Toda flows, gradient flows and the generalized Flaschka map

Anthony Bloch

- Dissipation and Radiation Induced Instability
- Toda and gradient flows, metric and metriplectic flows
- The Pukhanzky condition
- Generalized Flaschka Map
- Dispersionless Toda

Toda Flow:

$$\dot{X} = [X, \Pi_S X]$$

Double Bracket Flow:

$$\dot{X} = [X, [X, N]]$$

– gradient but special case yields Toda. (with Brockett and Ratiu)

$$\dot{P} = [P, [P, \Lambda]$$

(Bloch, Bloch, Flashcka and Ratiu, Total Least Squares).

Heat equation

$$u_t = u_{xx}$$

Kahler flow:

$$u_t = (-\Delta)^{1/2} u$$

(with Morrison and Ratiu)

Dispersionless Toda flow

$$\dot{x} = \{x, \{x, z\}\}$$

(Bloch, Flaschka, Ratiu)

$$\dot{X} = [X, \nabla H] + [X, [X, N]]$$

– double bracket dissipation (Bloch, Krishnaprasad, Marsden and Ratiu)

•Radiation Damping

See Hagerty, Bloch and Weinstein [1999], [2002].

Important early work: Lamb [1900]. Related recent work may be found in Soffer and Weinstein [1998a,b] [1999] and Kirr and Weinstein [2001].

• Original Lamb model an oscillator is physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence the oscillator loses energy and is effectively damped by the string.

• Lamb model

$w(x,t)$ **displacement of the string. with mass density** ρ**, tension** T**. Assuming a singular mass density at** $x = 0$**, we couple dynamics of an oscillator,** q**, of mass** M**:**

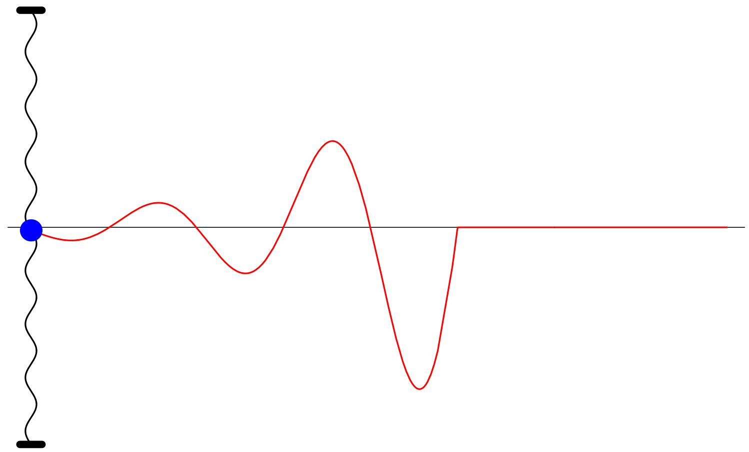

Figure 0.1: **Lamb model of an oscillator coupled to a string.**

$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= c^2 \frac{\partial^2 w}{\partial x^2} \\
M\ddot{q} + Vq &= T[w_x]_{x=0} \\
q(t) &= w(0,t).
\end{aligned}
$$

$[w_x]_{x=0} = w_x(0+,t) - w_x(0-,t)$ **is the jump discontinuity of the slope of the string. Note that this is a Hamiltonian system.**

Can solve for w **and reduce:**

• **Obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,**

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0.$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ **in the dynamics of the oscillator.**

Gyroscopic systems:
See Bloch, Krishnaprasad, Marsden and Ratiu [1994].
Linear systems of the form

$$M\ddot{q} + S\dot{q} + \Lambda q = 0$$

where $q \in \mathbb{R}^n$, M **is a positive definite symmetric** $n \times n$ **matrix,** S **is skew, and** Λ **is symmetric and indefinite.**

This system Hamiltonian with $p = M\dot{q}$**, energy function**

$$H(q,p) = \frac{1}{2} p M^{-1} p + \frac{1}{2} q \Lambda q$$

and the bracket

$$\{F, K\} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S_{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j}.$$

Aarise from simple mechanical systems via reduction; normal form of the linearized equations when one has an *abelian* **group.**

Dissipation induced instabilities—abelian case: Under the above conditions, if we modify the equation to

$$M\ddot{q} + (S + \epsilon R)\dot{q} + \Lambda q = 0$$

for small $\epsilon > 0$, where R is symmetric and positive definite, then the perturbed linearized equations

$$\dot{z} = L_\epsilon z,$$

where $z = (q, p)$ are spectrally unstable, *i.e.*, at least one pair of eigenvalues of L_ϵ is in the right half plane.

- **Gyroscopic systens connected to wave fields.**

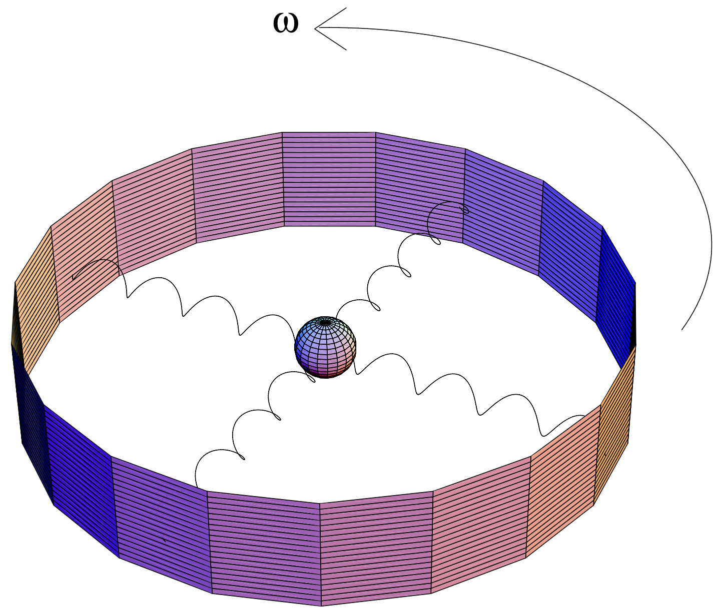

Figure 0.2: **Rotating plate with springs.**

In Hagerty, Bloch and Weinstein [2002] we describe a gyroscopic version of the Lamb model coupled to a standard non-

dispersive wave equation and to a dispersive wave equation. Show that instabilities will arise in certain mechanical systems. In the dispersionless case, the system is of the form

