Randomization and existence of large data solutions at critical and supercritical regimes

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Introduction

- This talk is mostly based on joint work with A. Nahmod (UMass). This work is concerned with long time existence of solutions to certain evolution equations obtained by randomizing the initial data.
  At the end of the talk I will also present a new result by D. Mendelson (MIT) who instead considered the question of *non-squeezing* for the flow of a certain NLKG equation which is only defined almost surely.
- The main theme of the talk is the following: Proving long time existence and stability for large data for evolution equation that are critical or supercritical is very hard. I will explain how introducing techniques that exploit randomness can help in this context.
- Note: The words **critical, supercritical, randomness** and **non-squeezing** will be introduced below in details.

Some Backgroud

Consider the Cauchy IVP for the p-NLS equation:

$$\begin{cases} iu_t + \Delta u = \pm |u|^{p-1} u, \\ u(x,0) = u_0(x) \in H^s \qquad x \in \mathbb{R}^n \text{ or } \mathbb{T}^n \end{cases}$$

Scaling: The scale invariant norm is $s_c := \frac{n}{2} - \frac{2}{(p-1)}$.

H^s data with $s > s_c$ is subcritical; $s = s_c$ is critical; $s < s_c$ is supercritical.

- Lots of progress in the last 20 years in the study of nonlinear dispersive and wave equations.
- The thrust of this body of work has focused on deterministic aspects of wave phenomena.
- Yet there remain some important open questions especially in the supercritical case.
  - Defocusing case: does blow up occur? (unknown despite strong ill-posedness results (*"norm explosion"*) by Christ, Colliander and Tao.)

p-NLS: Deterministic GWP Results on $\mathbb{R}^n$

- **Critical Data Results** :
  - Global well-posedness and scattering for energy-critical ($s_c = 1$) NLS in $\mathbb{R}^n$
    - *Defocusing*: Bourgain; Grillakis; Colliander-Keel-Staffilani-Takaoka-Tao; Killip-Visan, X. Zhang, Dodson.
    - *Focusing*: Kenig-Merle (concentrated compactness /rigidity method) and Killip-Visan.
  - Global well-posedness and scattering for mass-critical ($s_c = 0$) NLS in $\mathbb{R}^n$
    - *radial*: Tao, Visan-Killip, X. Zhang.
    - *nonradial*: Dodson
  - Global well-posedness and scattering for other critical regularities s_c *under the assumption of a uniform in time bound on the scale invariant norm*
    - For $s_c > 1$ for defocusing NLW and NLS by Kenig-Merle; Killip-Visan; Bulut.
    - For $0 < s_c < 1$ for defocusing NLS by Kenig-Merle ($s_c = \frac{1}{2}$, 3D); J. Murphy (14').
    - Assumption is in spirit of Escauriaza, Seregin and Sverak work on the Navier-Stokes equation.
- **Supercritical Data Results:** (?)

p-NLS: Deterministic GWP Results on $\mathbb{T}^n$

- **Critical Data Results**:
  - Global well-posedness for energy-critical NLS
    - *Defocusing and* $n = 3$: Ionescu-Pausader (large data, based on a work by Ionescu-Pausader-Staffilani); and previously Herr-Tzvetkov-Tataru (small data).
  - Global well-posedness for mass-critical NLS
    - (?) *In fact there are no even local results at the L^2 level!*

Deterministic $\rightarrow$ Nondeterministic Approach

Bourgain considered the L^2-critical[1]:

Theorem (Rational Torus; Bourgain(96'))

$$\begin{cases} iu_t + \Delta u = |u|^2 u \\ \end{cases}$$

[1] In 93' Bourgain had proved LWP for $s > 0$ and GWP in $H^1(\mathbb{T}^2)$ for cubic NLS

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is almost sure globally well-posed **below** L^2*; i.e. for* **supercritical data** $u_0 \in H^{-\varepsilon}$.

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Very informal definition of almost sure well-posedness

Given μ a probability measure on the space of initial data X (eg. $X = H^s$)

There exists $Y \subset X$, with $\mu(Y) = 1$ and such that for any $u_0 \in Y$ there exist $T > 0$ and a unique solution u to the IVP in $C([0,T], X)$ that is also stable in the appropriate topology.

