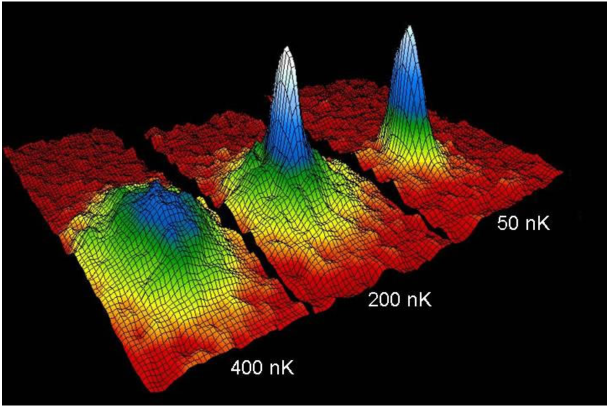
Bose-Einstein condensation and limit theorems

Kay Kirkpatrick, UIUC

2015

Bose-Einstein condensation: from many quantum particles to a quantum “superparticle”



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The big challenge: making physics rigorous

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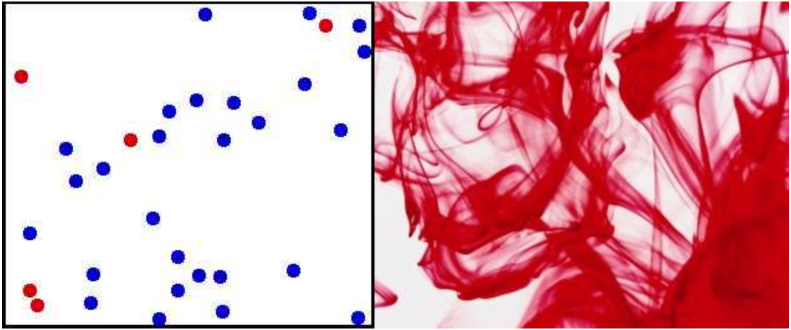
microscopic first principles $\rightsquigarrow$ zoom out $\rightsquigarrow$ MACROSCOPIC STATES



Courtesy Greg L and Digital Vision/Getty Images

1925: predicting Bose-Einstein condensation (BEC)

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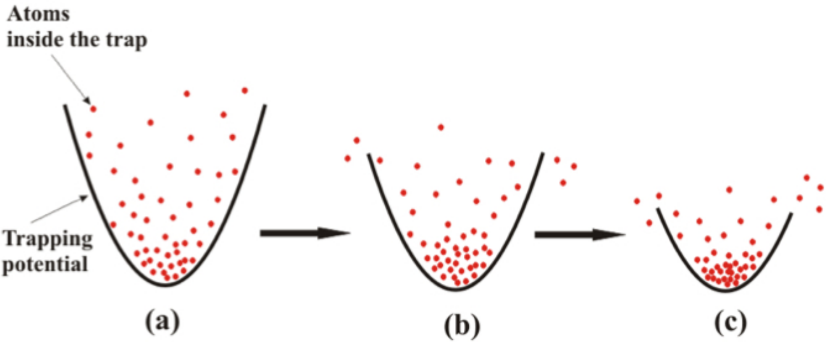
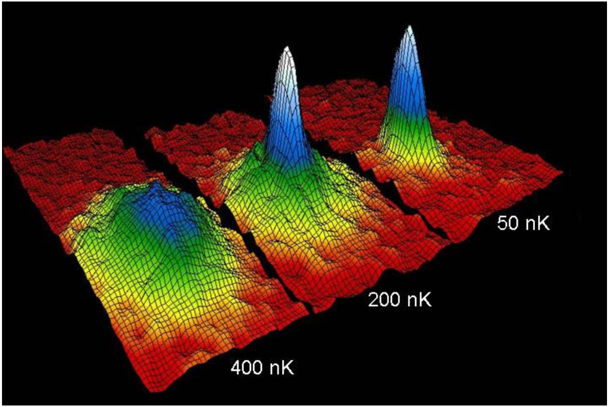
1995: Cornell-Wieman and Ketterle experiment



Courtesy U Michigan

After the trap was turned off

BEC stayed coherent like a single macroscopic quantum particle.