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\left[\frac{\partial \mathbf{w}}{\partial z}\right]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t), \end{aligned}$$

$\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t) \end{bmatrix}^T$ **is the displacement of the string in the first** n **dimensions and** $[\frac{\partial \mathbf{w}}{\partial z}]_{z=0}$ **is the jump discontinuity in the slope of the string.**

- **Can reduce dynamics to essentially:**

$$M\ddot{\mathbf{q}}(t) = -\,S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) - \frac{2T}{c}\dot{\mathbf{q}}(t),$$

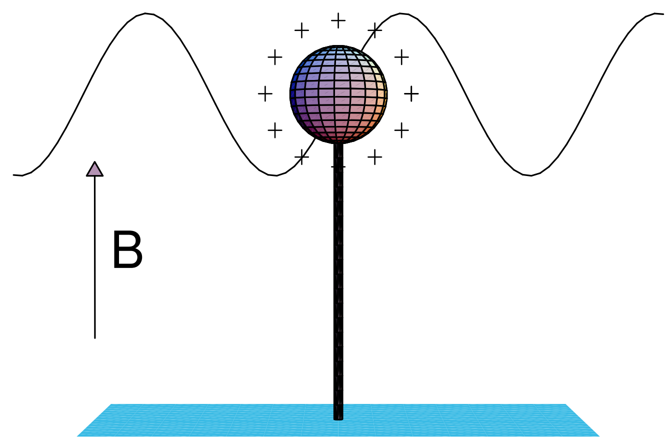

Figure 0.3: **Gyroscopic Lamb coupling to a spherical pendulum.**

Metriplectic Systems.

A *metriplectic system* consists of a smooth manifold P, two smooth vector bundle maps $\pi, \kappa : T^*P \to TP$ covering the identity, and two functions $H, S \in C^\infty(P)$, the *Hamiltonian* or *total energy* and the *entropy* of the system, such that

(i) $\{F, G\} := \langle \mathbf{d}F, \pi(\mathbf{d}G)\rangle$ **is a Poisson bracket; in particular** $\pi^* = -\pi$**;**

(ii) $(F, G) := \langle \mathbf{d}F, \kappa(\mathbf{d}G)\rangle$ **is a positive semidefinite symmetric bracket, i.e.,** $(\,,\,)$ **is** $\mathbb{R}$**-bilinear and symmetric, so** $\kappa^* = \kappa$**, and** $(F, F) \geq 0$ **for every** $F \in C^\infty(P)$**;**

(iii) $\{S, F\} = 0$ **and** $(H, F) = 0$ **for all** $F \in C^\infty(P) \iff \pi(\mathbf{d}S) = \kappa(\mathbf{d}H) = 0$**.**

Metriplectic Dynamics:

$$\frac{d}{dt}F = \{F, H + S\} + (F, H + S) = \{F, H\} + (F, S)$$

Application to Loop Groups...

An important and beautiful mechanical system that describes the interaction of particles on the line (i.e., in one dimension) is the Toda lattice. We shall describe the nonperiodic finite Toda lattice following the treatment of Moser.

This is a key example in integrable systems theory.

The model consists of n particles moving freely on the x-axis and interacting under an exponential potential. Denoting the position of the kth particle by x_k, the Hamiltonian is given by

$$H(x,y) = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{(x_k - x_{k+1})}.$$

The associated Hamiltonian equations are

$$\dot{x}_k = \frac{\partial H}{\partial y_k} = y_k \,, \tag{0.1}$$

$$\dot{y}_k = -\frac{\partial H}{\partial x_k} = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}} \,, \tag{0.2}$$

where we use the convention $e^{x_0-x_1} = e^{x_n-x_{n+1}} = 0$**, which corresponds to formally setting** $x_0 = -\infty$ **and** $x_{n+1} = +\infty$**.**

This system of equations has an extraordinarily rich structure. Part of this is revealed by Flaschka's (Flaschka 1974) change of variables given by

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad \textbf{and} \quad b_k = -\frac{1}{2} y_k \,. \tag{0.3}$$

In these new variables, the equations of motion then become

$$\dot{a}_k = a_k(b_{k+1} - b_k)\,, \quad k = 1, \ldots, n-1\,, \tag{0.4}$$

$$\dot{b}_k = 2(a_k^2 - a_{k-1}^2)\,, \quad k = 1, \ldots, n\,, \tag{0.5}$$

with the boundary conditions $a_0 = a_n = 0$**. This system may be written in the following Lax pair representation:**

$$\frac{d}{dt}L = [B, L] = BL - LB, \tag{0.6}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}.$$

Can show system is integrable.