[1] In 93' Bourgain had proved LWP for $s > 0$ and GWP in $H^1(\mathbb{T}^2)$ for cubic NLS

Bourgain's interest was to construct an **invariant Gibbs measure** derived from the PDE above viewed as an infinite dimension Hamiltonian system[2]:

1. Established local well posedness for 'typical elements' in the support of the measure; i.e. for random data in $H^{-\varepsilon}(\mathbb{T}^2)$, (an 'almost sure' -in the sense of probability- LWP in $H^{-\varepsilon}(\mathbb{T}^2)$).
2. Constructed an **invariant** Gibbs measure and use it to extend the local result to a global one in the almost sure sense.
   - The **invariance** of the Gibbs measure is used in lieu of conserved quantities- to **control the growth in time** of those solutions in its support.

Furthermore, Bourgain shows that almost surely in ω the nonlinear part

$$w := u - S(t)\phi^\omega$$

is smoother than the linear part.

[2] after Lebowitz, Rose and Speer's and Zhidkov's works.

On Randomized Data

In Bourgain's case, for the cubic NLS on $\mathbb{T}^2$, the typical element in the support of the Gibbs measure consists of **randomized data**:

$$\phi^{\omega}(x) = \sum \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle} \in H^{-\varepsilon}(\mathbb{T}^2),$$

where $\{g_n(\omega)\}_n$ are i.i.d. standard (complex/real) centered (Gaussian) random variables on a probability space $(\Omega, \mathcal{F}, P)$.

Remark

Note that

$$\phi(x) := \sum_{n\in\mathbb{Z}^2} \frac{1}{|n|} e^{i\langle x,n\rangle} \qquad H^{-\epsilon}$$

and that $\phi^{\omega}(x)$ defines almost surely in ω a function in $H^{-\epsilon}$;

but not *in H^s, $s \geq 0$*

Randomization does not improve regularity in terms of derivatives!

Randomization = Better estimates

The improvement is with respect to L^p spaces *almost surely*.

Key Point: Consider the randomized initial data $\phi^\omega(x)$. Although this initial data is in a **rough space** its linear flow $S(t)\phi^\omega(x)$ enjoys *almost surely* improved L^p bounds. These bounds yield improved nonlinear estimates *almost surely* arising in the analysis of

$$w(t,x) = u(t,x) - S(t)\phi^\omega(x),$$

where u is the solution of the equation at hand and as a consequence w **solves a difference equation**.

Why does Randomization Helps?

It is a phenomena akin to Kintchine inequalities used in Littlewood-Paley theory. Classical results of **Rademacher, Kolmogorov**, **Paley** and **Zygmund** show that random series enjoy better L^p bounds than deterministic ones.

Let $\{g_n(\omega)\}$ be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$ and $(c_n) \in \ell^2$.

Define

$$F(\omega) := \sum_n c_n g_n(\omega)$$

Then, there exists $C > 0$ such that for every $q \geq 2$ and every $(c_n)_n \in \ell^2$,

$$\left\| \sum_n c_n g_n(\omega) \right\|_{L^q(\Omega)} \leq C\sqrt{q} \left(\sum_n c_n^2 \right)^{\frac{1}{2}}.$$

More generally one uses the following, where k would represent the number of random terms in the multilinear estimate at hand.

Proposition (Large Deviation-type)

Let $d \geq 1$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Let $\{g_n\}_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex centered L^2 normalized independent Gaussians. For $k \geq 1$ denote by $A(k,d) := \{(n_1, \ldots, n_k) \in \{1, \ldots, d\}^k,\ n_1 \leq \cdots \leq n_k\}$ and

$$F_k(\omega) = \sum_{A(k,d)} c(n_1, \ldots, n_k) g_{n_1}(\omega) \ldots g_{n_k}(\omega).$$

Then for $p \geq 2$

$$\|F_k\|_{L^p(\Omega)} \lesssim \sqrt{k+1}(p-1)^{\frac{k}{2}} \|F_k\|_{L^2(\Omega)}.$$

As a consequence from Chebyshev's inequality for every $\lambda > 0$,

$$\mathbb{P}(\{\omega \,:\, |F_k(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\,\lambda^{\frac{2}{k}}}{\|F(\omega)\|_{L^2(\Omega)}^{\frac{2}{k}}}\right).$$

The large deviation result above with -say -

$$\lambda = \delta^{-\frac{k}{2}}\|F_k(\omega)\|_{L^2(\Omega)}$$

so that in a set Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta}}$ we can replace

$$|F_k(\omega)|^2 = \left|\sum_{A(k,d)} c(n_1,\dots,n_k) g_{n_1}(\omega)\dots g_{n_k}(\omega)\right|^2 \quad \text{by}$$

$$\delta^{-k}\|F_k(\omega)\|_{L^2(\Omega)}^2 = \delta^{-k}\sum_{A(k,d)} c(n_1,\dots,n_k)^2\left(\int_\Omega g_{n_1}(\omega)\dots g_{n_k}(\omega)\,d\omega\right)^2$$

.