Momentum is concentrated after release at 50 nK. (Atomic Lab)

The mathematics of BEC

Gross and Pitaevskii, 1961: a good model of BEC is the cubic nonlinear Schrödinger equation (NLS):

$$i\partial_t \varphi = -\Delta \varphi + \mu |\varphi|^2 \varphi$$

Fruitful NLS research: competition between two RHS terms

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Can we rigorously connect the physics and the math?

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Can we rigorously connect the physics and the math?

Yes!

The outline (w/ G. Staffilani, B. Schlein, G. Ben Arous)

microscopic first principles $\rightsquigarrow$ $\rightsquigarrow$ **Macroscopic states**

1. N bosons $\rightsquigarrow$ mean-field limit $\rightsquigarrow$ **Hartree equation**
2. N bosons $\rightsquigarrow$ localizing limit $\rightsquigarrow$ **NLS**
3. Quantum probability and CLTs

A quantum “particle” is really a wavefunction

For each t, $\psi(x,t) \in L^2(\mathbb{R}^d)$ solves a Schrödinger equation

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- $-\Delta = -\sum_{i=1}^{d} \partial_{x^i x^i} \geq 0$
- external trapping potential V_{ext}
- solution $\psi(x,t) = e^{-iHt}\psi_0(x)$

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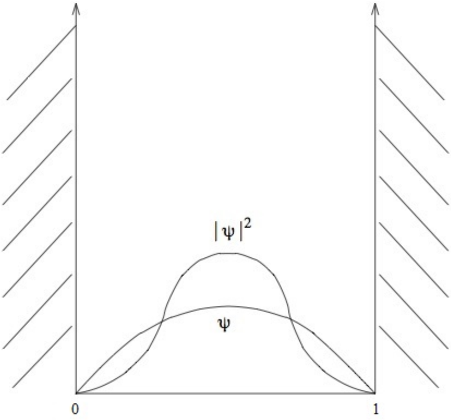
- $\int |\psi_0|^2 = 1 \implies |\psi(x,t)|^2$ is a probability density for all t. Exercise: why?

Particle in a box



$V_{ext} =$ "$\infty \cdot \mathbf{1}_{[0,1]^c}$" has ground state $\psi(x) = \sqrt{2}\sin(\pi x)$

The microscopic N-particle model

Wavefunction $\psi_N(\mathbf{x}, t) = \psi_N(x_1, ..., x_N, t) \in L^2(\mathbb{R}^{dN}) \; \forall t$
solves the N-body Schrödinger equation:

$$i\partial_t \psi_N = \sum_{j=1}^{N} -\Delta_{x_j} \psi_N + \sum_{i<j}^{N} U(x_i - x_j)\psi_N =: H_N \psi_N$$

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- pair interaction potential U
- solution $\psi_N(\mathbf{x}, t) = e^{-iH_N t} \psi_N^0(\mathbf{x})$
- joint density $|\psi_N(x_1, \ldots, x_N, t)|^2$

More assumptions

For N bosons, ψ_N is symmetric (particles are exchangeable):

$$\psi_N(x_{\sigma(1)}, ..., x_{\sigma(N)}, t) = \psi_N(x_1, ..., x_N, t) \text{ for } \sigma \in S_N.$$

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Initial data is factorized (particles i.i.d.):

$$\psi_N^0(\mathbf{x}) = \prod_{j=1}^{N} \varphi_0(x_j) \in L_s^2(\mathbb{R}^{3N}).$$

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But interactions create correlations for $t > 0$.

Mean-field pair interaction $U = \frac{1}{N} V$

Weak: order $1/N$. Long distance: $V \in L^\infty(\mathbb{R}^3)$.

$$i\partial_t \psi_N = \sum_{j=1}^{N} -\Delta_{x_j}\psi_N + \frac{1}{N}\sum_{i<j}^{N} V(x_i - x_j)\psi_N.$$

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Spohn, 1980: If ψ_N is initially factorized and approximately factorized for all t, i.e., $\psi_N(\mathbf{x}, t) \simeq \prod_{j=1}^{N} \varphi(x_j, t)$,
then "$\psi_N \to \varphi$" and φ solves the Hartree equation:

$$i\partial_t \varphi = -\Delta\varphi + (V * |\varphi|^2)\varphi.$$

Convergence "$\psi_N \to \varphi$" means in the sense of marginals:

$$\left\| \gamma_N^{(1)} - |\varphi\rangle\langle\varphi| \right\|_{Tr} \xrightarrow{N\to\infty} 0,$$

