Free Probability and Random Matrices: from isomorphisms to universality

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Joint works with F. Bekerman, Y. Dabrowski, A.Figalli, E. Maurel-Segala, J. Novak, D. Shlyakhtenko.

Free Probability $\quad \leftrightarrows \quad$ Classical Probability

Operator algebra $\quad\quad$ Commutative algebra

$\searrow\!\!\nwarrow \qquad\qquad \nearrow\!\!\swarrow$

Large Random Matrices

Free probability $\leftrightarrows$ Classical Probability

Isomorphisms between C^* and W^* algebras

Transport maps
Optimal transport

$\searrow\nwarrow$ $\nearrow\swarrow$

Random Matrices

Universality

Outline

Free probability

Random matrices

Transport maps
- The isomorphism problem
- Proofs : Monge-Ampère equation

Approximate transport and universality

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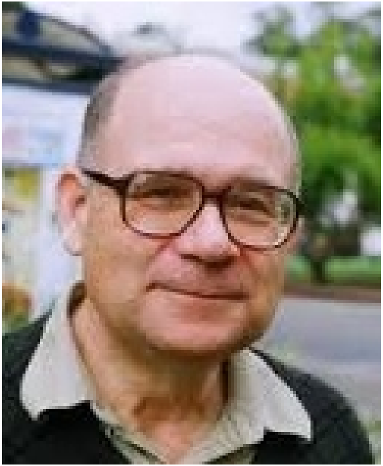
Approximate transport and universality

Free probability theory = Non-Commutative probability theory +Notion of Freeness



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It should satisfy

- Positivity : $\tau(PP^*) \geq 0$ for all P, $(zX_{i_1} \cdots X_{i_k})^* = \bar{z} X_{i_k} \cdots X_{i_1}$,
- Mass : $\tau(1) = 1$,
- Traciality : $\tau(PQ) = \tau(QP)$ for all P, Q.

Free probability : non-commutative law+freeness

$\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$ are free under τ iff for all polynomials $P_1 \ldots P_\ell$ and $Q_1, \cdots Q_\ell$ so that $\tau(P_i(\mathbf{X})) = 0$ and $\tau(Q_i(\mathbf{Y})) = 0$

$$\tau\left(P_1(\mathbf{X})Q_1(\mathbf{Y}) \cdots P_\ell(\mathbf{X})Q_\ell(\mathbf{Y})\right) = 0.$$

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- τ is uniquely determined by $\tau(P(\mathbf{X}))$ and $\tau(Q(\mathbf{Y}))$, Q, P polynomials.
- Let G be a group with free generators $g_1, \cdots, g_m$, neutral e

$$\tau(g) = 1_{g=e}, \quad \text{for } g \in G$$

is the law of m free variables.

Laws and representations as bounded linear operators

Let τ be a non-commutative law, that is a linear form on $\mathbb{C}\langle X_1, \dots, X_d\rangle$ so that

$$\tau(PP^*) \geq 0, \quad \tau(1) = 1, \quad \tau(PQ) = \tau(QP),$$

which is bounded, i.e. and for all $i_k \in \{1, \dots, d\}$, all $\ell \in \mathbb{N}$,

$$|\tau(X_{i_1} \cdots X_{i_\ell})| \leq R^\ell .$$

By the Gelfand-Naimark-Segal construction, we can associate to τ a Hilbert space H, $\Omega \in H$, and $a_1, \dots, a_d$ bounded linear operators on H so that for all P

$$\tau(P(X_1, \dots, X_d)) = \langle \Omega, P(a_1, \dots, a_d)\Omega\rangle_H .$$

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Examples of non-commutative laws :

τ linear, $\tau(PP^*) \geq 0$, $\tau(1) = 1$, $\tau(PQ) = \tau(QP)$

- Let $(X_1^N, \cdots, X_d^N)$ be d $N \times N$ Hermitian random matrices,

$$\tau_{X^N}(P) := \mathbb{E}[\frac{1}{N}\mathrm{Tr}\left(P(X_1^N, \cdots, X_d^N)\right)]$$

Here $\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii}$.