Randomization without invariant measure (a.s LWP)

In this vein, consider

$$\text{(IVP)} \qquad \begin{cases} u_t + P(D)u = F(u) & x \in M, \ t > 0 \\ u(x,0) = \phi(x), \end{cases}$$

and assume that $\phi \in X^s$, with s small, and let $\widehat{\phi}(n) = a_n$.

- Randomize ϕ as $\phi^\omega := \sum_{n\in\mathbb{Z}^d} a_n\, g_n(\omega)\, e^{i\langle x,n\rangle}$.
- Assume v^ω is the **linear evolution** with initial datum ϕ^ω.
- Use the fact that v^ω has better L^p estimates than ϕ almost surely to show that $w = u - v^\omega$ solves a **difference equation** that lives in a **smoother** space than X^s. Obtain for w a *deterministic* local well-posedness.

Remark (Important)

The difference equation that w solves is not back to merely being at a 'smoother' level but rather it is a hybrid equation with nonlinearity $=$
$=$ supercritical (but random) $+$ deterministic (smoother).

Probabilistic Local to Global: known mechanisms

- Invariant Gibbs or weighted Wiener measures- when available.

  The use of the invariance of the **measure** has limitations since in **higher dimensions** ($d \geq 3$) can't renormalize canonical construction; also its support (data) would live on extremely **rough spaces** (multilinear analysis so far not possible). **Constructions in higher dimensions (eg. ball) are under radial assumptions**.

- Sometimes may use energy methods (eg. Burq-Tzvetkov and Pocovnicu for NLW; Nahmod-Pavlovic-S. for Navier Stokes)

- Sometimes may use adaptation to this setting of Bourgain's *high-low method* (eg. Colliander-Oh and Poiret-Roberts-Thomann for NLS, Luerhmann-Mendelson and Bulut, NLW...)

  These methods also have limitations!

- Randomization techniques have now been used with or without the help of the invariant measure in several contexts and regimes:

After Bourgain's work in 94-96'; in 07-08 work by Burq-Tzvetkov (NLW, supercritical), T. Oh's (coupled KdV system, subcritical) and Tzvetkov (NLS, subcritical). Lots of work followed:

  - Schrödinger Equations: **Bourgain**, **Tzvetkov**, **Thomann**, **Thomann-Tzvetkov**, **A.N.-Oh-Rey-Bellet-Staffilani**, **Nahmod-Rey-Bellet-Sheffield-S.**, **Burq-Thomann-Tzevtkov**, **Colliander-Oh**, **Y. Deng**, **Burq-Lebeau**, **Bourgain-Bulut**, **Nahmo.- S.**, **Poiret-Robert-Thomann**, **Bényi- Oh- Pocovnicu**, ...
  - KdV Equations: **Bourgain**, **T. Oh** and **Richards**.
  - NLW Equations: **Burq-Tzvetkov**, **de Suzzoni**, **Bourgain-Bulut**, **Luehrmann-Mendelson**, **Pocovnicu**. See also **S. Xu** for the construction of an invariant measure on $\mathbb{R}$.
  - Benjamin-Ono Equations: **Y. Deng**, **Tzvetkov-Visciglia**. and **Y. Deng-Tzvetkov-Visciglia**.
  - Navier-Stokes Equations: **Nahmod-Pavlovic-S.** (infinite 'energy' global (weak) sols in $\mathbb{T}^2, \mathbb{T}^3$, global energy bounds, uniqueness in $\mathbb{T}^2$). Also work by **C.Deng-Cui** and **Zhang-Fang**

To sum up:

- When deterministic statements about existence, uniqueness and stability of solutions to certain evolution equations are **not** feasible/available:

  $\rightarrow$ turn to a more probabilistic point of view

  $\rightarrow$ within reach at this time: investigate these problems from a nondeterministic viewpoint; e.g. for random data.