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where $|\varphi\rangle\langle\varphi|(x_1, x_1') = \overline{\varphi}(x_1)\varphi(x_1')$ and

one-particle marginal density $\gamma_N^{(1)} := Tr_{N-1}|\psi_N\rangle\langle\psi_N|$ has kernel

$$\gamma_N^{(1)}(x_1; x_1', t) := \int \overline{\psi}_N(x_1, \mathbf{x}_{N-1}, t)\psi_N(x_1', \mathbf{x}_{N-1}, t)d\mathbf{x}_{N-1}.$$

Other mean-field limit theorems

Erdös and Yau, 2001: Convergence of marginals for Coulomb interaction, $V(\mathbf{x}) = 1/|\mathbf{x}|$, not assuming approximate factorization.

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Preview of localizing interactions: $(V_N * |\varphi|^2)\varphi \to (\delta * |\varphi|^2)\varphi$
Erdös, Schlein, Yau, K., Staffilani, Chen, Pavlovic, Tzirakis...

Definition of BEC at zero temperature

Almost all particles are in the same one-particle state:

$\{\psi_N \in L^2_s(\mathbb{R}^{3N})\}_{N\in\mathbb{N}}$ exhibits **Bose-Einstein condensation** into one-particle quantum state $\varphi \in L^2(\mathbb{R}^3)$ iff

one-particle marginals converge in trace norm:

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Generalizes factorized: $\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j)$ is BEC into φ.

BEC limit theorems with parameter $\beta \in (0,1]$

Now localized strong interactions: $N^{d\beta} V(N^{\beta}(\cdot)) \to b_0 \delta$.

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^{N} N^{d\beta} V(N^{\beta}(x_i - x_j)).$$

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Theorems (Erdös-Schlein-Yau 2006-2008 $d = 3$
K.-Schlein-Staffilani 2009 $d = 2$ plane and rational tori):
Systems that are initially BEC remain condensed for all time, and the macroscopic evolution is the NLS:

$$i\partial_t\varphi = -\Delta\varphi + b_0|\varphi|^2\varphi.$$

Our limit theorems make the physics of BEC rigorous

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^{N} N^{d\beta} V(N^{\beta}(x_i - x_j))$$

$$\begin{array}{lccccl} & & N\text{-body Schrod.} & & \\ \textit{micro}: & \psi_N^0 & \longrightarrow & \psi_N & \\ \text{init. BEC} & \downarrow & & \downarrow & \textbf{marg.} \\ \textit{MACRO}: & \varphi_0 & \longrightarrow & \varphi & \\ & & \textbf{NLS evolution} & & \end{array}$$

$$i\partial_t \varphi = -\Delta\varphi + b_0|\varphi|^2\varphi.$$

A taste of quantum probability $(\mathcal{H}, \mathcal{P}, \varphi)$

Hilbert space $\mathcal{H}$, set of projections $\mathcal{P}$, and state φ.

Quantum random variables (RVs) or observables: operators on $\mathcal{H}$.

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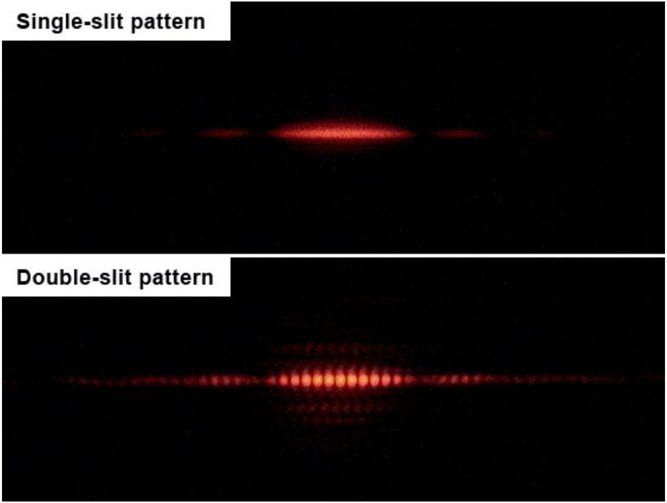
Quantum random variables (RVs) or observables: operators on $\mathcal{H}$.

