Representation Theory (Fall 2004)
Lecture 15-16

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Recall definitions:

$$G = SL_2(\mathbb{F}_p), \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}, \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G \right\} \simeq \mathbb{F}_p,$$

and if $\varphi : \mathbb{F}_p^\times \to \mathbb{C}^\times$ is a character, $\quad V_\varphi = Ind_B^G \varphi$.

Also recall that:

- V_φ is irreducible if $\varphi^2 \neq 1$.
- $V_1 =$ trivial $\oplus$ Steinberg

We are working on the case where $\varphi =: \tau$ is a quadratic character ($\tau^2 = 1$). We claim that $V_\tau = V_\tau^+ \oplus V_\tau^-$ and $V_\tau^\pm$ are each of dimension $(p+1)/2$.

Recall that $V_\tau = \{f : G \to \mathbb{C}$ with $f(nag) = \tau(a) f(g)\}$, where $n \in N$ and

$$a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{F}_p^\times.$$

Recall the definition of U: $(Uf)(g) = \sum_{n \in N} f(w^{-1} n g)$.

$U : V_\varphi \overset{\sim}{\to} V_{\varphi^{-1}}$ is G-linear so if $\tau = \tau^{-1}$ (so $U : V_\tau \to V_\tau$) then $V_\tau^\pm$ are the eigenspaces of U. Let's verify that U is G-linear:

$$\begin{aligned} (g_1(Uf))(g) &= (Uf)(gg_1) \\ &= \sum_{n \in N} f(w^{-1} n g g_1) \\ &= (U(g_1 f))(g) \end{aligned}$$

because $(g_1 f)(w^{-1} n g) = f(w^{-1} g g_1)$.

$Uf \in V_\tau$; does $(Uf)(n_1 a g) = \tau(a)(Uf)(g)$?

$(Uf)(n_1 a g) = \sum_{n \in N} f(w^{-1} n n_1 a g) = \sum_{n_2 \in N} f(w^{-1} n_2 a g)$; if $n_2 = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ then $n_2 a = a \begin{pmatrix} 1 & x/a^2 \\ 0 & 1 \end{pmatrix}$, which we call n_3. So ... $= \sum_{n_3 \in N} f(w^{-1} a n_3 g) \ = \sum_{n_3 \in N} f(a^{-1} w^{-1} n_3 g)$, (since $w^{-1} a = a^{-1} w^{-1}$) $= \tau(a^{-1}) \sum_{n_3 \in N} f(w^{-1} n_3 g) = \tau(a)(Uf)(g)$.

Notes by Kyle Schalm ; edited by Nick Leger

Claim: U is an isomorphism.

Idea: Restrict U to a subgroup and diagonalize U. Consider V_τ as a representation of N. The characters of N are

$$\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) \mapsto \psi(x)$$

where $\psi(x)$ is an additive character of $\mathbb{F}_p$. Fix a nonzero ψ; then any other character of $\mathbb{F}_p$ is of the form $\psi_t(x) := \psi(tx)$, for some $t \in \mathbb{F}_p$ (if $t = 0$, ψ is the trivial character).

We will show $V_\tau = \bigoplus_{t=0}^{p-1} V_\tau^t$ where V_τ^t is the ψ_t-component of V_τ, meaning that if $f_t \in V_\tau^t$, $f_t \neq 0$, then $f_t(gn) = \psi_t(n) f_t(g)$. Also, $f_t(n_1 a g n_2) = \tau(a)\psi_t(n_2) f_t(g)$.

First, we claim that f_t is determined by its value at $1, w$.

Let's use the Bruhat decomposition $G = B \sqcup BwN$: either $g \in B$, in which case $g = na$ and $f_t(g) = f_t(na) = \tau(a) f_t(1)$, or $g \in BwN$, in which case $g = n_1 a w n_2$ and $f_t(g) = f_t(n_1 a w n_2) = \tau(a)\psi_t(n_2) f_t(w)$.