Situations when such a point of view is desirable include:

- supercritical regime
- when certain type of ill-posedness is present,
- when there still remains a gap between local and global wellposedness (subcritical regime relative to the scaling threshold),

Setting could be $\mathbb{T}^d$, $\mathcal{M}^d$, or $\mathbb{R}^d$. In the latter, there are also probabilistic scattering results.

The Quintic NLS in $\mathbb{T}^3$ (Rational)

We consider the energy-critical quintic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda u|u|^4 & x \in \mathbb{T}^3 \\ u(0,x) = \phi(x) & \in \; H^\gamma(\mathbb{T}^3), \end{cases}$$

below $H^1(\mathbb{T}^3)$ (ie. for some $\gamma < 1$) and where $\lambda = \pm 1$

- Herr, Tzvetkov and Tataru (10') proved small data global well posedness in $H^1(\mathbb{T}^3)$.
- Ionescu and Pausader (12') proved *large data* global well posedness in $H^1(\mathbb{T}^3)$ in the defocusing case
  - Rely on large data GWP in $\mathbb{R}^3$ for the energy-critical quintic NLS (by Colliander-Keel-Staffilani-Takaoka-Tao (03')).

Our interest is first to establish a local almost sure well posedness for random data *below* $H^1(\mathbb{T}^3)$ that is in the **supercritical** regime relative to scaling, and then address g.w.p.

The Initial Data

The problem we are considering here is the analogue of the supercritical well-posedness result proved by **Bourgain** for the periodic mass critical cubic NLS in 2D; that is - a.s for data in $H^{-\epsilon}(\mathbb{T}^2)$, $\epsilon > 0$ mentioned above.

In our problem we consider data $\phi \in H^{1-\alpha-\varepsilon}(\mathbb{T}^3)$ for any $\varepsilon > 0$ of the form

$$\phi(x) = \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{\frac{5}{2}-\alpha}} e^{in\cdot x} \xrightarrow{\text{randomization}} \phi^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{\frac{5}{2}-\alpha}} e^{in\cdot x}$$

where $(g_n(\omega))_{n\in\mathbb{Z}^3}$ is a sequence of complex i.i.d centered Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$.

Heart of the matter

- Assume u solves our IVP, then we define $w := u - S(t)\phi^\omega$, where $S(t)\phi^\omega$ is the linear evolution of the initial profile ϕ^ω.
- We study the IVP for w which solves a difference equation with nonlinearity

$$\tilde{N}(w) := |w + S(t)\phi^\omega|^4 (w + S(t)\phi^\omega).$$

We expect to prove that w belongs to H^s for some $s > 1$.

- The heart of the matter is to prove multilinear estimates for $\tilde{N}(w)$ to then be able to set up a contraction method to obtain well-posedness.
- When the NLS equation is considered, **multilinear estimates for $\tilde{N}(w)$** can be carried out only after having removed certain resonant terms involved in the nonlinear part of the equation.
  - If the nonlinearity is **cubic** a Wick ordering of the Hamiltonian is needed (see **Bourgain (96'), Colliander-Oh (12')**).

Analogies and Difficulties for Local Well-posedness

There are four major complications in the work that we present here compared to the work of Bourgain:

- a quintic nonlinearity increases quite substantially the different cases that needs to be analyzed,
- the counting lemmata in a $3D$ integer lattice are much less favorable than in a $2D$ lattice, (here only **rational torus!**),
- the Wick ordering is not sufficient to remove certain bad *resonant* frequencies.
- We work with the (critical) atomic function spaces X^s as in **Herr-Tataru-Tzvetkov** whose norm is not invariant if one replaces the Fourier transform with its absolute value.

What Needs to be Removed?

To understand the problem, we write the Fourier coefficients of the quintic expression $|u|^4u(x)$ and identify the part that needs to be removed so that a more regular estimate holds after randomization. Once we separate those terms, we are left with nonlinear terms that are manageable.

Here we consider the linear evolution of randomized data that barely misses to be in $H^1(\mathbb{T}^3)$; ie. $\widehat{\phi^\omega}(n) := \frac{g_n(\omega)}{\langle n \rangle^\beta}$ for $\beta < \frac{5}{2}$.

The randomness coming from $(g_n(\omega))$ will allow us to say that in a certain space the nonlinearity increases its regularity so that it can hold a bit more than one derivative.

We realize immediately that terms containing **"too many" pairs of equal frequencies have no chance** to improve their regularity because **they are simply linear** with respect to a_n, as we will see later.

