Norm inflation for incompressible Euler

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Nov 21–23, 2014,
TexAMP Symposium

Outline

Intro

Folklore

Evidence

Results

Proof

Incompressible Euler

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \nabla \cdot u = 0, \\ u\big|_{t=0} = u_0, \end{cases}$$

where dimension $d \geq 2$,

- velocity: $u(t,x) = (u_1(t,x), \cdots, u_d(t,x))$;
- pressure: $p(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$

- Well-known: LWP in $H^s(\mathbb{R}^d)$, $s > s_c := 1 + d/2$
- (Old) Folklore problem: $s = s_c$?

The appearance of critical index $s_c = 1 + d/2$

Typical energy estimate (+usual regularization/mollification arguments):

$$\frac{d}{dt}\left(\|u(t,\cdot)\|^2_{H^s_x(\mathbb{R}^d)}\right) \leq C_{s,d}\|Du(t,\cdot)\|_{L^\infty_x(\mathbb{R}^d)} \cdot \|u(t,\cdot)\|^2_{H^s_x(\mathbb{R}^d)}$$

To close the H^s estimate, need

$$\|Du\|_{L^\infty_x} \leq const \cdot \|u\|_{H^s_x}.$$

Thus

$$s > 1 + d/2 =: s_c$$

A doomed attempt

What about closing estimates in $\|u\|_X = \|u\|_{H^s} + \|Du\|_\infty$ and still hope $s \leq s_c$?
Equation for Du (after eliminating pressure), roughly

$$\partial_t(Du) + \underbrace{(u \cdot \nabla)(Du)}_{\text{OK}} + \underbrace{(Du \cdot \nabla)u}_{\text{OK}} + \underbrace{R_{ij}(Du \otimes Du)}_{\text{came from pressure}} = 0.$$

Due to Riesz transform R_{ij},

$$\begin{aligned} \|R_{ij}(Du \otimes Du)\|_\infty &\lesssim \|(Du)(Du)\|_{H^{\frac{d}{2}+\epsilon}} \\ &\lesssim \|u\|_{H^{\frac{d}{2}+1+\epsilon}}. \end{aligned}$$

Again need

$$s > s_c (= d/2 + 1)$$

Issues with criticality

- $s > s_c$ can also be seen through vorticity formulation
- Similar questions arise in other function spaces
- An extensive literature on wellposedness results in "non-critical" spaces

Outline

Classical results: partial list

- Lichtenstein, Gunther (LWP in $C^{k,\alpha}$)
- Wolibner (GWP of 2D Euler in Hölder), Chemin
- Ebin-Marsden (LWP of Euler in $H^{d/2+1+\epsilon}$ on general compact manifolds, C^{∞}-boundary allowed)
- Bouruignon-Brezis $W^{s,p}$ ($s > d/p + 1$).

Wellposedness results: Sobolev

- Kato 75': LWP in $C_t^0 H_x^m(\mathbb{R}^d)$, integer $m > d/2 + 1$.
- Kato-Ponce 88': LWP in $W^{s,p}(\mathbb{R}^d)$, real $s > d/p + 1$, $1 < p < \infty$
- Kato-Ponce commutator estimate: $J^s = (1 - \Delta)^{s/2}$, $s \geq 0$, $1 < p < \infty$:

$$\|J^s(fg) - fJ^s g\|_p \lesssim_{d,s,p} \|Df\|_\infty \|J^{s-1} g\|_p + \|J^s f\|_p \|g\|_\infty,$$

Wellposedness results: Sobolev

In Sobolev spaces $W^{s,p}(\mathbb{R}^d)$, you need

$$s > d/p + 1$$

Not surprisingly, it came from

$$\|Du\|_\infty \leq const\|u\|_{W^{d/p+1+\epsilon,p}}$$

Wellposedness results: Besov

- Vishik '98: GWP of 2D Euler in $B^{2/p+1}_{p,1}(\mathbb{R}^2)$, $1 < p < \infty$.
- Chae '04: LWP in $B^{d/p+1}_{p,1}(\mathbb{R}^d)$, $1 < p < \infty$.
- Pak-Park '04: LWP in $B^1_{\infty,1}(\mathbb{R}^d)$.

