Representation Theory (Fall 2004)
Lectures 3 and 4

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Representations of Abelian Groups

Claim. $Hom(V, W) \simeq V^* \bigotimes W$

Proof: Examine the map

$$v^* \bigotimes w \longmapsto (v \longmapsto v^*(v)w)$$

Let $v_1, v_2, ..., v_n$ be a basis for V, and $v_1^*, v_2^*, ..., v_n^*$ a basis for V^* so that $v_i^*(v_j) = \delta_{ij}$. Let $w_1, w_2, ..., w_m$ be a basis for W. Then by using our map to identify tensors in $V^* \bigotimes W$ as homomorphisms from V to W, we have

$$(v_i^* \bigotimes w_j)(v_k) = v_i^*(v_k)w_j = \begin{cases} w_j & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

As matrices, we can depict this as follows:

$$\begin{array}{c} \text{col i} \\ \text{row j} \begin{pmatrix} 0 & \dots & & 0 & & \dots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \dots & & 0 & & \dots & 0 \end{pmatrix} \end{array} \text{row k} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

This defines a G-action on Hom(V,W). Let $\varphi \in$ Hom(V,W). Then

$$\varphi \longleftrightarrow v^* \bigotimes w \Rightarrow (g\varphi) \longleftrightarrow gv^* \bigotimes gw$$

$$\begin{aligned} \Rightarrow (g\varphi)(v) &= (gv^*)(v)(gw) \\ &= v^*(g^{-1}v)(gw) \\ &= (gv^*)(g^{-1}v)w \\ &= g(\varphi(g^{-1}v)) \end{aligned}$$

So,

$$(g\varphi)(v) = g(\varphi(g^{-1}v))$$

Notes by Bill Kalahurka; edited by John Garza.

Or,

$$(g\varphi)(gv) = g(\varphi(v))$$

So, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{g} & W \\ \varphi\downarrow & & \downarrow g\varphi \\ W & \xrightarrow{g} & W \end{array}$$

Claim. $Hom(V,W)^G = Hom_G(V,W)$

Proof: $\varphi \in Hom(V,W)^G \Leftrightarrow \varphi(gv) = g\varphi(v)$for all $v \in V \Leftrightarrow \varphi$ is G-linear $\Leftrightarrow \varphi \in Hom_G(V,W)$

Theorem (Schur's Lemma). *If $\varphi : V \longrightarrow W$, φ is G-linear and V and W are both irreducible representations, then*

1. *φ is either an isomorphism, or the zero map*
2. *If V=W, then $\varphi = \lambda id_V$, where $\lambda \in \boldsymbol{C}$*

Proof:

1. If either $Im(\varphi) = 0$ or $ker(\varphi) = V$, then $\varphi = 0$. If $\varphi \neq 0$, then $Im(\varphi) = W$, and $ker(\varphi) = 0$. Thus, φ is an isomorphism.
2. Let λ be an eigenvalue of φ. Then $0 \subsetneq ker(\varphi - \lambda id_V)$. So, $ker(\varphi - \lambda id_V) = V$. Therefore, $\varphi = \lambda id_V$ $\square$

Corollary *If G is abelian and V is an irreducible representation, then dim(V) = 1.*
Proof: Let $\rho : G \longrightarrow GL(V)$. Then $\rho(g) : V \longrightarrow V$ is G-linear.
By Schur's lemma,

$$\rho(g) = \lambda_g id_V \qquad \text{where}(\lambda_g \in (C))$$

Thus, every subspace of V is G-stable. So, $dim(V) = 1$. $\square$

Claim. *Let V be a finite dimensional representation of a finite group over $\mathbb{C}$. Then*

$$V = V_1^{\bigoplus a_1} \bigoplus \cdots \bigoplus V_N^{\bigoplus a_N}$$

Where $V_1, ..., V_N$ are irreducible representations, $a_1, ..., a_N \in \mathbb{Z}_{\geq 0}$, and $V_i^{\bigoplus a} = \bigoplus_{j=1}^{a} V_i$.

Proof: *Exercise.* (Use Schur's Lemma).