Generalizations: Lie algebras, rigid body on the Toda orbit (with Gay-Balmaz and Ratiu)

More structure in this example. For instance, if N is the matrix $\mathrm{diag}[1, 2, \ldots, n]$, the Toda flow (0.6) may be written in the following double bracket form:

$$\dot{L} = [L, [L, N]] \, . \tag{0.7}$$

See Bloch [1990], Bloch, Brockett and Ratiu [1990], and Bloch, Flaschka and Ratiu [1990]. This double bracket equation restricted to a level set of the integrals is in fact the gradient flow of the function $\mathrm{Tr} LN$ with respect to the so-called normal metric.

From this observation it is easy to show that the flow tends asymptotically to a diagonal matrix with the eigenvalues of $L(0)$ on the diagonal and ordered according to magnitude, recovering the observation of Moser, Symes.

• **Four-Dimensional Toda. Here we simulate the Toda lattice in four dimensions. The Hamiltonian is**

$$H(a,b) = a_1^2 + a_2^2 + b_1^2 + b_2^2 + b_1 b_2 \,. \tag{0.8}$$

and one has the equations of motion

$$\begin{aligned} \dot a_1 &= -a_1(b_1 - b_2) \qquad & \dot b_1 &= 2a_1^2 \,, \\ \dot a_2 &= -a_2(b_1 + 2b_2) \qquad & \dot b_2 &= -2(a_1^2 - a_2^2) \,. \end{aligned} \tag{0.9}$$

(setting $b_1 + b_2 + b_3 = 0$**, for convenience, which we may do since the trace is preserved along the flow). In particular,** $\mathrm{Trace}\, LN$ **is, in this case, equal to** b_2 **and can be checked to decrease along the flow.**

Figure 0.4 exhibits the asymptotic behavior of the Toda flow.

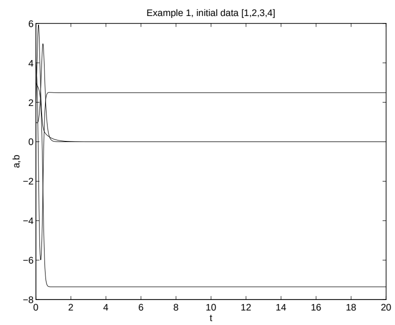

Figure 0.4: Asymptotic behavior of the solutions of the four-dimensional Toda lattice.

It is also of interest to note that the Toda flow may be written as a different double bracket flow on the space of rank one projection matrices. The idea is to represent the flow in the variables $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ where the λ_i are the (conserved) eigenvalues of L and r_i, $\sum_i r_i^2 = 1$ are the top components of the normalized eiqenvectors of L (see Moser). Then one can show (Bloch (1990)) that the flow may be written as

$$\dot{P} = [P, [P, \Lambda]] \tag{0.10}$$

where $P = \mathbf{r}\mathbf{r}^T$ and $\Lambda = \mathbf{diag}(\lambda)$.

This flow is a flow on a simplex The Toda flow in its original variables can also be mapped to a flow convex polytope (see Bloch, Brockett and Ratiu, Bloch, Flaschka and Ratiu).

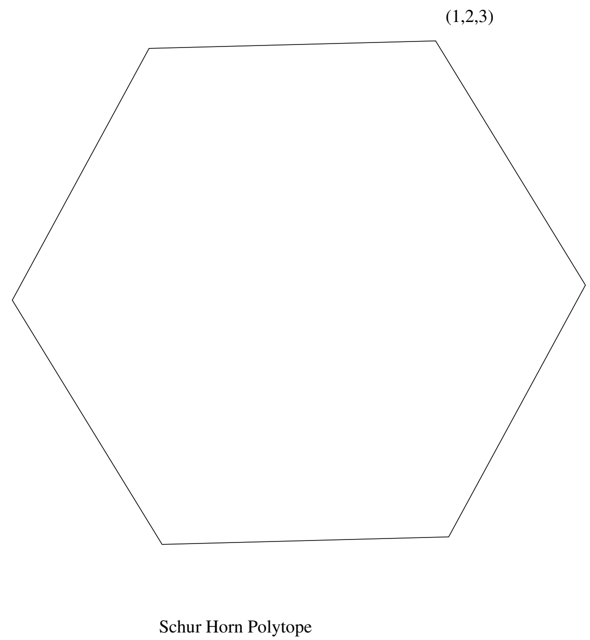

Figure 0.5: Image of Toda Flow

Toda rigid body:

$$h(c_1, c_2, a_1, a_2) = 3\Big((3D_1^2 - 2\lambda)c_1^2 + 2dc_1c_2 + (3D_2^1 - 2d)c_2^2\Big) + 6\left(a_1^2\frac{2B_1 - B_2}{2A_1 - A_2} + a_2^2\frac{2B_2 - B_1}{2A_2 - A_1}\right)$$

and the associated equations of motion are

$$\dot{c}_1 = -2a_1^2\frac{2B_1 - B_2}{2A_1 - A_2}\,, \qquad \dot{a}_1 = a_1\left((3D_1^2 - 2d)c_1 + dc_2\right),$$

$$\dot{c}_2 = -2a_2^2\frac{2B_2 - B_1}{2A_2 - A_1}\,, \qquad \dot{a}_2 = a_2\left((3D_2^1 - 2d)c_2 + dc_1\right).$$

Metrics on finite-dimensional orbits

Let $\mathfrak{g}_u$ be the compact real form of a complex semisimple Lie algebra $\mathfrak{g}$ and consider the flow on an adjoint orbit of $\mathfrak{g}_u$ given by

$$\dot{L}(t) = [L(t), [L(t), N]] \,. \tag{0.11}$$

Consider the gradient flow with respect to the "normal" metric. Explicitly this metric is given as follows.

