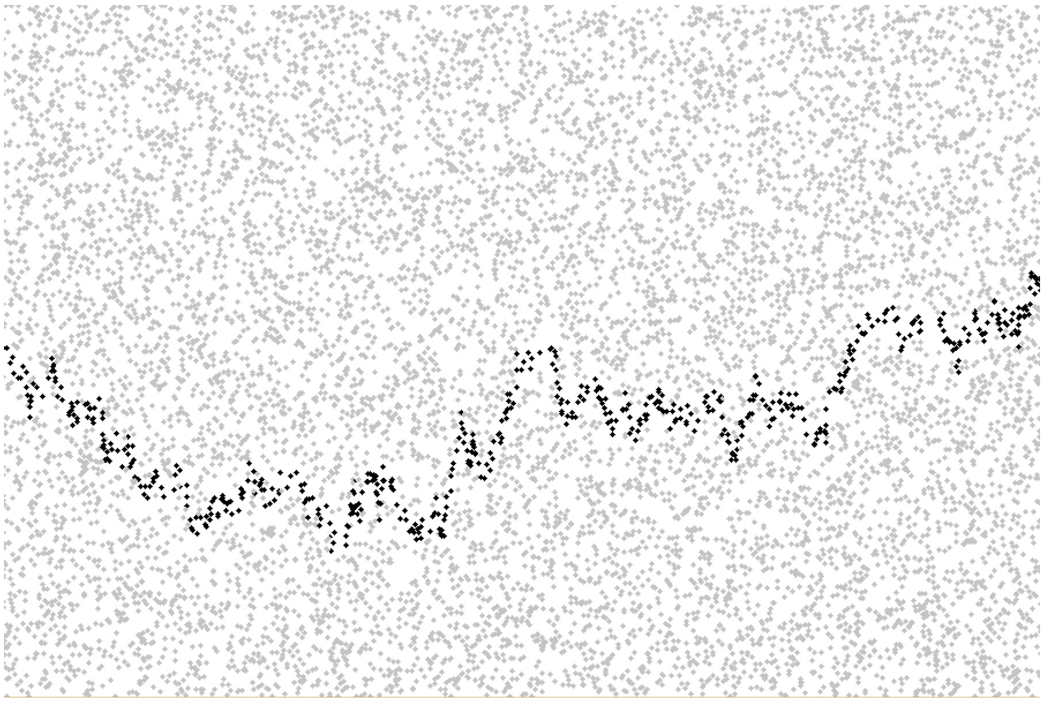
- **Relies on Lemma 1: under** $\mathbb{P}_0^g$**,** I **is of probability 1; under** $\mathbb{P}_0$**, it is of probability 0.**

Sample of the Point Process under its Palm Probability



Grey level proportional to the ”age” of the point w.r.t. the PM

Sample of the Point Process unde its Point Map Probability



A point is black if it has pre-images of all orders and grey otherwise

Mecke's Invariant Measure Equation

- **Consider the Cesàro sums**

$$\widetilde{\mathbb{P}}_0^{g,n} := \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{P}_0^{g,i}, \quad n \in \mathbb{N}.$$

When the limit of $\widetilde{\mathbb{P}}_0^{g,n}$ exists (w.r.t. the topology of $M^1(N^0)$), let

$$\widetilde{\mathbb{P}}_0^{g} := \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{P}_0^{g,i}.$$

In general, $\widetilde{\mathbb{P}}_0^{g}$ is not a g probability.

- **Theorem 3**

  Assume

  – **there exists a subsequence** $\{\widetilde{\mathbb{P}}_0^{g,n_i}\}_{i=1}^{\infty}$ **converging to a probability** $\widetilde{\mathbb{P}}_0^g$

  – $(\theta_g)_*$ **is continuous at** $\widetilde{\mathbb{P}}_0^g$

Then $\widetilde{\mathbb{P}}_0^g$ **solves Mecke's Invariant Measure Equation**

- **Proposition 1**

  If θ_g **is** $\widetilde{\mathbb{P}}_0^g$**-almost surely continuous, then** $(\theta_g)_*$ **is continuous at** $\widetilde{\mathbb{P}}_0^g$

- **Proposition 2**

  If g **is** $\widetilde{\mathbb{P}}_0^g$**-almost surely continuous, then** $(\theta_g)_*$ **is** $\widetilde{\mathbb{P}}_0^g$**-continuous**

Construction of Point Map Probabilities

- **Tightness**
- **Markov**

Tightness Approach

- **Tightness Lemma Kallenberg 1986:**
  The sequence of point processes with distributions Q_n is sequentially compact iff for all bounded Borel $B \subset I\!R^d$,

$$\lim_{r\to\infty} \limsup_{n\to\infty} Q_n(\Phi(B) > r) = 0.$$

- **The tightness lemma gives a necessary and sufficient condition for the existence of converging subsequences of $I\!P_0^{f,n}$.**
- **Example: Any point-map on the Matérn hard core model**

Markovian Approach

- **Illustrated by the Strip PS**
- **Proposition 3**
  If $f = f_g$ is the strip PS on the stationary Poisson point process on $\mathbb{R}^2$, then