Case: $t \neq 0$. $f_t(1) = f_t(n \cdot 1) = f_t(1 \cdot n) = \psi_t(n) f_t(1)$, since ψ_t is not trivial, there is an n for which $\psi_t(n) \neq 1$ so that $f_t(1) = 0$. Hence f_t is completely determined by its value at w. Normalize by letting $f_t(w) = 1$.

Case: $t = 0$. Let

$$f_0 = \begin{cases} 1 \mapsto 0 \\ w \mapsto 1 \end{cases} \qquad f_\infty = \begin{cases} 1 \mapsto 1 \\ w \mapsto 0. \end{cases}$$

These span V_τ^0 (which is therefore 2-dimensional).

To show V_τ^t are stable by U:

If $t \neq 0$: $Uf_t = \lambda_t f_t$ for some $\lambda_t \in \mathbb{C}$. To compute this λ_t, $(Uf_t)(w) = \lambda_t f_t(w) = \lambda_t$ and $(Uf_t)(w) = \sum_{n \in N} f_t(w^{-1} n w)$, where

$$w^{-1} n w = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right).$$

If $x \neq 0$, this is

$$\left(\begin{array}{cc} 0 & -x^{-1} \\ 1 & 0 \end{array}\right) \left(\begin{array}{cc} x^{-1} & 0 \\ 0 & x \end{array}\right) \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \left(\begin{array}{cc} 1 & -x^{-1} \\ 0 & 1 \end{array}\right),$$

so $f_t(w^{-1} n w) = \psi_t(-x^{-1})\tau(x^{-1}) f_t(w) = \psi_t(-x^{-1})\tau(x^{-1})$, (since $w \in N$).

Hence

$$\begin{aligned} \lambda_t &= \sum_{x \in \mathbb{F}_p^\times} \psi_t(-x^{-1})\tau(x^{-1}) \\ &= \sum_{x \in F_p^\times} \tau(-x)\psi(tx) \quad (x \mapsto -x^{-1}) \\ &= \tau(-t) \sum_{x \in \mathbb{F}_p^\times} \tau(x)\psi(x) \quad (tx \mapsto x) \\ &=: g(\tau), \end{aligned}$$

this last sum is a Gauss sum and equal to p, so $g(\tau)^2 = \tau(-1)p$ and $g(\tau) = \pm\sqrt{\tau(-1)p}$. (Gauss actually proved $g(\tau) = \begin{cases} \sqrt{p} & p \equiv 1 \mod 4 \\ i\sqrt{p} & p \equiv 3 \mod 4 \end{cases}$).

If $t = 0$ (continuing our computation of λ_t):

Using the fact that V_τ^0 is spanned by f_0, f_∞ and these are determined by their values at 1 and w,

$Uf_0(1) = \sum_{n\in N} f_0(w^{-1}n \cdot 1) = \sum_{n \in N} \psi_0(n) f_0(w^{-1}) = f_0(-w)|N|$ (as $w^{-1} = -w$) $= \tau(-1)p$;

and (with n as before),
$Uf_0(w) = \sum_{n\in N} f_0(w^{-1}nw) = \sum_{x \in \mathbb{F}_p} \tau(x) f_0(w) = \sum_{x\in\mathbb{F}_p} \tau(x) = 0$;

$Uf_\infty(1) = \sum_{n\in N} f_\infty(w^{-1}n) = pf_\infty(-w) = \tau(-1)pf_\infty(w) = 0$;

$Uf_\infty(w) = \sum_{x\neq 0} \tau(x) f_\infty(w) + f_\infty(1) = 1$ since $f_\infty(w) = 0$.