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- Let $(X_1^N, \cdots, X_d^N)$ be d $N \times N$ Hermitian random matrices for $N \geq 0$ so that

$$\tau(P) := \lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}\left(P(X_1^N, \cdots, X_d^N)\right)]$$

exists for all polynomial P.

The Gaussian Unitary Ensemble

X^N follows the GUE iff it is a $N \times N$ matrix so that

- $(X^N)^* = X^N$,
- $(X^N_{k\ell})_{k\leq\ell}$ are independent,
- With $g_{k\ell}, \tilde{g}_{k\ell}$ iid centered Gaussian variables with variance one

$$X^N_{k\ell} = \frac{1}{\sqrt{2N}}(g_{k\ell} + i\tilde{g}_{k\ell}),\ k < \ell, \quad X^N_{kk} = \frac{1}{\sqrt{N}} g_{kk}$$

In other words, the law of the GUE is given by

$$d\mathbb{P}(X^N) \quad = \quad \frac{1}{Z^N} \exp\{-\frac{N}{2}\mathrm{Tr}((X^N)^2)\} dX^N$$

with $dX^N = \prod_{k\leq\ell} d\Re X^N_{k\ell} \prod_{k<\ell} d\Im X^N_{k\ell}$.

The GUE and the semicircle law

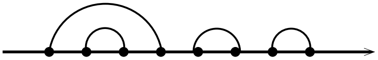
Let X^N be a matrix following the Gaussian Unitary Ensemble, that is a

$$d\mathbb{P}(X^N) \quad = \quad \frac{1}{Z^N}\exp\{-\frac{N}{2}\mathrm{Tr}((X^N)^2)\}dX^N$$

Theorem (Wigner 58') *With σ the semicircle distribution,*

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] \quad = \quad \int x^p \sigma(dx) \qquad \forall p \in \mathbb{N}$$

$\int x^p d\sigma(x)$ is the number of non-crossing pair partitions of p points.



The law of free semicircle variables

Let $X_1^N, \cdots, X_d^N$ be independent GUE matrices, that is

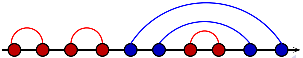
$$\mathbb{P}\left(dX_1^N, \cdots, dX_d^N\right) = \frac{1}{(Z^N)^d} \exp\{-\frac{N}{2}\mathrm{Tr}(\sum_{i=1}^{d}(X_i^N)^2)\} \prod dX_i^N .$$

Theorem (Voiculescu 91')

For any polynomial $P \in \mathbb{C}\langle X_1, \cdots, X_d\rangle$

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P(X_1^N, \cdots, X_d^N))] = \sigma(P)$$

σ is the law of d free semicircle variables. If $P = X_{i_1}X_{i_2}\cdots X_{i_k}$, $\sigma(P)$ is the number of non-crossing color wise pair partitions build over points of color $i_1, i_2 \dots$



More general laws

Let $V \in \mathbb{C}\langle X_1, \ldots, X_d\rangle$ and set

$$\mathbb{P}_V^N(dX_1^N, \ldots, dX_d^N) = \frac{1}{Z_V^N} \exp\{-N\mathrm{Tr}(V(X_1^N, \ldots, X_d^N))\} dX_1^N \cdots dX_d^N$$

Theorem (G–Maurel Segala 06' and G–Shlyakhtenko 09')

Assume that V satisfies a "local convexity property " [e.g $V = 2^{-1}\sum X_i^2 + W$, W small]. Then there exists a non-commutative law τ_V so that for any polynomial P

$$\tau_V(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}_V^N(X_1^N, \ldots, X_d^N)$$

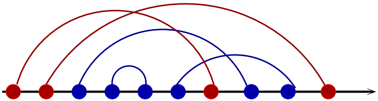
Example ; q-Gaussian variables [Bozejko and Speicher 91']

A d-tuple of q-Gaussian variables is such that

$$\tau_{q,d}(X_{i_1}\cdots X_{i_p}) = \sum_{\pi} q^{i(\pi)} \qquad \forall i_k \in \{1,\cdots,d\}$$

where the sum runs over pair partitions of colored dots whose block contains dots of the same color and $i(\pi)$ is the number of crossings.