Decompose orthogonally, relative to $-\kappa(\ ,\ ) = \langle\ ,\ \rangle$, $\mathfrak{g}_u = \mathfrak{g}_u^L \oplus \mathfrak{g}_{uL}$ where $\mathfrak{g}_{uL}$ is the centralizer of L and $\mathfrak{g}_u^L =$ Im $ad\, L$. For $X \in \mathfrak{g}_u$ denote by $X^L \in \mathfrak{g}_u^L$ the orthogonal projection of X on $\mathfrak{g}_u^L$. Then set the inner product of the tangent vectors $[L, X]$ and $[L, Y]$ to be equal to $\langle X^L, Y^L \rangle$. Denote this metric by $\langle\ ,\ \rangle_N$. Then we have

Proposition 0.1. *The flow (0.11) is the gradient vector field of $H(L) = \kappa(L, N)$, κ the Killing form, on the adjoint orbit $\mathcal{O}$ of $\mathfrak{g}_u$ containing the initial condition $L(0) = L_0$, with respect to the normal metric $\langle\ \ ,\ \ \rangle_N$ on $\mathcal{O}$.*

Proof. We have, by the definition of the gradient,

$$dH \cdot [L, \delta L] = \langle \operatorname{grad} H, [L, \delta L] \rangle_N \tag{0.12}$$

where $\cdot$ denotes the natural pairing between 1-forms and tangent vectors and $[L, \delta L]$ is a tangent vector at L. Set $\operatorname{grad} H = [L, X]$. Then (0.12) becomes

$$-\langle [L, \delta L], N \rangle = \langle [L, X], [L, \delta L] \rangle_N$$

or

$$\langle [L, N], \delta L \rangle = \langle X^L, \delta L^L \rangle \, .$$

Thus

$$X^L = ([L, N])^L = [L, N]$$

and

$$\operatorname{grad} H = [L, [L, N]]$$

as required. $\blacksquare$

For L and N as above obtain the Toda lattice flow. Full Toda may be also obtained with a modified metric.

Now in addition to the normal metric on an orbit there exist two other natural metrics, the induced and Kahler metrics.

• There is the natural metric b on G/T induced from the invariant metric on the Lie algebra –this is the induced metric.

• There is the normal metric described above which, following Atiyah we call b_1, which comes from viewing G/T as an adjoint orbit.

• Finally identifying the adjoint orbit with a coadjoint orbit we obtain the Kostant Kirilov symplectic structure which, together the fact that G/T is a complex manifold defines a Kahler metric b_2.

If we define b_1 and b_2 in terms of positive self-adjoint operators A_1 and A_2, $A_1 = A_2^2$. In fact b is just $Tr(AB)$, b_1 is $Tr(A^L B^L)$ and b_2 is essentially the square root of b_1.

The group of area preserving diffeomorphisms of the annulus and its (co)adjoint orbit structure:

Consider the geometry of the group $\mathrm{SDiff}(\mathcal{A})$ **of area (but not necessarily orientation) preserving diffeomorphisms of the annulus**

$$\mathcal{A} \stackrel{\mathbf{def}}{=} \{0 \leq z \leq 1\} \times \{\exp(2\pi i\theta) \mid 0 \leq \theta \leq 1\}\,.$$

The Lie algebra $\mathfrak{g} = \mathbf{sDiff}(\mathcal{A})$ **of** $\mathrm{SDiff}(\mathcal{A})$ **is the algebra of divergence-free vector fields tangent to the boundary of** $\mathcal{A}$**.**

These vector fields are Hamiltonian with respect to the area form $dz \wedge d\theta$ **and their Hamiltonian functions** $x(z, \theta)$ **satisfy**

$$\partial x(z_0, \theta)/\partial\theta = 0 \quad \textbf{for} \quad z_0 = 0, 1\,.$$

We will identify the Lie algebra $\mathfrak{g}$ with the Poisson algebra $\mathfrak{p}$ of functions obeying the boundary conditions (2.1). The two algebras are in fact not the same: $\mathfrak{p}$ is a trivial extension of $\mathfrak{g}$, or equivalently, $\mathfrak{g} = \mathfrak{p}/\{$ constant functions$\}$. However, it is easier to work with $\mathfrak{p}$.

The adjoint representation of $G = \mathrm{SDiff}(\mathcal{A})$ on its Lie algebra $\mathfrak{g}$ is then the map $P_g = F \mapsto F \circ g$ for $g \in \mathrm{SDiff}(\mathcal{A})$ and $F \in \mathfrak{g}$. This may be seen as follows. Let g_t be the flow of the Hamiltonian vector field X_H on $\mathcal{A}$. Then

$$\left.\frac{d}{dt}\right|_{t=0} g_t^* F = \mathcal{L}_{X_H} F = \langle dF, X_H \rangle \ = \left\langle F_z dz + F_\theta d\theta \,,\ H_\theta \frac{\partial}{\partial z} - H_z \frac{\partial}{\partial \theta} \right\rangle \{F, H\}\,, \tag{0.13}$$

where g_t^* denotes pull-back and $\langle \quad , \quad \rangle$ is the natural pairing between 1-forms and tangent vectors on $\mathcal{A}$.

In particular, we see that the tangent vector to an adjoint orbit $\mathcal{O}$ of G at the point $F \in \mathcal{O}$ is of the form $\{F, H\}$, $H \in \mathfrak{g}$.