The key idea of Besov refinements:
you can push regularity down to critical $s = d/p + 1$, but you pay summability! Example:

$$H^1(\mathbb{R}^2) = B^1_{2,2}(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$$

But

$$B^1_{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$$

The Besov l^1-cheat

- If you insist on having critical regularity $s_c = d/p + 1$, then you need

$$B^{s_c}_{p,q}$$

  with $q = \mathbf{1}$!(to accommodate L^∞ embedding)
- NO wellposedness results were known for $1 < q \leq \infty$.

A common theme

Find a Banach space X such that (e.g. $f = \nabla \times u$, $X = B^{d/p}_{p,1}$)

- $\|f\|_\infty + \|R_{ij} f\|_\infty \lesssim \|f\|_X$
- some version of Kato-Ponce commutator estimate holds in X.

The inevitable

- Completely breaks down for critical (say) $H^{\frac{d}{2}+1}(\mathbb{R}^d)$ spaces.
- Two fantasies:
  - super-good commutator estimates?
  - divergence-free may save the day?
- **NO**!
- Takada '10: **divergence-free** counterexamples of Kato-Ponce-type commutator estimates in critical $B^{d/p+1}_{p,q}(\mathbb{R}^d)$ $(1 \leq p \leq \infty, 1 < q \leq \infty)$ and $F^{d/p+1}_{p,q}(\mathbb{R}^d)$ $(1 < p < \infty, \leq q \leq \infty$ or $p = q = \infty)$ space.

The slightest clue

Consider 2D Euler in vorticity form: $\omega = -\partial_{x_2} u_1 + \partial_{x_1} u_2$,

$$\partial_t \omega + \nabla^\perp \Delta^{-1} \omega \cdot \nabla \omega = 0.$$

Critical space: $H^2(\mathbb{R}^2)$ for u.

$$\frac{1}{2}\frac{d}{dt}\|\partial_{x_1}\omega\|_2^2 = -\underbrace{\int_{\mathbb{R}^2} (\partial_1 \nabla^\perp \Delta^{-1}\omega \cdot \nabla\omega)\partial_1 \omega dx}_{\text{Can be made very large when } \omega \in H^1 \text{ only}}$$

Not difficult to show: no $C_t^1 H^2$ wellposedness (for velocity u).
But this does not rule out $C_t^0 H^2$, $L_t^\infty H^2$, and so on!

Folklore problem

Conjecture: The Euler equation is "illposed" for a class of initial data in $H^{d/2+1}(\mathbb{R}^d)$

- Rem: analogous versions in $W^{d/p+1,p}$, Besov, Triebel-Lizorkin...
- Part of the difficulty: How even to formulate it?
- Infinitely worse: How to approach it?
- Need a deep understanding of how critical space typology changes under the Euler dynamics

Outline

Explicit solutions

Two and a half dimensional shear flow (DiPerna-Majda '87):

$$u(t,x) = (f(x_2), 0, g(x_1 - tf(x_2))), \quad x = (x_1, x_2, x_3),$$

f and g are given $1D$ functions.

- Solves 3D Euler with pressure $p = 0$
- DiPerna-Lions: $\forall\, 1 \leq p < \infty$, $T > 0$, $M > 0$, exist

$$\|u(0)\|_{W^{1,p}(\mathbb{T}^3)} = 1, \ \|u(T)\|_{W^{1,p}(\mathbb{T}^3)} > M$$

- Bardos-Titi '10: $u(0) \in C^\alpha$, but $u(t) \notin C^\beta$ for any $t > 0$, $1 > \beta > \alpha^2$ (illposedness in $F^1_{\infty,2}$ and $B^1_{\infty,\infty}$).
- Misiołek and Yoneda: illposedness in LL_α, $0 < \alpha \leq 1$:

$$\|f\|_{\mathrm{LL}_\alpha} = \|f\|_\infty + \sup_{0<|x-y|<\frac{1}{2}} \frac{|f(x) - f(y)|}{|x-y||\log|x-y||^\alpha} < \infty.$$