Under this decomposition, $V_i^{\bigoplus a_i}$ is called the *V_i-isotypical component of V*, and a_i is called the *multiplicity* of V_i in V. Also, $a_i = dim(Hom_G(V_i, V))$. Since,

$$\begin{aligned} & V = \bigoplus_{j=1}^{N} V_j^{\bigoplus a_i} \\ \Rightarrow\quad & Hom_G(V_i, V) = \bigoplus_{j=1}^{N} Hom_G(V_i, V_j)^{\bigoplus a_j} \\ \Rightarrow\quad & dim(Hom_G(V_i, V)) = \sum_{j=1}^{N} a_j Hom_G(V_i, V_j) = a_i. \end{aligned}$$

The last equality follows from the fact that when V and W are irreducible,

$$dim(Hom(V,W)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}$$

If V is a representation of the finite group G, then the map $\varphi = \frac{1}{|G|}\sum_{g\in G} \in End(V)$ is G-linear. Since For all $h \in G$, $\sum_{g\in G} h^{-1}gh = \sum_{g\in G} g$

Proposition 1. *$\varphi : V \longleftarrow V^G \subseteq V$ is a projection, and V^G is the trivial isotypical component of V.*

Proof: Let $v = \varphi(w)$. Then

$$hv = h\varphi(w) = \frac{1}{|G|}\sum_{g\in G} hg(w) = \varphi(w) = v$$

This shows $v \in V^G$. On the otherhand, if $v \in V^G$, then $\varphi(v) = v$. So, $\varphi \circ \varphi = \varphi$. □

So, $tr(\varphi) = dim(V^G)$. But also, $tr(\varphi) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)$. So,

$$(*) \boxed{dim(V^G) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)}$$

In particular, if V is an irreducible, non-trivial representation, then

$$\sum_{g\in G} \chi_V(g) = 0$$

Character table for S_3:

	1	3	2
S_3	1	(12)	(123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

Notice,

$$1 \times 1 - 1 \times 3 + 1 \times 2 = 0$$
$$2 \times 1 + 0 \times 3 - 1 \times 2 = 0$$

Now, Recall $\chi_{Hom(V,W)} = \overline{\chi_V} \cdot \chi_W$. Let V and W be irreducible, and apply (*) to $Hom(V,W)$.

$$\begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases} = dim(Hom(V,W)^G) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi}_V(g)\chi_W(g)$$

Notice

$$\{f : G \longrightarrow \mathbb{C}\} \cong \mathbb{C}^{|G|} \supseteq \text{class functions}$$

Define an inner product on this space:

$$(\alpha, \beta) \equiv \frac{1}{|G|}\sum_{g\in G} \overline{\alpha}(g)\beta(g)$$

So, if V and W are irreducible then

$$(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$$

i.e. the irreducible representations of a group are an orthonormal set.

Corollary 1. *A representation V is determined, up to isomorphism, by its character*

Proof: If $\chi_V = \sum_i a_i \chi_{V_i}$, then the a_i's are uniquely determined by $a_i = (\chi_V, \chi V_i)$
Furthermore, $(\chi_V, \chi_V) = \sum_{i=1}^N a_i^2$, which establishes:

Corollary 2. *V is an irreducible representation $\Leftrightarrow (\chi_V, \chi_V) = 1$.*

Example: Looking at the character table of S_3 we can verify that V is irreducible:

$$\frac{1}{6}(2^2 \times 1 + 0^2 \times 3 + (-1)^2 \times 2) = \frac{6}{6} = 1$$

Notice that, dim{class functions} = the number of conjugacy classes, which establishes that

Corollary 3. *# irreducible representations $\leq$ # conjugacy classes*

Corollary 4. *The multiplicity of V_i in V_{reg} is $dimV_i$*

This is because:

$$\chi_{reg}(g) = \begin{cases} |G| & g = id \\ 0 & otherwise \end{cases}$$

So,

$$(\chi_{V_i}, \chi_{V_{reg}}) = \frac{1}{|G|}(|G| \times dimV_i) = dimV_i$$

Furthermore,

$$V_{reg} = \bigoplus_{i=1}^N V_i^{\text{dim}V_i} \Rightarrow |G| = \sum_{i=1}^N (\text{dim}V_i)^2$$

More generally,

$$\begin{cases} 1 & g = 1 \\ 0 & otherwise \end{cases} = \delta_1(g) = \chi_{reg}(g) = \sum_i (\text{dim}V_i)\chi(g) = \sum_\chi \chi(1)\chi(g)$$

Exercise challenge: $\chi(1) \mid |G|$
In the case where we have an abelian group A, we have

$$\frac{1}{|A|}\sum_{a \in A} \overline{\chi}(a)\chi'(a) = \begin{cases} 1 & \chi = \chi' \\ 0 & \chi \neq \chi' \end{cases}$$

And here, χ and χ' are group homomorphisms from A to $\mathbb{C}^\times$.