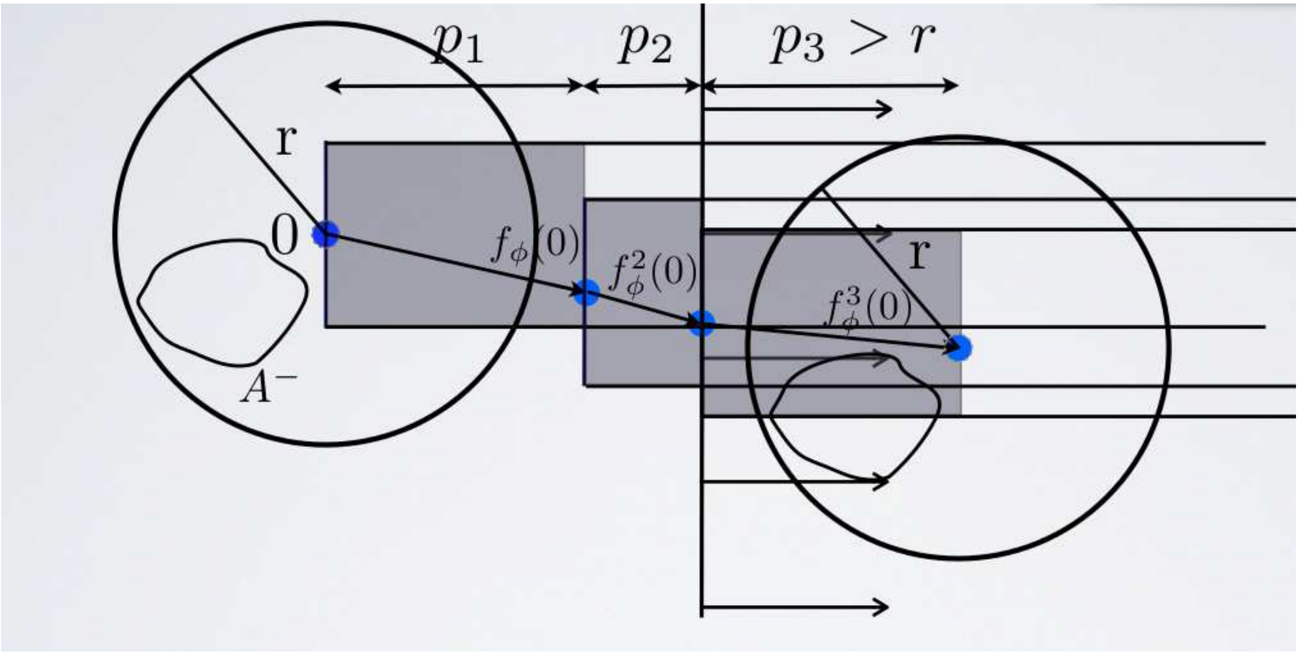
  – **the sequence $\mathbb{P}_0^{g,n}$ is convergent**

  – **θ_g is $\mathbb{P}_0^g$-a.s. continuous**

Sketch of Proof

- $B_r^+(0)$: **right open half disk of radius** r
- $B_r^-(0)$: **left closed half disk of radius** r
- **It is sufficient to show the convergence in** $B_r^+(0)$ **and** $B_r^-(0)$ **for all** r**.**
- $A^+ \subset B_r^+(0)$ **and** $A^- \subset B_r^-(0)$**.**

Sketch of Proof



+ Strong Markov property of the Poisson point process.