Understanding the solution operator

- Kato '75: the solution operator for the Burgers equation is not Hölder continuous in $H^s(\mathbb{R})$, $s \geq 2$ norm for any prescribed Hölder exponent.
- Himonas and Misiołek '10: the data-to-solution map of Euler is not uniformly continuous in H^s topology
- Inci '13: nowhere locally uniformly continuous in $H^s(\mathbb{R}^d)$, $s > d/2 + 1$.
- Cheskidov-Shvydkoy '10: illposedness of Euler in $B^s_{r,\infty}(\mathbb{T}^d)$, $s > 0$ if $r > 2$; $s > d(2/r - 1)$ if $1 \leq r \leq 2$.
- Yudovich (Hölder $\leq e^{-Ct}$), Bahouri-Chemin '94 (not Hölder $\leq e^{-t}$), Kelliher '10 (no Hölder)

No bearing on the critical case!

Outline

Strong illposedness of Euler

Roughly speaking, the results are as follows:

Theorem[Bourgain-L '13]. Let the dimension $d = 2, 3$. The Euler equation is **indeed illposed** in the Sobolev space $W^{d/p+1,p}$ for any $1 < p < \infty$ or the Besov space $B^{d/p+1}_{p,q}$ for any $1 < p < \infty$, $1 < q \leq \infty$.

Situation worse/better than we thought

- Usual scenario (you would think):

  Initially $\|u(0)\|_X \ll 1$
  Later $\|u(T)\|_X \gg 1$

- **Here**: $\|u(0)\|_X \ll 1$, but

$$\text{ess-sup}_{0<t<T} \|u(t)\|_X = +\infty$$

- Rem: kills even $L^\infty_t X$!

"Generic" Illposedness

- "Strongly illposed": any smooth u_0, once can find a nearby v_0, s.t.

$$\|v_0 - u_0\|_X < \epsilon,$$

but

$$\text{ess-sup}_{0<t<t_0} \|v(t)\|_X = +\infty, \quad \forall t_0 > 0.$$

- Illposedness (Norm inflation) is **dense** in critical X-topology!

Vorticity formulation

To state more precisely the main results, recall:
2D Euler: $\omega = \nabla^{\perp} \cdot u$:

$$\begin{cases} \partial_t \omega + (\Delta^{-1} \nabla^{\perp} \omega \cdot \nabla)\omega = 0, \\ \omega\Big|_{t=0} = \omega_0. \end{cases}$$

3D Euler: $\omega = \nabla \times u$,

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \\ u = -\Delta^{-1} \nabla \times \omega, \\ \omega\Big|_{t=0} = \omega_0. \end{cases}$$

Coming up: 2D Euler

- TWO cases for 2D Euler:
  a Initial vorticity ω is *not* compactly-supported
  b Compactly supported case
- 3D Euler a lot more involved (comments later)

2D Euler non-compact data

Theorem 1: For any given $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ and any $\epsilon > 0$, there exist C^∞ perturbation $\omega_0^{(p)} : \mathbb{R}^2 \to \mathbb{R}$ s.t.

1. $\|\omega_0^{(p)}\|_{\dot{H}^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \epsilon$.
2. Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. The initial velocity $u_0 = \Delta^{-1}\nabla^\perp \omega_0$ has regularity $u_0 \in H^2(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.
3. There exists a unique classical solution $\omega = \omega(t)$ to the 2D Euler equation (in vorticity form) satisfying

$$\max_{0 \le t \le 1} \left(\|\omega(t,\cdot)\|_{L^1} + \|\omega(t,\cdot)\|_{L^\infty} + \|\omega(t,\cdot)\|_{\dot{H}^{-1}} \right) < \infty.$$