$i(\pi) = 4$



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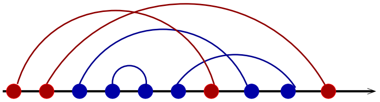

Theorem (Dabrowski 10')

If dq small, there exists $V_{q,d} = 1/2\sum X_i^2 + W_{q,d}$ with $W_{q,d}$ small so that

$$\tau_{q,d}(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N,\ldots,X_d^N))d\mathbb{P}^N_{V_{q,d}}(X_1^N,\ldots,X_d^N)$$

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The isomorphism problem

Let τ, μ be two non-commutative laws of d (resp. m) variables $X = (X_1, \ldots, X_d)$ (resp. $Y = (Y_1, \ldots, Y_m)$).

Can we find "transport maps" $T = (T_1, \ldots, T_m)$ and $T' = (T'_1, \ldots, T'_d)$ of d (resp. m) variables so that for all polynomials P, Q

$$\begin{aligned} \tau(P(X)) &= \mu(P(T_1(Y), \ldots, T_d(Y))) \\ \mu(Q(Y)) &= \tau(Q(T'_1(X), \ldots, T'_m(X))) \end{aligned}$$

We denote $\tau = T\sharp\mu$ and $\mu = T'\sharp\tau$

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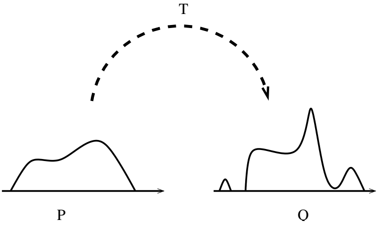
The free group isomorphism problem : Does there exists transport maps from τ to μ, the law of d (resp. m) free variables with $d \neq m$?

Classical transport

Let P, Q be two probability measures on $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. A transport map from P to Q is a measurable function $T : \mathbb{R}^d \to \mathbb{R}^m$ so that for all bounded continuous function f

$$\int f(T(x))dP(x) = \int f(x)dQ(x) .$$



We denote $T\#P = Q$.

Fact (von Neumann [1932]) : If $P, Q \ll dx$, T exists.

Free transport (G. and Shlyakhtenko 12')

Recall that

$$\tau_W(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_d^N))d\mathbb{P}_V^N(X_1^N, \ldots, X_d^N)$$

with

$$V = \frac{1}{2}\sum X_i^2 + W \quad \text{with} \quad W \text{ self-adjoint, small}$$

Theorem

There exists F^W, T^W smooth transport maps between $\tau_W, \sigma = \tau_0$ so that for all polynomial P

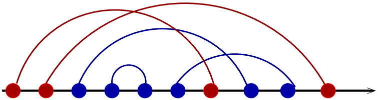
$$\tau_W = T^W \sharp \tau_0 \quad \tau_0 = F^W \sharp \tau_W$$

In particular the related C^* algebras and von Neumann algebras are isomorphic.

Isomorphisms of q-Gaussian algebras

Let $\tau_{q,d}$ be the law of d q Gaussian

$$\tau_{q,d}(X_{i_1}\cdots X_{i_p}) = \sum_{\pi} q^{i(\pi)} \qquad \forall i_k \in \{1,\cdots,d\}$$



Theorem (G–Shlyakhtenko 12')

For qd small enough, there exists smooth transport maps between $\tau_{q,d}$ and $\tau_{q,0} = \sigma$. In particular the $C*$-algebra and von Neumann algebras of q-Gaussian laws, q small, are isomorphic to that of free semicircle law σ.