So U acts on V_τ as $\begin{pmatrix} 0 & 1 \\ \tau(-1)p & 0 \end{pmatrix}$ (with respect to the basis $\langle f_0, f_\infty \rangle$) with eigenvalues $\pm g(\tau)$.
So U diagonalizes as $\frac{p+1}{2}$ $g(\tau)$'s followed by $\frac{p+1}{2}$ $-g(\tau)$'s.
Let $V_\tau^\pm = \{f \in V_\tau : Uf = \pm g(\tau) f\}$ be the eigenspaces of U.
G preserves each space $V_\tau = V_\tau^+ \oplus V_\tau^-$ (each of dimension $\frac{p+1}{2}$).
Summary: take any character $\varphi : \mathbb{F}_p^\times \to \mathbb{C}^\times$ and decompose V_φ to obtain the principle series:

	Number	Space	Dim.
$\varphi^2 \neq 1$	$\frac{p-3}{2}$	V_φ	$p+1$
$\varphi = \tau$	2	V_τ^+, V_τ^-	$\frac{p+1}{2}$
$\varphi = 1$	2	trivial, Steinberg	$1, p$

This gives us $\frac{p-3}{2} + 4 = \frac{p+5}{2}$ representations; we need $p+4$.
(There is some stuff I don't understand here, and then ....)
Example, with $p = 3$. $|G| = 3 \cdot (3^2 - 1) = 24$. $PSL_2(\mathbb{F}_p) = SL_2(\mathbb{F}_p)/\{\pm 1\}$ ($\{\pm 1\}$ is the center). This is simple for $p > 3$. For $p = 3$, $PSL_3 \simeq A_4$.
Conjugacy classes look like

	Triv.	*St.*	V_τ^+	V_τ^-
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	3	2	2
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	3	-2	-2
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1	0	$\frac{1\pm\sqrt{-3}}{2}$	$\frac{1\pm\sqrt{-3}}{2}$
$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	1	0	$\frac{1\pm\sqrt{-3}}{2}$	$\frac{1\pm\sqrt{-3}}{2}$
$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	1	0	$\frac{1\pm\sqrt{-3}}{2}$	$\frac{1\pm\sqrt{-3}}{2}$
$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$	1	0	$\frac{1\pm\sqrt{-3}}{2}$	$\frac{1\pm\sqrt{-3}}{2}$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	1	-1	0	0
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	1	-1	0	0

We should be able to find some representation of A_4 from here. The first two come from PSL_2.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow (12)(34)$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \leftrightarrow (123)$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \leftrightarrow (132)$$

(There are still two representations of A_4 we haven't found.)

Example: $p = 5$, $PSL_2(\mathbb{F}_5) \simeq A_5$.

	St.	v_t^+	v_t^-
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	5	3	3
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	5	3	3
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	0	ε	ε'
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	0	ε'	ε
$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	0	ε	ε'
$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$	0	ε'	ε
$\begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix}$	1	-1	-1
$\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, x^2 \neq 1$	-1	0	0

where $\epsilon = \frac{1+\sqrt{5}}{2}$, $\epsilon' = \frac{1-\sqrt{5}}{2}$. Some correspondences with A_5 are

$$(12345) \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$(21345) \leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$(123) \leftrightarrow \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$(12)(34) \leftrightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2^{-1} \end{pmatrix}$$

$PSL_2(\mathbb{F}_4) \simeq A_5$, $PSL_2(\mathbb{F}_9) \simeq A_6$, $PSL_4(\mathbb{F}_2) \simeq A_8$ these are the only isomorphisms between the PSL family and the alternating family.

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Let $H = \mu_p \times L \times L$, $(p > 2)$ where L is a vector space over $\mathbb{F}_p$ of dimension 2. Let $B : L \times L \to \mathbb{F}_p$ be a nondegenerate symmetric bilinear form. Let

$$\Psi = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)$$

and

$$\Phi = \left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right).$$

This is a non-degenerate skew symmetric form on $L \times L$.

$\Psi((l_1, l_2), (l_1', l_2')) = l_1 B l_2'$ and $\Psi((l_1, 0), (0, l_2)) = B(l_1, l_2)$.

$\psi = e^{2\pi i/p}\Psi$, $\varphi = e^{2\pi i/p}\Phi$, $b = e^{2\pi i}B$.

The whole point is the connection

$G = SL_2(\mathbb{F}_p) \hookrightarrow \{Sp(U) : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to (u \mapsto u\sigma)\}$, where $U = L \times L$, $Sp(U) = \{\mu : U \to U$ preserves $\Phi\}$.