   Here $\omega(t) \in C^\infty$, $u(t) = \Delta^{-1}\nabla^\perp \omega(t) \in C^\infty \cap L^2 \cap L^\infty$ for each $0 \le t \le 1$.
4. For any $0 < t_0 \le 1$, we have

$$\text{ess-sup}_{0 < t \le t_0} \|\nabla \omega(t,\cdot)\|_{L^2(\mathbb{R}^2)} = +\infty.$$

Comments

- The $\dot{H}^{-1}$ assumption on the vorticity data $\omega_0^{(g)}$ is actually not needed
- In our construction, although the initial velocity u_0 is $C^\infty \cap L^\infty$, $\|\nabla u_0\|_{L^\infty(\mathbb{R}^2)} = +\infty$.
- Classical C^∞-solutions! (No need to appeal to Yudovich theory)
- Kato '75 introduced the uniformly local Sobolev spaces $L^p_{ul}(\mathbb{R}^d)$, $H^s_{ul}(\mathbb{R}^d)$ which contain $H^s(\mathbb{R}^d)$ and the periodic space $H^s(\mathbb{T}^d)$. We can refine the result to

$$\text{ess-sup}_{0<t\leq t_0} \|\nabla\omega(t,\cdot)\|_{L^2_{ul}(\mathbb{R}^2)} = +\infty.$$

2D compactly supported case

- Next result: the compactly supported data for the 2D Euler equation.
- Carries over (with simple changes) to the periodic case as well
- For simplicity consider vorticity functions having one-fold symmetry: g is odd in x_1

$$g(-x_1, x_2) = -g(x_1, x_2), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

- Preserved by the Euler flow

2D Euler compact data

Theorem 2: Let $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ be any given vorticity function which is odd in x_2. For any any $\epsilon > 0$, we can find a perturbation $\omega_0^{(p)} : \mathbb{R}^2 \to \mathbb{R}$ s.t.

1. $\omega_0^{(p)}$ is compactly supported (in a ball of radius ≤ 1), continuous and

$$\|\omega_0^{(p)}\|_{\dot{H}^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \epsilon.$$

2. Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. Corresponding to ω_0 there exists a unique time-global solution $\omega = \omega(t)$ to the Euler equation satisfying $\omega(t) \in L^\infty \cap \dot{H}^{-1}$. Furthermore $\omega \in C_t^0 C_x^0$ and $u = \Delta^{-1}\nabla^\perp \omega \in C_t^0 L_x^2 \cap C_t^0 C_x^\alpha$ for any $0 < \alpha < 1$.
3. $\omega(t)$ has additional local regularity in the following sense: there exists $x_* \in \mathbb{R}^2$ such that for any $x \neq x_*$, there exists a neighborhood $N_x \ni x$, $t_x > 0$ such that $\omega(t, \cdot) \in C^\infty(N_x)$ for any $0 \leq t \leq t_x$.

Conti.

- For any $0 < t_0 \leq 1$, we have

$$\text{ess-sup}_{0<t\leq t_0} \|\omega(t,\cdot)\|_{\dot{H}^1} = +\infty.$$

More precisely, there exist $0 < t_n^1 < t_n^2 < \frac{1}{n}$, open precompact sets Ω_n, $n = 1, 2, 3, \cdots$ such that $\omega(t) \in C^\infty(\Omega_n)$ for all $0 \leq t \leq t_n^2$, and

$$\|\nabla\omega(t,\cdot)\|_{L^2(\Omega_n)} > n, \quad \forall\, t \in [t_n^1, t_n^2].$$