Idea of the proof : Monge-Ampère equation

Let P, Q be probability measures on $\mathbb{R}^d$ that have smooth densities

$$P(dx) = e^{-V(x)}dx \qquad Q(dx) = e^{-W(x)}dx\,.$$

Then $T\#P = Q$ is equivalent to

$$\begin{aligned}\int f(T(x))e^{-V(x)}dx &= \int f(x)e^{-W(x)}dx \\ &= \int f(T(y))e^{-W(T(y))}JT(y)dy\end{aligned}$$

with JT the Jacobian of T. Hence, it is equivalent to the Monge-Ampère equation

$$V(x) = W(T(x)) - \log JT(x)\,.$$

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There exists a free analogue to Monge-Ampère equation. For $W - V$ small it has a unique solution.

Non-perturbative transport maps

$$\mathbb{P}_V^N(dX_1^N,\ldots,dX_d^N) = \frac{1}{Z_N} e^{-N\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod dX_i^N$$

$$\tau_W(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N,\ldots,X_d^N))d\mathbb{P}^N_{\frac{1}{2}\sum X_i^2+W}(X_1^N,\ldots,X_d^N)$$

Theorem (WIP with Y-Dabrowski and D-Shlyakhtenko 14')

Assume that "$V = \frac{1}{2}\sum X_i^2 + W$ is strictly convex", then there exists $(F_i)_{1\le i\le d} \in (\overline{\mathbb{C}\langle X_1,\ldots,X_d\rangle})^d$ so that

$$\tau_W = F\#\tau_0 .$$

The Poisson approach

Transport $\mu_V(dx) = e^{-V(x)}dx$ to $\mu_W(dx) = e^{-W(x)}dx$ by interpolation. Define a flow $T_{s,t}$ so that $T_{s,t}\#\mu_{V_s} = \mu_{V_t}$, $V_t = (1-t)V + tW$. Let

$$\phi_t = \lim_{s\to t} T_{s,t} = \partial_t T_{0,t} \circ T_{0,t}^{-1} \tag{1}$$

If $\phi_t = \nabla\psi_t$, Monge-Ampère equation becomes

$$L_t\psi_t = W - V \tag{2}$$

with $L_t = \Delta - \nabla V_t.\nabla$ infinitesimal generator.
Program : Solve Poisson equation (2) by

$$\psi_t = -\int_0^\infty e^{sL_t}(W-V)ds$$

and then deduce $T_{0,t}$ solution of the transport equation (1) driven by $\nabla\psi_t$.

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and then deduce $T_{0,t}$ solution of the transport equation (1) driven by $\nabla\psi_t$. Generalizes by using free Stochastic differential equation.

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Local fluctuations in RMT

$$dP^N_{\beta,V}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z^N_{\beta,V}} 1_{\lambda_1<\lambda_2<\cdots<\lambda_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N\beta\sum\lambda_i^2} \prod d\lambda_i$$

- For $\beta = 1, 2, 4$, Tracy and Widom (93)showed that for each $E \in [-2, 2]$, any compactly supported function f

$$\lim_{N\to\infty} \mathbb{E}_{P^N_{\beta,V}}[\sum_{i_1<\cdots<i_k} f(N(\lambda_{i_1} - E), \ldots, N(\lambda_{i_k} - E))] = \rho^k_\beta(f)$$

Tao showed (12) that $N(\lambda_i - \lambda_{i-1})$ converges towards the Gaudin distribution. Moreover, Tracy -Widom (93) proved

$$\lim_{N\to\infty} \mathbb{E}_{P^N_{\beta,V}}[f(N^{2/3}(\lambda_N - 2))] = \mathrm{TW}_\beta(f)\,.$$

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- For $\beta \geq 0$, this was extended by Ramirez,Rider, Virag at the edge (06) and by Valko,Virag (07) at the edge.