- Formally, in the work of Bourgain and of Colliander-Oh, who consider the cubic NLS, the nonlinear term that they can control is

$$v|v|^2 - v \underbrace{\left(\int_{\mathbb{T}^d} |v|^2 \, dx \right)}_{\text{mass}} \qquad d = 1, 2;$$

and this is achieved by Wick ordering the Hamiltonian. An important ingredient in making this successful is that the mass is independent of time.

- In our case we can estimate

$$v|v|^4 - 3v \left(\int_{\mathbb{T}^3} |v|^4 \, dx \right)$$

but **Wick ordering is not helpful** since it does not remove the problematic term involving $\int_{\mathbb{T}^3} |u|^4(t,x)dx$, which moreover *is not constant in time!*

The Gauged Equation: well-posedness set up

We prove our almost sure local well-posedness result in two steps.

Step 1) We consider the initial value problem:

$$\begin{cases} iv_t + \Delta v = \lambda \left(v|v|^4 - 3v \left(\int_{\mathbb{T}^3} |v|^4 \, dx \right) \right) & x \in \mathbb{T}^3 \\ v(0, x) = \phi^\omega(x), \end{cases}$$

where $\lambda = \pm 1$ and as above

$$\phi^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{\frac{5}{2} - \alpha}} e^{in \cdot x}.$$

and prove almost sure local well-posedness in a certain Banach space $(X, \| \cdot \|)$.

Step 2) We then consider the following **gauged solution**:

$$u(t,x) := e^{i\lambda \int_0^t \beta_v(s)\,ds} v(t,x).$$

where

$$\beta_v(t) = 3 \int_{\mathbb{T}^3} |v(t,x)|^4 \, dx.$$

Observe that u solves the original quintic NLS **with the same data.**

A similar gauge transformation was used by S. (97') in the context of gKdV.

Back from the gauge

Step 3): We transfer the well-posedness result for the gauged initial value problem in the space $(X, \|\cdot\|)$ to a well-posedness result for the original initial value problem by using a metric space given by (X, d) where

$$d(u, v) := \|e^{-i\lambda \int_0^t \beta_u(s)\, ds} u(t, x) - e^{-i\lambda \int_0^t \beta_v(s)\, ds} v(t, x)\|.$$

One can show that this is indeed a metric by using the properties of the norm $\|\cdot\|$ and the fact that if

$$e^{-i\lambda \int_0^t \beta_u(s)\, ds} u(t, x) = e^{-i\lambda \int_0^t \beta_v(s)\, ds} v(t, x)$$

then $\beta_v(t) = \beta_u(t)$ and hence $u = v$.

Main Results: The Gauged Problem

We have:

Theorem (Gauged IVP)

Let $0 < \alpha < \frac{1}{12}$, $s = s(\alpha) > 1$ *and* ϕ^ω *as above. Then there exists,* $0 < \delta_0 \ll 1$ *and* $r = r(s, \alpha) > 0$ *s.t. for any* $\delta < \delta_0$, *there exists* $\Omega_\delta \in A$ *with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each $\omega \in \Omega_\delta$ *there exists a unique solution* v *of the gauged quintic NLS in the space*

$$S(t)\phi^\omega + X^s([0, \delta)),$$

with initial condition ϕ^ω.

Back to the NLS: Main Result

Following the remarks above we denote by $X^s([0,\delta))_d$ the metric space $(X^s([0,\delta)), d)$ where d is the metric defined above, we obtain

Theorem (Nahmod–S.)

Let $0 < \alpha < \frac{1}{12}$, $s = s(\alpha) > 1$ *and* ϕ^ω *as above. Then there exists* $0 < \delta_0 \ll 1$ *and* $r = r(s,\alpha) > 0$ *s.t. for any* $\delta < \delta_0$, *there exists* $\Omega_\delta \in A$ *with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each $\omega \in \Omega_\delta$ *there exists a unique solution* u *of the (original) quintic NLS in the space*

$$S(t)\phi^\omega + X^s([0,\delta))_d,$$

with initial condition ϕ^ω.