Lemma 1. *If V is irreducible and W is one dimensional, then $V \bigotimes W$ is irreducible.*

Proof. $\chi_{V \bigotimes W} = \chi_V \cdot \chi_W$

$$\Rightarrow (\chi_{V \bigotimes W}, \chi_{V \bigotimes W}) = \sum_{g \in G} |\chi_V(g)|^2 |\chi_W(g)|^2 = (\chi_V, \chi_V) = 1 \qquad \square$$

Now, we try to determine the character table for S_4. We always have the trivial representation, and for S_n, we always have the alternating representation:

	1	6	8	6	3
S_4	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1

We can look for other representations by looking for permutation representations. The natural action of S_4 on 1,2,3,4 gives a representation with character $(4, 2, 1, 0, 0)$. By subtracting the standard representation, we get a representation V, with character $(3, 1, 0, -1, -1)$ which we can quickly determine to be irreducible:

$$\frac{1}{24}(3^2 \cdot 11 + 1^2 \cdot 16 + 0^2 \cdot 18 + (-1)^2 \cdot 16(-1)^2 \cdot 13) = 1$$

(In general, the permutation representation of S_n acting on n things is a direct sum of the trivial representation and another irreducible representation, which is called the standard representation). By the lemma, $V \bigotimes U'$ is irreducible with character $(3, -1, 0, 1, -1)$. We now have exactly four irreducible representations, and five conjugacy classes, so we must have at most one more representation to be found. Using the formulae after corollary 4 above, we can deduce the characters of the last representation W,e.g.,

$$1^2 + 1^2 + 3^2 + 3^2 + (\dim W)^2 = 24 \Rightarrow \dim W = 2.$$

And,

$$1 \cdot 1 + 1 \cdot 1 + 3 \cdot (-1) + 3 \cdot (-1) + 2 \cdot \chi_W((12)(34)) = 0 \Rightarrow \chi_W((12)(34)) = 2.$$

So, we have

S_4	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V'	3	1	0	-1	-1
$V \bigotimes U' = V'$	3	-1	0	1	-1
W	2	0	-1	0	2

Another permutation representation is the one induced by the conjugation action of S_4 on its two cycles, i.e, if $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$, then S_4 acts on $X = V_4 \setminus \{1\}$. To find the character of this representation, we want to find out how many points of X are fixed by elements of each conjugacy class of S_4

	(12)(34)	(13)(24)	(14)(23)
1	fixed	fixed	fixed
(12)	fixed	(14)(23)	(13)(24)
(123)	(14)(23)	(12)(34)	(13)(24)
(1234)	(14)(23)	fixed	(12)(34)
(12)(34)	fixed	fixed	fixed

So, if χ is the character of this representation, we see that
$\chi = (3,1,0,1,3) = (1,1,1,1,1) + (2,0,-1,0,2)$, which shows that the permutation representation on $X = U \bigoplus W$. We now turn our attention to A_4. Note that the 2-cycles in A_4 now split into two distinct conjugacy classes.

	1	4	4	3
A_4	1	(123)	(132)	(12)(34)
U	1	1	1	1

Any representation of a group G is also a representation of any subgroup of G. In particular, the standard representation V of S_4, has character $(3,0,0,1)$ in A_4, and in fact, it remains irreducible in A_4. Since the sum of the squares of the dimensions of the irreducibles must equal 12, there must be exactly two more irreducibles, each of dimension 1. So, these representations are homomorphisms from A_4 into $\mathbb{C}^\times$. The commutator subgroup $A_4{}'$, of A_4 is V_4. Since $A_4/V_4 \simeq \mathbb{Z}/3\mathbb{Z}$, the character of a one dimensional representation must consist only of cube roots of unity:

	1	4	4	3
A_4	1	(123)	(132)	(12)(34)
U	1	1	1	1
	1	ζ_3	ζ_3^2	1
	1	ζ_3^2	ζ_3	1
V	3	0	0	1

Proposition 2. *Let $\alpha : G \longrightarrow \mathbb{C}$, and define*

$$\varphi_\alpha \equiv \sum_{g \in G} \alpha(g) g \in End(V)$$

Then φ_α is G-linear for all representations V if and only if α is a class function.

Proof: First assume α is a class function, then

$$h^{-1}\varphi_\alpha h = \sum_{g \in G} \alpha(g) h^{-1} g h = \sum_{g \in G} \alpha(hgh^{-1}) g = \sum_{g \in G} \alpha(g) g = \varphi_\alpha$$

Conversely, suppose $\sum_{g \in G} \alpha(hgh^{-1})g = \sum_{g \in G} \alpha(g)g$, for all V. Take $V = V_{reg}$ - the regular representation. Let e_g be the vector associated with the element g in the regular representation, now apply both sides to e_1:

$$\sum_{g \in G} \alpha(hgh^{-1}) e_g = \sum_{g \in G} \alpha(g) e_g$$

$$\Rightarrow \alpha(hgh^{-1}) = \alpha(g) \text{for all } g \in G \qquad \square$$

Another way to say this is that the center of $\mathbb{C}[G] = \{\sum_{g \in G} \alpha(g) g | \alpha \text{is a class function}\}$.