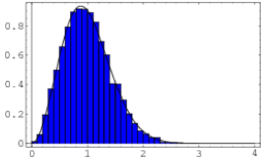
Universality for β-models

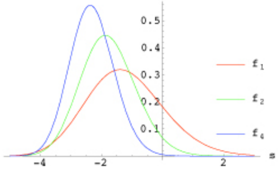
$$dP^N_{\beta,V}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z^N_{\beta,V}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i \,.$$

$$\text{Then } \lim_{N\to\infty} \frac{1}{N} \sum f(\lambda_i) = \int f(x) d\mu_V(x)$$

Theorem (Bourgade, Erdös, Yau (1104.2272, 1306.5728))

Assume $\beta \geq 1$, V $C^4(\mathbb{R})$, μ_V with a connected support vanishing as a square root at the boundary, the local fluctuations of the eigenvalues are as in the case $V = \beta x^2$.





Approximate transport for β-models

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Theorem (Bekerman–Figalli–G 2013)

$\beta \geq 0$. Assume V, W $C^{37}(\mathbb{R})$, with equilibrium measures μ_V, μ_W with connected support. Assume V, W are off-critical. Then, there exists $T_0 : \mathbb{R} \to \mathbb{R}$ C^{19}, $T_1 : \mathbb{R}^N \to \mathbb{R}^N$ C^1 so that

$$\|(T_0^{\otimes N} + \frac{T_1}{N})\# P^N_{\beta,V} - P^N_{\beta,W}\|_{TV} \leq const.\sqrt{\frac{\log N}{N}},$$

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$$\|(T_0^{\otimes N}+\frac{T_1}{N})\#P^N_{\beta,V}-P^N_{\beta,W}\|_{TV}\leq const.\sqrt{\frac{\log N}{N}},$$

$$P^N_{\beta,V}\left(\sup_{1\leq k\leq N}\|T_1^{N,k}\|_{L^1(\mathbb{P}^V_N)}+\sup_{k,k'}\frac{|T_1^{N,k}-T_1^{N,k'}|}{\sqrt{N}|\lambda_k-\lambda_{k'}|}\geq C\log N\right)\leq N^{-c}.$$

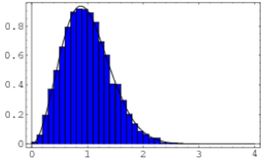
Hence, universality holds.

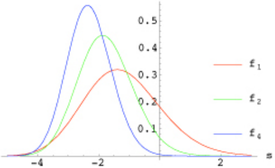
Universality in several matrix models

$$d\mathbb{P}_N^V(X_1^N, \ldots, X_d^N) = \frac{1}{Z_V^N} e^{-\frac{N}{2}\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod_i 1_{\|X_i^N\| \le R} dX_i^N$$

Theorem

Assume $V = V^* = \frac{1}{2}\sum X_i^2 + \sum t_i q_i$, t_i small. The law of the spacing $Nc_j^i(\lambda_j^i - \lambda_{j+1}^i)$ of the eigenvalues of X_i^N converges to the Gaudin distribution, and that of $N^{2/3}c_i(\max_j \lambda_j^i - C_i)$ to the Tracy-Widom law.





Universality for polynomial in several random matrices [Figalli-G 14']

Let P be a self-adjoint polynomial in d indeterminates and let $X_1^N, \ldots, X_d^N$ be d independent GUE or GOE matrices. Haagerup and Thorbjornsen proved that the largest eigenvalue of $P(X_1^N, \ldots, X_d^N)$ converges a.s to its free limit.

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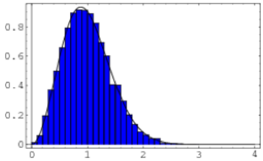
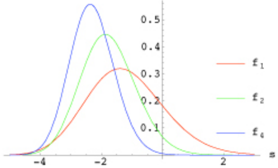
$$Y^N = X_1^N + \epsilon P(X_1^N, \ldots, X_d^N)$$

fluctuates locally as when $\epsilon = 0$, that is the spacings follow Gaudin distribution in the bulk and the Tracy-Widom law at the edge.