Resonances Revisited

Let us define

$$N(v) := v|v|^4 - 3v\left(\int_{\mathbb{T}^3} |v|^4\, dx\right).$$

After taking Fourier transform in space one can show that if $\hat{v}(n,t) = a_n(t)$

$$\begin{aligned}
\mathcal{F}N(v)(n,t) &= \sum_{k=1}^{7} J_k(a_n(t)) \\
&= \sum_{\substack{n=n_1-n_2+n_3-n_4+n_5;\\ n_1,n_3,\, n_5\neq n_2,\, n_4}} a_{n_1}\overline{a_{n_2}}a_{n_3}\overline{a_{n_4}}a_{n_5} \\
&+ 6m \sum_{\substack{n=n_1-n_2+n_3;\\ n_1,n_3,\neq n_2}} a_{n_1}\overline{a_{n_2}}a_{n_3} \\
&+ 6 \sum_{\substack{n=n_1-n_2+n_3;\\ n_1,n_3,\neq n_2}} |a_{n_1}|^2 a_{n_1}\overline{a_{n_2}}a_{n_3} - 3 \sum_{\substack{n=n_1-n_2+n_3\\ n_1,n_3,\neq n_2}} a_{n_1}|a_{n_2}|^2\overline{a_{n_2}}a_{n_3} \\
&+ 2 \sum_{n=2n_1-n_2} |a_{n_1}|^2 a_{n_1}^2\overline{a_{n_2}} \\
&+ \quad \text{extra 'cubic' terms} \ldots
\end{aligned}$$

- We are interested in solving (not for v but) for

$$w = v - v_0^\omega \quad \text{where} \quad v_0^\omega = S(t)\phi^\omega$$

since we expect that w is a smoother function.

- We use a "duality" result in [HTT] and **several** trilinear and quintilinear estimates involving random and deterministic functions.

- We use a contraction method to prove well-posedness of the **difference** IVP (zero data) in the atomic space X^s thus we need to estimate the Duhamel term.

  - The atomic spaces were introduced in the context of critical dispersive equations by **Koch-Tataru (05'-07')** and **Hadac-Herr-Kock (09')** and then used by **[HTT]** and **[IP]** in the periodic 3D quintic NLS that we are treating here.

On Global Solutions

Extending these solutions globally in time is hard since the invariant measure is supported only on $H^s, s < -\frac{1}{2}$! Other possible routes could be:

- High-Low method of Bourgain and randomization. Work by **Colliander-Oh; Luehrmann-Mendelson; Poiret-Roberts-Thomann** (mass-subcritical, energy-subcritical).

  We are *energy-critical* so cannot implement at the moment: one would need to use the global result in H^1 by Ionescu-Pausader where the bounds for the "Strichartz norm" of the solution are tower-exponential in the energy.
- Recent *conditional* argument of **Bényi- Oh- Pocovnicu** for the energy critical NLS in $\mathbb{R}^3$. They assume that the solutions to the difference equation have uniformly bounded critical Sobolev norms and then they use a perturbation lemma, first introduced by Colliander, Keel, Staffilani, Takaoka and Tao, to show that the solution can be extended.
  - Randomization is used to prove that in small time intervals the random term in the difference equation is small.
- Recent result of **Pocovnicu** similar to the one above but for energy critical NLW in $\mathbb{R}^4, \mathbb{R}^5$. Here she is able to remove the conditional assumption by using a "probabilistic" energy bound on the difference equation.

On global solutions (long time)

Fix $T > 0$, let $\alpha > 0$ be as in the a.s. LWP and consider randomized initial data:

$$\phi^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{\frac{5}{2}-\alpha}} e^{in \cdot x} \qquad \text{in} \quad H^{1-\epsilon}(\mathbb{T}^3).$$

Theorem (Nahmod–S.)

Let $0 < \alpha < \frac{1}{12}$, $s = s(\alpha) > 1$ *and* ϕ^ω *as above. Fix a large interval of time* $[0, T]$. *Then there exists* $0 < \delta \sim T^{-\frac{1}{4}}$ *and there exists* $\Omega_\delta \in A$ *with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\delta}$$

and for each $\omega \in \Omega_\delta$ *there exists a unique solution u of the (original) quintic NLS in the space*

$$S(t)\phi^\omega + X^s([0, T))_d,$$

with initial condition ϕ^ω.

In short: There exists $\sigma = \sigma(T, \alpha) > 0$ and a set Ω_σ, with $\mathbb{P}(\Omega_\sigma) > \sigma$ such that for any $\omega \in \Omega_\sigma$, we have that ϕ^ω evolves globally to time T.

Remark

- *This is a large data result.*
- *As $T \to \infty$ the size of the set of initial data giving rise to solutions on the whole interval $[0, T]$ shrinks to zero.*

Idea of the proof: It is a combination of an iterated continuity argument and the fact that the random term can be made small.