$u = (l_1, l_2)$, $u\sigma = (l_1, l_2)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha l_1 + \gamma l_2, \beta l_1 + \delta l_2)$.

(l_1 and l_2 are themselves vectors of dimension 2.)

$$\sigma \mapsto \left(\begin{array}{c|c} \alpha I_2 & \beta I_2 \\ \hline \gamma I_2 & \delta I_2 \end{array}\right).$$

σ preserves Φ, i.e.

$$\left(\begin{array}{c|c} \alpha I_2 & \gamma I_2 \\ \hline \beta I_2 & \delta I_2 \end{array}\right)\left(\begin{array}{c|c} 0 & B \\ \hline -B & 0 \end{array}\right)\left(\begin{array}{c|c} \alpha I_2 & \beta I_2 \\ \hline \gamma I_2 & \delta I_2 \end{array}\right) = \Phi \text{ using } \det(\sigma) = 1.$$

We can embed $Sp(U)$ in $Aut_0(H)$ (these preserve the center).

$\sigma \mapsto ((\zeta, u) \mapsto (\zeta \mathcal{V}_\sigma(u), u\sigma))$ where $\mathcal{V}_\sigma(u) = (\frac{\psi(u\sigma, u\sigma)}{\psi(u,u)})^{1/2}$.

We have the Schrödinger representation of H,

$$\rho : H \to GL(V)$$

where $V = \{f : L \to \mathbb{C}\}$, defined by $\rho(\zeta, 0) \mapsto \zeta$. The center acts by multiplication; $\zeta \in \mu_p$ acts by sending f to ζf.

- $(1, (l_1, 0))$ acts by $l \mapsto f(l + l_1)$
- $(1, (0, l_2))$ acts by $l \mapsto f(l)\Psi(l, l_2) = f(l)b(l, l_2)$.

If $\sigma \in Aut_0(H)$, then the composition $\rho^\sigma := \rho \circ \sigma$ acts on the center in the same way.

By Stone-von Neumann, $R(\sigma)^{-1}\rho R(\sigma) = \rho^\sigma$ for some $R(\sigma) \in GL(V)$. Pick such an $R(\sigma)$ for each σ. It's well defined up to a scalar by Schur. It follows that $c(\sigma_1, \sigma_2)R(\sigma_1\sigma_2) = R(\sigma_1)R(\sigma_2)$ for some $c(\sigma_1, \sigma_2) \in \mathbb{C}^\times$.

$R : G \to PGL(V)$ is a projective representation. Can we lift it to $GL(V)$? Plan: let's try to understand R for some $\sigma \in N, T$ and $\sigma = w$, and then extend with Bruhat.

Case (i) $\sigma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$

let $h = (1, u) \in H$ for notation.

Subcase (1): $u = (l_1, 0)$ so $u\sigma = (al_1, 0)$. For $f \in V$, $(h^\sigma f)(l) = f(l + al_1)$, $(hf)(l) = f(l + l_1)$.

Subcase (2): $u = (0, l_2)$, $u\sigma = (0, a^{-1}l_2)$. $(h^\sigma f)(l) = b(a^{-1}l_2, l)f(l)$, $(hf)(l) = b(l_2, l)f(l)$.

Define $(R(\sigma)f)(l) = f(al)$. We claim that this works (satisfies $R(\sigma)^{-1}\rho R(\sigma) = \rho^\sigma$).

Proof: $(R(\sigma)^{-1}hR(\sigma))f)(l) = (R(\sigma)^{-1}hf)(al) = R(\sigma)^{-1}f(a(l + l_1)) = f(l + al_1) = (h^\sigma f)(l)$.