Note that the function

$$\operatorname{Tr} F = \int_{\mathcal{A}} F dz d\theta$$

is invariant under the adjoint action. Thus we can define a weakly nondegenerate invariant inner product on $\mathfrak{g}$ by

$$\langle F, H \rangle = \operatorname{Tr} FH \,, \qquad F, G \in \mathfrak{g} \,. \tag{0.14}$$

Proof of invariance: if $g \in \mathrm{SDiff}(\mathcal{A})$ we have for any $F, G \in \mathfrak{g}$

$$\langle F \circ g, G \circ g \rangle = \int_{\mathcal{A}} (F \circ g)(G \circ g) dz d\theta = \int_{\mathcal{A}} ((FG) \circ g) dz d\theta \tag{0.15}$$

$$= \int_{\mathcal{A}} FG dz d\theta = \langle F, G \rangle \tag{0.16}$$

since g is area preserving.

The infinitesimal version of this relation reads

$$\langle \{F, G\}, H\rangle = \langle F, \{G, H\}\rangle \quad \textbf{for all } F, G, H \in \mathfrak{g}.$$

This can be proved, as usual, by taking a derivative relative to g **at the identity.**

Hence we may regard the space $\mathfrak{g}$ **as its own (algebraic) dual and identify the co-adjoint action with the adjoint action.**

The Lie–Poisson bracket on $\mathfrak{g}$ **is given by**

$$\{\{f, g\}\}(F) = \left\langle F, \left\{\frac{\delta f}{\delta F}, \ \frac{\delta g}{\delta F}\right\} \right\rangle \tag{0.17}$$

where $\frac{\delta f}{\delta F}$ **denotes the functional derivative. Restricted to an adjoint orbit in** $\mathfrak{g}$ **this corresponds to the orbit symplectic form.**

An adjoint orbit of $G = \mathrm{SDiff}(\mathcal{A})$ **carries a natural metric, which is the analogue of the finite-dimensional "normal metric".**

Lemma 0.2. *Let $x(z,\theta) \in \mathfrak{g} = sDiff(\mathcal{A})$. Then, relative to the L^2 inner product $\langle \ , \ \rangle$ on $L^2(\mathcal{A})$, $\overline{\mathfrak{g}} := L^2(\mathcal{A})$ may be decomposed orthogonally as*

$$\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_x \oplus \overline{\mathfrak{g}}^x$$

where the closures are taken in $L^2(\mathcal{A})$ and

$$\mathfrak{g}_x = \{y(z,\theta) \in \mathfrak{g} \mid \{x(z,\theta), y(z,\theta)\} = 0\} \tag{0.18}$$

$$\mathfrak{g}^x = \{w(z,\theta) \in \mathfrak{g} \mid w(z,\theta) = \{x(z,\theta), u(z,\theta)\}, u \in \mathfrak{g}\}. \tag{0.19}$$

We can now define the normal metric $\langle \ , \ \rangle_N$ on adjoint orbits of $\mathrm{SDiff}(\mathcal{A})$:

Definition 0.3. *Let $\{x,u\}$ and $\{x,w\}$ be two tangent vectors to the orbit $\mathcal{O}$ at x. Then $\langle \ , \ \rangle_N$ is given by*

$$\langle \{x,u\}\,,\ \{x,w\} \rangle_N = \langle u^x, w^x \rangle\,, \tag{0.20}$$

where u^x denotes the $\overline{\mathfrak{g}}^x$- component of u in the decomposition given by Lemma 2.1.

The single and double bracket equations on orbits of $\mathrm{SDiff}(\mathcal{A})$

Now consider two natural partial differential equations associated with $\mathrm{SDiff}(\mathcal{A})$: the Hamiltonian flow with respect to the orbit symplectic form, and the gradient flow with respect to the normal metric, of a linear functional restricted to an adjoint orbit of $\mathrm{SDiff}(\mathcal{A})$. These flows, as well as being interesting in their own right, are central to our interpretation of the Toda lattice flow.

Lemma 0.4. *The Hamiltonian flow of*

$$H(x(z,\theta)) = -\langle x(z,\theta), z\rangle = -\int\int x(z,\theta)z\, dzd\,\theta$$

under the Lie–Poisson bracket above is

$$x_t(z,\theta,t) = \{x(z,\theta,t), z\} = -x_\theta\,. \tag{0.21}$$

Proof. Let $f : \mathfrak{g} \to \mathbb{R}$ be arbitrary. Since $\frac{\delta H}{\delta x} = -z$, we have

$$\left\langle x_t, \frac{\delta f}{\delta x} \right\rangle = Df(x) \cdot x_t = \frac{d}{dt} f(x(z, \theta, t)) \tag{0.22}$$

$$= \{\{f, H\}\}(x) = \left\langle x, \left\{ \frac{\delta f}{\delta x}, -z \right\} \right\rangle = \left\langle \{x, z\}, \frac{\delta f}{\delta x} \right\rangle \tag{0.23}$$

the last equality following from the invariance of $\langle , \rangle$ under the adjoint action.

■

Now consider the gradient flow of $-\langle x(z, \theta), z \rangle$ with respect to the normal metric.

Proposition 0.5. *The gradient flow of $H(x(z, \theta)) = -\langle x(z, \theta), z \rangle$ on an adjoint orbit of* $\mathrm{SDiff}(\mathcal{A})$ *with respect to the normal metric is given by*

$$\begin{aligned} x_t(z, \theta, t) &= \{x(z, \theta, t), \{x(z, \theta, t), z\}\} \\ &= x_{\theta z} x_\theta - x_z x_{\theta\theta} \,. \end{aligned} \tag{0.24}$$