Proof : Beta-models and Monge-Ampère

$$dP^N_{\beta,V}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z^N_{\beta,V}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i$$

$T\#P^N_{\beta,V} = P^N_{\beta,W}$ satisfies the Monge-Ampère equation $P^N_{\beta,V}$-a.s. :

$$\beta \sum_{i<j} \log \frac{T_i(\lambda) - T_j(\lambda)}{\lambda_i - \lambda_j} = N\sum(V(T_i(\lambda)) - W(\lambda_i) - \log \partial_{X_i} T_i(\lambda)/N)$$

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If $V - W$ small, it can be solved by implicit function theorem.
Goal : Take $V_t = tV + (1-t)W$. Define $T_{0,t}\#P^N_{\beta,V_0} = P^N_{\beta,V_t}$.
Then, $\phi_t = \partial_t T_{0,t} \circ T^{-1}_{0,t}$ satisfies

$$M_t \phi_t = W - V$$

with

$$M_t f = \beta \sum_{i<j} \frac{f_i(\lambda) - f_j(\lambda)}{\lambda_i - \lambda_j} + \sum \partial_{X_i} f_i - N \sum V'(\lambda_i) f_i$$

Ansatz and approximate transport

Take $V_t = tV + (1-t)W$. Aim : Build ϕ_t, $\partial_t T_{0,t} \circ (T_{0,t})^{-1} = \phi_t$ so that

$$R_t^N = M_t \phi_t - (V - W)$$

goes to zero in $L^1(\mathbb{P}_N^V)$.Then $T_{0,t}$ solution of $\partial_t T_{0,t} = \phi_t(T_{0,t})$ is an approximate transport. Ansatz :

$$\phi_t^i(\lambda) = \phi_{0,t}(\lambda_i) + \frac{1}{N}\phi_{1,t}(\lambda_i) + \frac{1}{N}\sum_j \phi_{2,t}(\lambda_i, \lambda_j).$$

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We find with $M_N = \sum(\delta_{\lambda_i} - \mu_{V_t})$

$$R_t^N = N \int [\Xi\phi_{0,t} + W - V](x) dM_N(x) + \cdots$$

$$\text{with} \quad \Xi f(x) = V_t'(x) f(x) - \beta \int \frac{f(x) - f(y)}{x - y} d\mu_{V_t}(y),$$

Ξ is invertible. Choose $\phi_{0,t}, \phi_{1,t}, \phi_{2,t}$ so that R_t^N small.

Approximate transport maps for several matrix-models

$$\mathbb{P}^N_{V_a}(dX^N_1,\ldots,dX^N_d)=\frac{1}{Z^N_V}e^{-aN\mathrm{Tr}(V(X^N_1,\ldots,X^N_d))}\prod_i e^{-N\mathrm{tr}W_i(X_i)}dX^N_i\,.$$

Then,the law of the eigenvalues $P^N_{V_a}$ of $X^1_N,\ldots,X^d_N$ under $\mathbb{P}^N_{V_a}$ is

$$P^N_{V_a}(d\lambda^i_j)=\frac{1}{\tilde{Z}^N_V}I^{aV}_N(\lambda^i_j)\prod_{i=1}^d\prod_{j<k}|\lambda^i_j-\lambda^i_k|^\beta e^{-N\sum W_i(\lambda^i_j)}d\lambda^i_j$$

$$I^{aV}_N(\lambda^i_j)=\int e^{-aN\mathrm{Tr}(V(U^N_1D(\lambda^1)(U^N_1)^*,\ldots,U^N_dD(\lambda^1)(U^N_d)^*)}dU^N_1\cdots dU^N_d$$

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$$= (1 + O(\frac{1}{N})) \exp\{(N^2 F^a_2 + NF^a_1 + F^a_0)(\frac{1}{N}\sum \delta_{\lambda^i_j}, 1 \le i \le d)\}$$

by G-Novak 13'.

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- These ideas are robust : they generalize to type III factors, planar algebras etc[B. Nelson] The main questions are now around smoothness of the transport maps and topology
- The main issue in several matrix models lies in the topological expansion : being able to carry it out in non-pertubative regimes would solve important questions in free probability (convergence of entropy etc)