Case (ii): $\sigma = w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Subcase (1): $u = (l_1, 0)$, $u\sigma = (0, l_1)$. $(h^\sigma f)(l) = b(l, l_1)f(l)$, $(hf)(l) = f(l + l_1)$.
Subcase (2): $u = (0, l_2)$, $u\sigma = (-l_2, 0)$. $(h^\sigma f)(l) = f(l - l_2)$, $(hf)(l) = b(l, l_2)f(l)$.
To find R, let's define a "Fourier transform":

$$\mathcal{F}f(l) = |L|^{-1/2} \sum_{l' \in L} f(l')b(l, -l')$$

and define $f_1(l) = f(l + l_1)$. Then

$$\begin{aligned}\mathcal{F}f_1(l) &= |L|^{-1/2} \sum_{l' \in L} f_1(l')b(l, -l') \\ &= |L|^{-1/2} \sum_{l' \in L} f_1(l' + l)b(l, -l') \\ &= |L|^{-1/2} \sum_{l'' \in L} f_1(l'')b(l, -l'' + l_1) \\ &= b(l, l_1)\mathcal{F}f(l).\end{aligned}$$

Now $\mathcal{F}h = h^\sigma \mathcal{F} \implies \mathcal{F}h\mathcal{F}^{-1} = h^\sigma \implies R(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)) = \mathcal{F}^{-1}$.

Case (iii): $\sigma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} x \in \mathbb{F}_p$.

Subcase (1): $u = (l_1, 0)$, $h^\sigma = (\nu_\sigma(u), (l_1, xl_1))$, $u\sigma = (l_1, xl_1)$;
$\Psi(u\sigma, u\sigma) = \Psi((l_1, xl_1), (l_1, xl_1)) = B(l_1, xl_1)$,
$\Psi(u, u) = \Psi((l_1, 0), (l_1, 0)) = 0$,
so $\nu_\sigma(u) = b(l_1, l_1)^{x/2}$;
$(h^\sigma f)(l) = b(l_1, l_1)^{x/2} b(l, xl_1)f(l + l_1) = b(l_1, l_1)^{x/2} b(l, l_1)^x f(l + l_1)$;
$(hf)(l) = f(l + l_1)$.
Claim $(R(\sigma)f)(l) := b(l, l)^{x/2} f(l) = e^{\pi i x B(l,l)/p} f(l)$ works:

$$\begin{aligned}(R(\sigma)^{-1}hR(\sigma)f)(l) &= (R(\sigma)^{-1}h)(f(l)b(l, l)^{x/2}) \\ &= R(\sigma)^{-1} f(l + l_1)b(l + l_1, l + l_1)^{x/2} \\ &= b(l, l)^{-x/2} b(l + l_1, l + l_1)^{x/2} f(l + l_1) \\ &= b(l, l_1)^x b(l_1, l_1)^{x/2} f(l + l_1) \\ &= (h^\sigma f)(l)\end{aligned}$$

Now use Bruhat to define $R(\sigma)$ for arbitrary σ ($\gamma \neq 0$):
If $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \alpha\gamma^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} 1 & \delta\gamma^{-1} \\ 0 & 1 \end{pmatrix}$,
define $R(\sigma) = R(...)R(...)R(...)$.

Verify: if $\sigma_1, \sigma_2, \sigma_1\sigma_2 \in G \setminus B$, then $c(\sigma_1, \sigma_2)R(\sigma_1\sigma_2) = R(\sigma_1)R(\sigma_2)$ with

$$c(\sigma_1, \sigma_2) = \begin{cases} +1 & \text{(I)} \to \text{done} \to \text{principal series} \\ -1 & \text{(II)} \to \text{replace R by } -\text{R on G} \setminus \text{B} \to \text{cuspidal series} \end{cases}$$

(Something missing here....)

Consider characters of the circle group

$$\lambda : K = \{\gamma \in \mathbb{F}_{q^2}^\times : \mathbb{N}\gamma = 1\} \to \mathbb{C}^\times.$$

If $\lambda^2 \neq 1$, V_λ is irreducible of dimension $p - 1$. If $\lambda^2 = 1$, V_λ splits into V_λ^+, V_λ^- each of dimension $\frac{p-1}{2}$. These give the missing representations. Case (I) recovers what we did before.