Proof. The proof parallels that in the finite-dimensional case (Bloch, Brockett, and Ratiu [1992]). Let $\{x, \delta x\}$ be a tangent vector to the orbit at x. Then by definition of the gradient,

$$DH(x) \cdot \{x, \delta x\} = \langle \operatorname{grad} H(x), \{x, \delta x\} \rangle_N,$$

where $\cdot$ denotes the natural pairing between 1-forms and tangent vectors. Set $\operatorname{grad} H(x) = \{x, y\}$. Then we have

$$-\langle \{x, \delta x\}, z \rangle = \langle \{x, y\}, \{x, \delta x\} \rangle_N$$

so that invariance of $\langle \ , \ \rangle$ under the adjoint action implies

$$\langle \{x, z\}, \delta x \rangle = \langle y^x, \delta x^x \rangle = \langle y^x, \delta x \rangle$$

for all $\delta x \in \mathfrak{g}$. Since $\{x, z\} \in \mathfrak{g}^x \subset \overline{\mathfrak{g}}^x$, this relation implies $y^x = \{x, z\}$ and hence $\operatorname{grad} H(x) = \{x, \{x, z\}\}$ as required. ■

We now consider the equilibria for the partial differential equations 0.21 and 0.24. Clearly we have:

Proposition 0.6. *All functions x which depend on z only are equilibria of 0.21 and 0.24.*

We also have:

Proposition 0.7. *All moments $\int_{\mathcal{A}} x^k$ are conserved along the flow of 0.21 and 0.24.*

Proof. Both 0.21 and 0.24are of the form $x_t = \{x, y\}$. Thus we get

$$\frac{d}{dt}\int_{\mathcal{A}} x(z,\theta)^k dzdt = \int_{\mathcal{A}} kx^{k-1}\{x,y\}dzd\theta = \langle kx^{k-1}, \{x,y\}\rangle \tag{0.25}$$

$$= k\langle\{x^{k-1}, x\}, y\rangle = 0. \tag{0.26}$$

∎

This is a "formal" assertion. Smooth solutions to these PDE's rarely exist for all time. Indeed, they are known to exhibit shocks in certain cases (see Brockett and Bloch [1990], Bloch and Kodama [1992]). For us, these *formal equilibria* were, nevertheless, the guide to our infinite-dimensional convexity theorem.

Two functions $f, g \in L^2([0,1]) \cap L^\infty([0,1])$ with the same moments are equimeasurable, i.e. $|\{z \mid f(z) > y\}| = |\{z \mid g(z) > y\}|$, where absolute value denotes Lebesgue measure on $[0,1]$. Hence the equilibria of the PDE's discussed above are equimeasurable rearrangements of one another. But within the smooth category there are very few of these. For example, if $x(z)$ is monotone decreasing its only smooth rearrangement is $1 - x(z)$. On the other hand, if we demand only that x belong to $L^2([0,1]) \cap L^\infty([0,1])$, we get an infinite number of equilibria – all the possible functions

on $[0,1]$ with the moments $I_p = \int_0^1 x^p \, dz$.

Since these functions are natural formal equilibria of these PDE's, one might want to enlarge the function space under consideration to include them. Important for convexity.

We now take the continuum limit of the Toda lattice equations by setting $n = \epsilon z$ with $0 \leq z \leq 1$, $\tau = \epsilon t$, and letting ϵ tend to zero. The functions $a_n(t)$, $b_n(t)$ approach functions $v(z,t)$, $u(z,t)$ of two variables, and the finite Toda lattice equations become

$$\frac{\partial v}{\partial t} = v\frac{\partial u}{\partial z}\,, \qquad \frac{\partial u}{\partial t} = 2\frac{\partial v^2}{\partial z}\,. \tag{0.27}$$

This is a quasilinear hyperbolic system, called the *dispersionless Toda equations*. **The naive continuum limits** I_p **of the constants of motion** $\operatorname{Tr} L^p$ **of the finite Toda system give constants of motion for the dispersionless Toda system. For example,**

$$\operatorname{Tr} L^2 = \sum (a_n^2 + a_{n-1}^2 + b_n^2)$$

becomes

$$I_2 = \int_0^1 (2v(z)^2 + u(z)^2) dz \,.$$

One can show that

$$I_p = \int_0^1 \int_0^1 (u(z) + v(z)e^{2\pi i\theta} + v(z)e^{-2\pi i\theta})^p dz d\theta \,.$$

We think of $(u(z) + v(z)e^{2\pi i\theta} + v(z)e^{-2\pi i\theta})$ **as the continuous analog of a tridiagonal matrix. The exponentials** $\exp(\pm 2\pi i\theta)$ **label the**

first super- and sub-diagonals. The variable z parametrizes the diagonal direction.

Proposition 0.8. *The system (0.27) has the bi-Hamiltonian structure*

$$J_0 = \begin{pmatrix} 0 & \partial_z v \\ v\partial_z & 0 \end{pmatrix} \tag{0.28}$$

with corresponding Hamiltonian

$$H_2 = \frac{1}{2}\int_0^1 (u^2(z) + 2v^2(z))dz$$

and

$$J_1 = \begin{pmatrix} 4v\partial_z v & u\partial_z v \\ v\partial_z u & v\partial_z v \end{pmatrix} \tag{0.29}$$

with corresponding Hamiltonian

$$H_1 = \int_0^1 u(z)dz$$

where $(\dot{u},\dot{v})^T = J\nabla H(u,v)$.

The Hamiltonian structure J_0 yields the Poisson bracket

$$\{F,H\}(u,v) = \int_0^1 \left(\frac{\delta F}{\delta u}\,,\frac{\delta F}{\delta v}\right)\begin{pmatrix} 0 & \partial_z v \\ v\partial_z & 0 \end{pmatrix}\left(\frac{\delta H}{\delta u}\,,\frac{\delta H}{\delta v}\right)^T dz$$
$$= \int_0^1 \left[\frac{\delta F}{\delta u}\frac{\partial}{\partial z}\left(v\frac{\delta H}{\delta v}\right) + v\frac{\delta F}{\delta v}\frac{\partial}{\partial z}\left(\frac{\delta H}{\delta u}\right)\right] dz\,. \tag{0.30}$$

For $H = H_2$ we get the dispersionless Toda equations.