Remark

A similar argument can be used to show almost sure global well-posedness for small data.

Question:

Can we do better in 3D? Most likely.... but not there yet $\longrightarrow$ new ingredients, ideas and some extra work (!).

Note:

- Poiret-Roberts-Thomann results for the cubic NLS on $\mathbb{R}^3$ are global (supercritical, but energy subcritical) but also on a set of positive measure.

- One can think of this probabilistic approach as a different method than the one -for example- used by Krieger-Schlag to exhibit large data global supercritical solutions to the septic NLW in $\mathbb{R}^3$ ($s_c = 7/6$).

Finite Dimensional Hamiltonian Systems

This last part of the talk deals with an infinite version of a **Non-squeezing Theorem**. We first recall the finite dimension version.

Hamilton's equations of motion have the antisymmetric form

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}(-\frac{\partial H}{\partial q_i}) = 0.$$

By defining $y := (q_1, \ldots, q_k, p_1, \ldots, p_k)^T \in \mathbb{R}^{2k}$ $(2k = d)$ we can rewrite the system in the compact form

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The Non-Squeezing Theorem in $\mathbb{R}^{2k}$

We recall a version of **Gromov's** famous theorem:

Theorem (Finite Dimensional Non-squeezing)

Assume that Φ_t is the flow generated by a finite dimensional Hamiltonian system just recalled and assume that there is an underlining symplectic structure compatible with the flow. Fix $y_0 \in \mathbb{R}^{2k}$ and let $B_r(y_0)$ be the ball in $\mathbb{R}^{2k}$ centered at y_0 and radius r. If

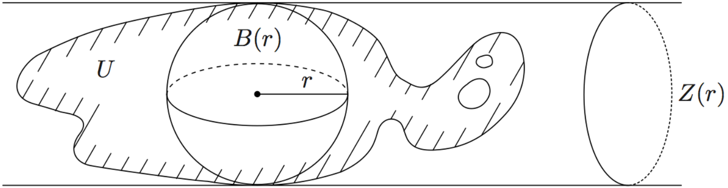
$$Z_R(z_0) := \{y = (q_1, \dots, q_k, p_1, \dots, p_k) \in \mathbb{R}^{2k} / |q_i - z_0| \leq R\},$$

is a cylinder of radius R, and

$$\Phi_t(B_r(y_0)) \subset Z_R(z_0),$$

it must be that $r \leq R$.

The Non-Squeezing Theorem



Can one generalize this theorem to the infinite dimensional setting given by a periodic dispersive equation written in Hamiltonian form?

The infinite dimensional Non-squeezing Theorem

Generalizing this kind of result in infinite dimensions has been a long time project of **Kuksin** who proved, roughly speaking, that compact perturbations of certain linear dispersive equations do indeed satisfy the non-squeezing theorem. Kuksin work tough does not apply for example to the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x), \ \text{where } x \in \mathbb{T}. \end{cases}$$

Using Strichartz estimates and the conservation of mass one can prove global well-posedness for data in L^2, see **Bourgain**. Hence one can define a global flow map

$$\Phi(t)u_0 := u(x,t).$$

It is easy to show that the L^2 space equipped with the form

$$\omega(f,g) = \langle if, g \rangle_{L^2}$$

is a symplectic space the global flow $\Phi(t)$ is a symplectomorphism.

The cubic, periodic, defocusing nonlinear Schrödingier Cauchy problem introduced above is not a compact linear perturbation. Nevertheless **Bourgain** proved the following theorem:

Theorem (Infinite Dimension Non-squeezing)

Assume that Φ_t is the flow generated by the cubic, periodic, defocusing NLS equation in L^2. If we identify L^2 with l^2 via Fourier transform, we let $B_r(y_0)$ be the ball in l^2 centered at $y_0 \in l^2$ and radius r,

$$Z_R(z_0) := \{(a_n) \in l^2 / |a_i - z_0| \leq R\}$$

a cylinder of radius R and

$$\Phi_t(B_r(y_0)) \subset Z_R(z_0),$$

at some time t, then it must be that $r \leq R$.

Idea of the Proof

The proof of this theorem is based on the following steps

- Use the projection operator P_N to project the Cauchy problem onto a finite dimensional Hamiltonian system.
- Use Gromov's Theorem.
- Show that the flow $\Phi_N(t)$ of the projected problem approximates well the flow $\Phi(t)$ of the original problem.