The expectation of an observable A in a pure state is

$$\mathbb{E}_{\varphi}[A] := \langle \varphi | A\varphi \rangle = \int \varphi(x)\overline{A\varphi}(x)dx.$$

Position observable is $X(\varphi)(x) := x\varphi(x)$ with density $|\varphi|^2$.

Only some probability facts have quantum analogues



Courtesy of Jordgette

The BEC limit theorems imply quantum LLNs

If A is a one-particle observable and

$$A_j = 1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots \otimes 1,$$

then for each $\epsilon > 0$,

$$\limsup_{N\to\infty} \mathbb{P}_{\psi_N} \left\{ \left| \frac{1}{N} \sum_{j=1}^{N} A_j \right.\right.$$

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$$\limsup_{N\to\infty} \mathbb{P}_{\psi_N} \left\{ \left| \frac{1}{N} \sum_{j=1}^{N} A_j - \langle \varphi | A\varphi \rangle \right| \geq \epsilon \right\} = 0.$$

BEC can explode as a bosenova

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We need a control theory of BEC

- Central limit theorem for BEC (Ben Arous-K.-Schlein, 2013)

  Our quantum CLT has correlations coming from interactions

- CLT for quantum groups (Brannan-K., 2015)

Our CLT for interacting quantum many-body systems

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σ_t^2 has $\varphi_0 \rightsquigarrow \varphi_t$.... and twisted by the Bogoliubov transform.

We studied freely independent RVs via quantum groups

(instead of random matrices) with Michael Brannan (Texas A&M)

Theorem (Brannan, K. 2015): Deformed quantum groups have an action

ASK MIKE FOR HIS ACTION FIGURE TEX CODE

on Free Araki-Woods factors

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$$\Gamma = \Gamma(\mathbb{R}^n, U_t)'' := \{\ell(\xi) + \ell(\xi)^* : \xi \in H_{\mathbb{R}}\}''$$

with free quasi-free state φ_Ω,

$$\alpha(c_i) = \sum u_{ij} \otimes c_j, \quad U_t = A^{it}, \quad \text{some } A > 0.$$

Usually a full type III_λ factor for $\lambda \in [0, 1]$. Best case: $\lambda = 1$!

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(exciting new development from MSRI...)

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$$(\Gamma, \varphi_\Omega).$$

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MH: quantum to classical; B-K: classical to quantum

How do physics, the world, and the universe work?

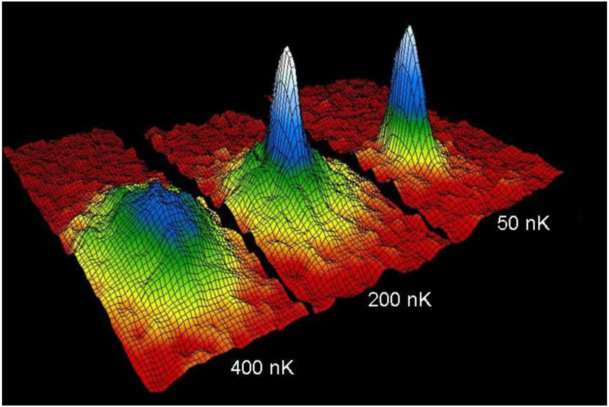
Physics

$\downarrow \qquad \uparrow$

Analysis

Thanks

NSF DMS-1106770, OISE-0730136, CAREER DMS-1254791



arXiv:0808.0505 (AJM), 1009.5737 (CPAM), 1111.6999 (CMP), 1505.05137(PJM)

Why do interactions become the cubic nonlinearity?

$$i\partial_t \psi_N = \sum -\Delta_{x_j} \psi_N + \frac{1}{N} \sum \sum V(x_i - x_j) \psi_N$$

Particle 1 sees

$$\begin{aligned} \frac{1}{N} \sum_{j=2}^{N} V(x_1 - x_j) &\simeq \frac{1}{N} \sum_{j=2}^{N} \int V(x_1 - y) |\varphi(y)|^2 dy \\ &= \frac{N-1}{N} \int V(x_1 - y) |\varphi(y)|^2 dy \\ &\xrightarrow{N \to \infty} (V * |\varphi|^2)(x_1) \end{aligned}$$