Comments: uniqueness

- Yudovich '63: existence and uniqueness of weak solutions to 2D Euler in bounded domains for L^∞ vorticity data.
- Yudovich '95: improved uniqueness result (for bounded domain in general dimensions $d \geq 2$) allowing vorticty $\omega \in \cap_{p_0 \leq p < \infty} L^p$ and $\|\omega\|_p \leq C\theta(p)$ with $\theta(p)$ growing relatively slowly in p (such as $\theta(p) = \log p$).
- Vishik '99 uniqueness of weak solutions to Euler in $\mathbb{R}^d$, $d \geq 2$, under the following assumptions:
  - $\omega \in L^{p_0}$, $1 < p_0 < d$,
  - For some $a(k) > 0$ with the property

    $$\int_1^\infty \frac{1}{a(k)} dk = +\infty,$$

    it holds that

    $$\left| \sum_{j=2}^{k} \|P_{2^j}\omega\|_\infty \right| \leq \text{const} \cdot a(k), \qquad \forall\, k \geq 4.$$

- Uniqueness OK in our construction: uniform in time L^∞ control of the vorticity ω

3D case

Without going into any computation, you realize:

- One of difficulty in 3D: lifespan of smooth initial data
- Technical issues: make judicious perturbation in $H^{5/2}$ while controlling the lifespan!
- Vorticity stretching

Comments (contin...)

- Critical norm $\dot{H}^{\frac{3}{2}}$ for vorticity ($H^{\frac{5}{2}}$ for velocity).
- A technical nuissance: nonlocal fractional differentiation operator $|\nabla|^{\frac{3}{2}}$
- Theorem 1-Theorem 4 can be sharpened significantly: e.g. $\|u_0 - u_0^{(g)}\|_{B^{d/p+1}_{p,q}} < \epsilon$,

$$\text{ess-sup}_{0<t<t_0} \|u(t,\cdot)\|_{\dot{B}^{d/p+1}_{p,\infty}} = +\infty$$

  for any $t_0 > 0$.
- Similarly: Sobolev $W^{d/p+1,p}$, Triebel-Lizorkin etc.

Outline

Consider 2D Euler in vorticity formulation:

$$\partial_t \omega + \Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = 0.$$

Critical space: $\dot{H}^1(\mathbb{R}^2)$ for ω (H^2 for velocity u)

Step 1: Creation of large Lagrangian deformation

- Flow map $\phi = \phi(t, x)$

$$\begin{cases} \partial_t \phi(t,x) = u(t, \phi(t,x)), \\ \phi(0,x) = x. \end{cases}$$

- For any $0 < T \ll 1$, $B(x_0, \delta) \subset \mathbb{R}^2$ and $\delta \ll 1$, choose initial (vorticity) data $\omega_a^{(0)}$ such that

$$\|\omega_a^{(0)}\|_{L^1} + \|\omega_a^{(0)}\|_{L^\infty} + \|\omega_a^{(0)}\|_{H^1} \ll 1,$$

  and

$$\sup_{0<t\leq T} \|D\phi_a(t,\cdot)\|_\infty \gg 1.$$

  ϕ_a is the flow map associated with the velocity $u = u_a$.
- Judiciously choose $\omega_a^{(0)}$ as a chain of bubbles concentrated near origin (respect H^1 assumption!)
- Deformation matrix Du remains essentially hyperbolic

Step 2: Local inflation of critical norm.

- The solution constructed in Step 1 does not necessarily obey $\sup_{0<t\leq T} \|\nabla\omega_a(t)\|_2 \gg 1$.
- Perturb the initial data $\omega_a^{(0)}$ and take

$$\omega_b^{(0)} = \omega_a^{(0)} + \frac{1}{k}\sin(kf(x))g(x),$$

  where k is a very large parameter.
- $\|g\|_2 \sim o(1)$, f captures $\|D\phi_a(t,\cdot)\|_\infty$.
- A perturbation argument in $W^{1,4}$ to fix the change in flow map
- As a result, in the main order the H^1 norm of the solution corresponding to $\omega_b^{(0)}$ is inflated through the Lagrangian deformation matrix $D\phi_a$.

- Rem: Nash twist, Onsager (De Lellis-Szekelyhidi)

Step 3: Gluing of patch solutions

- Repeat the local construction in infinitely many small patches which stay away from each other initially.
- To glue these solutions: consider two cases.

Noncompact case...

- Case 3a: noncompact data
- Add each patches sequentially and choose their mutual distance ever larger!