The third item is the most difficult to prove. The tools used are strong multilinear estimates based on the Strichartz estimates.

Remark

Unfortunately Bourgain's argument may not work for other kinds of dispersive equations. For example for the KdV problem, the lemma in Bourgain's work that gives the good approximation of the flow $\Phi(t)$ by $\Phi_N(t)$ does not hold. This has to do with the number of interacting waves in the nonlinearity. For the KdV problem one can still prove the non-squeezing theorem holds, but the existing proof is indirect and it has to go through the Miura transformation, see **Colliander-Keel-S-Takaoka- Tao**.

Non-squeezing and A.S. Well-posedness

- Periodic, cubic NLKG

$$\begin{cases} u_{tt} - \Delta u + u + u^3 = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^3 \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \in H^{\frac{1}{2}}(\mathbb{T}^3) \times H^{-\frac{1}{2}}(\mathbb{T}^3) =: \mathcal{H}^{1/2}(\mathbb{T}^3) \end{cases}$$

- Hamiltonian:

$$H(u) = \frac{1}{2} \int |\nabla u|^2 + |u|^2 + |u_t|^2 + \frac{1}{4} \int |u|^4.$$

- Symplectic phase space $\mathcal{H}^{\frac{1}{2}}(\mathbb{T}^d)$.
- Critical regularity for cubic NLKG (via NLW scaling $s_c = \frac{d}{2} - \frac{2}{p-1}$).

  $\Rightarrow$ no global flow and no control on local time of existence
- There is a global *almost sure* flow, **[Burq-Tzvetokov, Mendelson]**.

Statement of main results

Theorem (Mendelson, 2014)

Let Φ *denote the flow of the cubic nonlinear Klein-Gordon equation. Fix* $0 < R$, $k_0 \in \mathbb{Z}^3$, $z \in \mathbb{C}^2$, *and* $u_* \in \mathcal{H}^{1/2}(\mathbb{T}^3)$. *For all* $0 < \eta < R$, *there exists* $N \equiv N(\eta, \|u_*\|, R, k_0)$ *and* $\sigma \equiv \sigma(\eta, \|u_*\|, R, k_0) > 0$ *such that*

$$\Phi(\sigma)\big(P_{2N}\,\mathbf{B}_R(u_*)\big) \not\subseteq \mathbf{Z}_r(z; k_0)$$

for all $r < R - \eta$.

$$Z_r(z; k_0) = \Big\{(u_0, u_1) \in \mathcal{H}^s : \langle k_0\rangle^{1/2}|\widehat{u_0}(k_0) - z_1| + \langle k_0\rangle^{-1/2}|\widehat{u_1}(k_0) - z_2| < r\Big\}.$$

Remark

This theorem is proved by combining techniques that use randomization like explained in previous slides, and more deterministic tools, such as perturbation lemmas, definition of capacity etc. More on Dana's talk.

Some further questions

- Cubic NLS on $\mathbb{T}^3$ ($=\dot{H}^{\frac{1}{2}}$ critical) and on $\mathbb{T}^4$ ($\dot{H}^1$-critical) -work in progress-.
- Improve theorems on weak/wave turbulence:
  - Growth (lower/upper bounds) of (higher) Sobolev norms on tori.
    - After work by Colliander-Keel-S.-Takaoka-Tao, V. Sohinger, Z. Hani, Guardia-Kaloshin, Hani-Pausader-Tzvetkov-Visciglia. (c.f. related work by Faou-Germain-Hani, Bourgain-Demeter)
  - Construct non-equilibrium invariant measures for Hamiltonian PDE. **Work in Progress** by Z. Hani, A. Nahmod, L. Rey-Bellet and G. S.: Non-equilibrium invariant measures associated to cubic NLS.
    - Invariant measures with *entropy production* $\longleftrightarrow$ *Transfer of energy*
    - Ideas from stochastic PDE models: Work by L. Rey-Bellet *et al.* on non-equilibrium statistical mechanics of open classical systems. Work by M. Hairer and J. Mattingly in the case of anharmonic oscillator chains.
- Study the ergodicity of (invariant) measures associated to infinite dimensions Hamiltonian flows.
- More sophisticated Probabilistic notions of uniqueness (3D NS, etc).
  - e.g. Albeverio-Cruzeiro (90') for Euler.
- Probabilistic approaches to study properties of discrete versions of these equations (Chatterjee-Kirkpatrick, Chatterjee).