We observe that the dispersionless Toda flow gradient is with respect to the normal metric on a level set of integrals.

This can be seen to be true by considering the gradient flow

$$\dot{x} = \{x,\{x,z\}\}$$

in the case

$$x(z,\theta) = \frac{1}{4\pi^2}\left(u(z) + 2v(z)\mathbf{cos}2\pi\theta\right)\,. \tag{0.31}$$

There is also a natural infinite-dimensional polytope: essentially the convex hull of the measurable rearrangements of the "diagonal" x – ie x depending on z only.

Also: a version of the Flaschka map.

Symplectic Geometry of the Flaschka Map: we show that is is a momentum map.

(Recent work with F. Gay-Balmaz and Tudor Ratiu).

Basic question: when is it possible to introduce global Darboux coordinates on the coadjoint orbit of Lie group. This happens for instance when G is an exponential solvable Lie group – as is the case for the lower triangular matrices.

In particular we show that there is a remarkable equivalence relation on coadjoint orbits, related to the so-called Pukanszky's condition. The associated quotient space turns out to be the base space of a cotangent bundle diffeomorphic to the coadjoint orbit. Such a realization is possible for solvable Lie algebras.

Show how this situation occurs for the generalized Toda lattice flows on semisimple Lie algebras, which generalize the Toda lattice flow on Jacobi matrices. We analyze the situation both for the lattice in its normal real form and compact real form, as well as for dispersionless Toda.

Theorem 0.9. *Let G be a connected and simply connected solvable Lie group and $\mathcal{O}$ a connected and simply connected 2d-dimensional coadjoint orbit of G. Then there is a diffeomorphism $\Phi : \mathbb{R}^{2d} \to \mathcal{O}$ such that $\Phi^*\omega_{\mathcal{O}}$ is constant and hence equal to the standard symplectic form on $\mathbb{R}^{2d}$.*

– see Pukanszky and Bloch, Gay-Balmaz, Ratiu.

Let $\mathfrak{g}$ be a Lie algebra and $\mu_0 \in \mathfrak{g}^*$. Given a linear subspace $\mathfrak{a} \subset \mathfrak{g}$, define

$$\mathfrak{a}^{\perp_{\mu_0}} := \{\xi \in \mathfrak{g} \mid \langle \mu_0, [\xi, \eta] \rangle = 0, \ \forall\, \eta \in \mathfrak{a}\}.$$

Definition 0.10. *Let G be a Lie group and $\mu_0 \in \mathfrak{g}^*$. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called* ***real polarization associated to*** μ_0 *if*

(i) $\{\mathrm{Ad}_g \mathfrak{h} \mid g \in G_{\mu_0}\} = \mathfrak{h}$;

(ii) $\mathfrak{h}^{\perp_{\mu_0}} = \mathfrak{h}$.

Lemma 0.11 (Pukanszky's condition)**.** *Let G be a Lie group, $\mu_0 \in \mathfrak{g}^*$, and $\mathfrak{h} \subset \mathfrak{g}$ a real polarization associated to μ_0. Then the following are equivalent:*

(i) $\mu_0 + \mathfrak{h}^\circ \subseteq \mathcal{O}_{\mu_0}$*;*

(ii) $\{\mathrm{Ad}_h^* \mu_0 \mid h \in H\} = \mu_0 + \mathfrak{h}^\circ$*, for all* $h \in H$*;*

(iii) $\{\mathrm{Ad}_h^* \mu_0 \mid h \in H\}$ *is closed in* $\mathfrak{g}^*$*.*

If any of these equivalent conditions hold, we say that the real polarization $\mathfrak{h}$ associated to $\mu_0 \in \mathfrak{g}^*$ satsifies ***Pukanszky's condition****.*

Theorem 0.12 (Pukanszky's conditions and momentum maps)**.** *Let G be a Lie group, $\mu_0 \in \mathfrak{g}^*$, and denote by $\mathcal{O}_{\mu_0}$ the coadjoint orbit of μ_0. Let $\mathfrak{h} \subset \mathfrak{g}$ be a real polarization associated to μ_0 and define H as above. Let $\nu_0 := i_{\mathfrak{h}}^* \mu_0$. Then the following are equivalent:*

(i) $\mathfrak{h}$ *verifies Pukanszky's conditions;*

(ii) *The reduced momentum map $\mathbf{J}_R^{\nu_0} : (T^*(G/H), \omega_{can} - B_{\nu_0}) \to \mathfrak{g}^*$ is onto $\mathcal{O}_{\mu_0}$;*

(iii) *The symplectic action of G on $T^*(G/H)$ is transitive;*

(iv) $\mathbf{J}_R^{\nu_0} : (T^*(G/H), \omega_{can} - B_{\nu_0}) \to \left(\mathcal{O}_{\mu_0}, \omega_{\mathcal{O}_{\mu_0}}\right)$ *is a symplectic diffeomorphism, where $\omega_{\mathcal{O}_{\mu_0}}$ is the minus orbit symplectic form, i.e.,*

$$\omega_{\mathcal{O}_{\mu_0}}(\mu)(\mathrm{ad}_\xi^* \mu, \mathrm{ad}_\eta^* \mu) = -\langle \mu, [\xi, \eta] \rangle, \quad \mu \in \mathcal{O}_{\mu_0}, \quad \xi, \eta \in \mathfrak{g}.$$

For an arbitrary $\mu = \mathrm{Ad}^*_g \mu_0 \in \mathcal{O}_{\mu_0}$, define the Lie subalgebra

$$\mathfrak{h}(\mu) = \mathfrak{h}\left(\mathrm{Ad}^*_g \mu_0\right) := \mathrm{Ad}_{g^{-1}}\left(\mathfrak{h}(\mu_0)\right). \tag{0.32}$$

It easy to check that $\mathfrak{h}(\mu)$ is a real polarization associated to μ verifying Pukanszky's condition $\mu + \mathfrak{h}(\mu)^\circ \subset \mathcal{O}_\mu$.