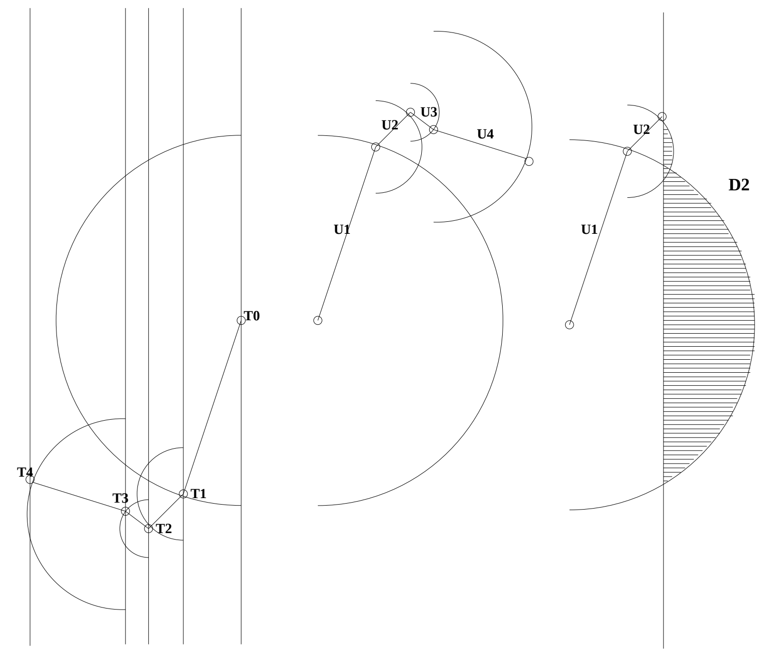
Directional PM

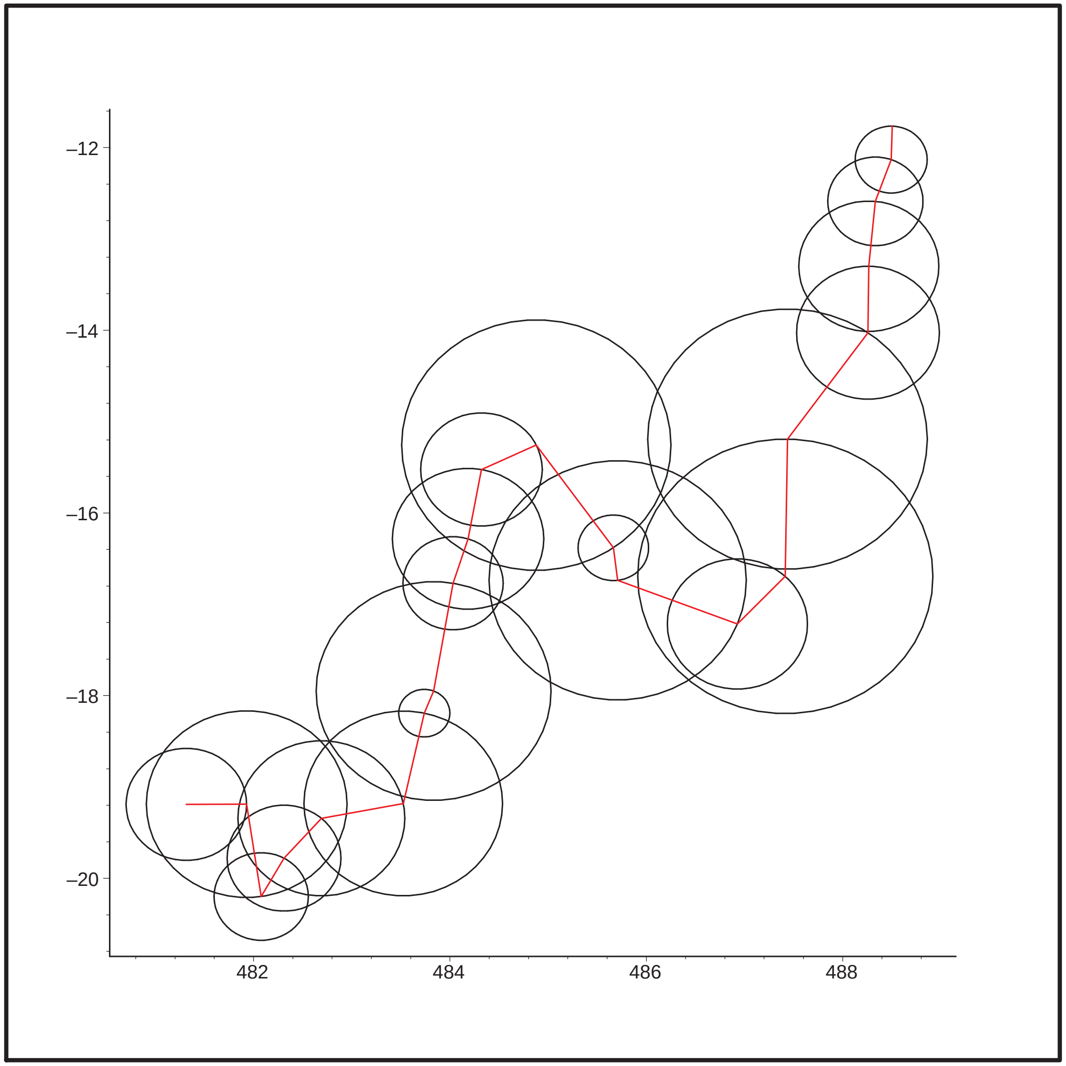
The edge process
$(U_n = T_n - T_{n+1})$
is not a Markov Chain.

There exists a sequence of
***finite* stopping times** (τ_k) **s.t.**

$$\Phi_k = (U_{\tau_k}, ..., U_{\tau_{k+1}-1})$$

is a Ψ-irreducible, aperiodic Markov Chain which admits a small set and is geometrically ergodic.





−12
−14
−16
−18
−20
482
484
486
488

Conclusions

- **Transformations associated with point shifts lead to a variety of topological and metrical dynamical systems on stationary point processes**
- **The main new object of this presentation, point map probabilities, generalize strictly Palm probabilities**

Future Research

- **Ergodicity and mixing properties of the measure preserving transformations: partial results available on Poisson point processes.**
- **Better understanding of the space of solutions of Mecke's invariant measure equation**
- **Random dynamical systems on point processes: partial results on random walks on this type of random medium**

Reference

F.B. and M.O. Mirsadeghi
Compactification of the Action of a Point-Map on the Palm Probability of a Point Process
Arxiv, 2014