  REM: this is the analogue of weakly interacting particles in Stat Mech.
- Key properties exploited here:
  - finite transport speed of the Euler flow;
  - spatial decay of the Riesz kernel.

Living on different scales...

- Case 3b: compact data
- Patches inevitably get close to each other!
- Need to take care of fine interactions between patches (Strongly interacting case!)

3b (contin...): a very involved analysis

- For each $n \geq 2$, define $\omega_{\leq n-1}$ the existing patch and ω_n the current (to be added) patch
- There exists a patch time T_n s.t. for $0 \leq t \leq T_n$, the patch ω_n has disjoint support from $\omega_{\leq n-1}$, and obeys the dynamics

$$\partial_t \omega_n + \Delta^{-1}\nabla^{\perp}\omega_{\leq n-1} \cdot \nabla \omega_n + \Delta^{-1}\nabla^{\perp}\omega_n \cdot \nabla \omega_n = 0.$$

- By a re-definition of the patch center and change of variable, $\tilde{\omega}_n$ satisfies

$$\partial_t \tilde{\omega}_n + \Delta^{-1}\nabla^{\perp}\tilde{\omega}_n \cdot \nabla \tilde{\omega}_n + b(t)\begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} \cdot \nabla \tilde{\omega}_n + r(t,y) \cdot \nabla \tilde{\omega}_n = 0,$$

  where $b(t) = O(1)$ and $|r(t,y)| \lesssim |y|^2$.
- Re-do a new inflation argument!

3b (conti...)

- Choose initial data for ω_n such that within patch time $0 < t \leq T_n$ the critical norm of ω_n inflates rapidly.
- As we take $n \to \infty$, the patch time $T_n \to 0$ and ω_n becomes more and more localized
- the whole solution is actually time-global.
- During interaction time T_n (Smoothness in limited patch time!) the patch ω_n produces the desired norm inflation since it stays well disjoint from all the other patches.

Difficulties in 3D: a snapshot

- First difficulty in 3D: lack of L^p conservation of the vorticity.
- Deeply connected with the vorticity stretching term $(\omega \cdot \nabla)u$
- To simplify the analysis, consider the axisymmetric flow without swirl

$$\partial_t \left(\frac{\omega}{r}\right) + (u \cdot \nabla)\left(\frac{\omega}{r}\right) = 0, \quad r = \sqrt{x_1^2 + x_2^2},\, x = (x_1, x_2, z).$$

- Owing to the denominator r, the solution formula for ω then acquires an additional metric factor (compared with 2D) which represents the vorticity stretching effect in the axisymmetric setting.
- Difficulty: control the metric factor and still produce large Lagrangian deformation.

Difficulties in 3D (conti..)

- The best local theory still requires $\omega/r \in L^{3,1}(\mathbb{R}^3)$
- The dilemma: need infinite $\|\omega/r\|_{L^{3,1}}$ norm to produce inflation.
- A new perturbation argument: add each new patch ω_n with sufficiently small $\|\omega_n\|_\infty$ norm (over the whole lifespan) such that the effect of the large $\|\omega_n/r\|_{L^{3,1}}$ becomes negligible.
- The spin-off: local solution with infinite $\|\omega/r\|_{L^{3,1}}$ norm!
- More technical issues...

In summary

- Our new strategy: *Large Lagrangian deformation induces critical norm inflation*
- Exploited both Lagrangian and Eulerian point of view
- A multi-scale construction!
- In stark contrast: H^1-critical NLS in $\mathbb{R}^3$:

$$i\partial_t u + \Delta u = |u|^4 u$$

  is *wellposed* for $u_0 \in \dot{H}^1(\mathbb{R}^3)$
- Flurry of more recent developments
  - Proof of endpoint Kato-Ponce (conjectured by Grafakos-Maldonado-Naibo)
  - C^m case: anisotropic Lagrangian deformation, flow decoupling Misolek-Yoneda, Masmoudi-Elgindi,$\cdots$ ...