Consider the relation $\sim$ on the coadjoint orbit $\mathcal{O}_{\mu_0}$ defined, for $\nu, \gamma \in \mathcal{O}_{\mu_0}$, by:

$$\nu \sim \gamma \quad \textbf{if and only if} \quad \nu \in \gamma + \mathfrak{h}(\gamma)^\circ. \tag{0.33}$$

This is an equivalence relation. The associated quotient space is denoted $N_{\mu_0} := \mathcal{O}_{\mu_0}/\sim$, with quotient map

$$\pi_{\mu_0} : \mathcal{O}_{\mu_0} \to N_{\mu_0}, \qquad \mu \mapsto \pi_{\mu_0}(\mu) =: [\mu]_\sim.$$

The abstract Flaschka map: **The** ***abstract Flaschka map*** $F : \mathcal{O}_{\mu_0} \to T^*N_{\mu_0}$ **is defined by its restrictions** $F|_{[\mu]_\sim}$ **to the equivalence classes** $[\mu]_\sim \subset \mathcal{O}_{\mu_0}$**, that is, by the collection of maps**

$$F|_{[\mu]_\sim} : [\mu]_\sim \to T^*_{[\mu]_\sim} N_{\mu_0}. \tag{0.34}$$

Given a section $s_{\mu_0} : N_{\mu_0} \to \mathcal{O}_{\mu_0}$**, the map** $F|_{[\mu]_\sim}$ **is, in turn, defined by**

$$\left\langle F|_{[\mu]_\sim}(s_{\mu_0}([\mu]_\sim) + \sigma), v_{[\mu]_\sim} \right\rangle := \langle \sigma, \xi \rangle \,, \tag{0.35}$$

where $\xi \in \mathfrak{g}$ **is such that**

$$v_{[\mu]_\sim} = T_{\bar{\mu}}\pi_{\mu_0}\left(\mathrm{ad}^*_\xi\, \bar{\mu}\right), \qquad \bar{\mu} := s_{\mu_0}([\mu]_\sim). \tag{0.36}$$

Theorem 0.13. *Let $\mu_0 \in \mathfrak{g}^*$ and $\mathfrak{h}$ a real polarization associated to μ_0 verifying the Pukanszky condition. Define $\nu_0 := i^*_{\mathfrak{h}}\mu_0 \in \mathfrak{h}^*$. Fix a suitable one-form $\alpha_{\nu_0} \in \Omega^1(G)$ and consider the abstract Flaschka transformation $F : \mathcal{O}_{\mu_0} \to T^*(G/H)$ associated to the section $s_{\mu_0} := \overline{\alpha}_{\nu_0} \circ \Sigma^{-1}$. Then F is a smooth diffeomorphism whose inverse is the reduced momentum map associated to the symplectic reduction of T^*G by H at ν_0, that is,*

$$F^{-1} = \mathbf{J}_R^{\nu_0} : T^*(G/H) \to \mathcal{O}_{\mu_0}.$$

Therefore, F is a symplectic diffeomorphism relative to the minus coadjoint orbit symplectic form on $\mathcal{O}_{\mu_0}$ and the magnetic form $\omega_{can} - B_{\nu_0}$ on $T^*(G/H)$.

Toda Case:

Proposition 0.14. *The Flaschka map* $F : \mathcal{O}_{\mu_0} \to T^*\mathbb{R}^r_+$ *is given by*

$$F\left(\sum_{i=1}^r c_i h_i + \sum_{i=1}^r a_i(e_{\alpha_i} + e_{-\alpha_i})\right) = \left(a_1, ..., a_r, -\frac{2}{|\alpha_1|^2}\frac{c_1}{a_1}, ..., -\frac{2}{|\alpha_r|^2}\frac{c_r}{a_r}\right), \tag{0.37}$$

where $|\alpha_i|^2 := \kappa(\alpha_i, \alpha_i)$. *The inverse is*

$$F^{-1}(u_1, ..., u_r, v_1, ..., v_r) = -\sum_{i=1}^r \frac{|\alpha_i|^2 u_i v_i}{2} h_i + \sum_{i=1}^r u_i(e_{\alpha_i} + e_{-\alpha_i}). \tag{0.38}$$

Flaschka Map for the diffeomorphism groups of the annulus:

$$F_v(2v\cos(2\pi\theta)+u)=\left(v(z),-\frac{1}{2\pi v(z)}\int_0^z u(s)ds\right).$$

This map F **is, formally, a symplectic diffeomorphism between** $\left(\mathcal{O}_{\nu_0},\omega_{\mathcal{O}_{\nu_0}}\right)$ **and the weak symplectic vector space** $(T^*\mathcal{F}([0,1],\mathbb{R}_+)=\mathcal{F}([0,1],\mathbb{R}_+)\times\mathcal{F}([0,1],\mathbb{R}),\ \Omega_{\mathrm{can}})$**, as can also be shown by a direct verification. This formula is the analogue of the formulae for the finite dimensional normal real form and for the compact real form.**