Corollary 1. *For any finite group G there are as many irreducible representations as conjugacy classes.*

Proof: $\{\chi_V | V is irreducible\} \subseteq \{\text{class functions}\}$. Taking dimensions shows that the number of irreducible representations is less than or equal to the number of conjugacy classes.
Let α be a class function such that $(\alpha, \chi_V) = 0$ for all irreducible V. $\varphi_\alpha = \sum_{g \in G} \overline{\alpha}(g) g$ is G linear

on any V. If V is irreducible, then by Schur's lemma, $\varphi_\alpha = \lambda id_V$. Taking traces, we have $0 = \sum_{g \in G} \overline{\alpha}(g)\chi_V(g) = \lambda \text{dim} V$. So $\lambda = 0$, therefore $\varphi_\alpha = 0$ on any V, in particular for V_{reg}, so $\alpha = 0$ $\square$

Let V and W be representations, with W irreducible, and let V_W be the isotypical component of V in W. Then,

$$(\frac{\text{dim} W}{|G|} \sum_{g \in G} \overline{\chi}_W(g) g) : V \longrightarrow V_W$$

$$= \sum \text{subrepresentations of V isomorphic to W}$$

In the case where G is abelian,

$$(\frac{1}{|G|} \sum_{g \in G} \overline{\chi}(g) g) : V \longrightarrow V_\chi$$

As an example, let $G = \langle \tau \rangle \simeq \mathbb{Z}/3\mathbb{Z}$, and let V be the three-dimensional representation achieved by cyclic permutation of coordinates, i.e.

$$\begin{array}{cccc} \tau : & (1,0,0) & \longmapsto & (0,1,0) \\ & (0,1,0) & \longmapsto & (0,0,1) \\ & (0,0,1) & \longmapsto & (1,0,0) \end{array}$$

Then we should have that:

$$\begin{aligned} \varphi_1 \equiv \frac{1}{3}(1 + \tau + \tau^2) \quad &: \quad V \longrightarrow V_1 \\ \varphi_{\zeta_3} \equiv \frac{1}{3}(1 + \zeta_3^2 \tau + \zeta_3 \tau) \quad &: \quad V \longrightarrow V_{\zeta_3} \\ \varphi_{\zeta_3^2} \equiv \frac{1}{3}(1 + \zeta_3 \tau + \zeta_3^2 \tau) \quad &: \quad V \longrightarrow V_{\zeta_3^2} \end{aligned}$$

And indeed this is correct, since φ_1 maps each basis vector to $(1/3)(1,1,1)$, φ_{ζ_3} maps each basis vector to $(1/3)(1, \zeta_3^2, \zeta_3)$ and $\varphi_{\zeta_3^2}$ maps each basis vector to $(1/3)(1, \zeta_3, \zeta_3^2)$.

We want to prove that $\chi(1) \mid |G|$. Given $t \in G$, define the set $U_t \subseteq G^k \ timesG^k$ as follows:

$$U_t = \{((x_1, ..., x_k), (y_1, ...y_k)) | [x_1, y_1] \ldots [x_k, y_k] t = 1\}$$

where $[x, y] = xyx^{-1}y^{-1}$. Define $N_k(t)$ to be the cardinality of U_t.

Claim.

$$N_k(t) = \sum_\chi (\frac{|G|}{\chi(1)})^{2k-1} \chi(t)$$

Proof: First, we need to show the following:

$$\frac{1}{|G|} \sum_{x \in G} \chi(xyx^{-1}z) = \frac{\chi(y)\chi(z)}{\chi(1)}$$

Let

$$\varphi_y = \frac{1}{|G|} \sum_{x \in G} xyx^{-1} \in \text{End} V$$

Then φ_y is G-linear:

$$z\varphi_y z^{-1} = \frac{1}{|G|}\sum_{x\in G} zxyxz^{-1} = \frac{1}{|G|}\sum_{x\in G} xyx^{-1} = \varphi_y$$

By Schur's lemma, $\varphi_y = \lambda id_V$.

$$\chi(y) = \frac{1}{|G|}\sum_{x\in G}\chi(xyx^{-1}) = \text{Trace}(\varphi_y) = \lambda \text{dim}V$$

(recall, $tr(xyx^{-1}) = tr(y)$). So,

$$\frac{1}{|G|}\sum_{x\in G}\chi(xyx^{-1}z) = \text{tr}(\varphi_y z) = \lambda tr(z) = \frac{\chi(y)\chi(z)}{\chi(1)}$$

Now, let $z = y^{-1}$, sum over y, and divide by $|G|$:

$$\frac{1}{|G|^2}\sum_{x,y\in G}\chi([x,y]) = \sum_{y\in G}\frac{\chi(y)\chi(y^{-1})}{\chi(1)} = \frac{(\chi,\chi)}{\chi(1)} = \frac{1}{\chi(1)}$$

Now, by letting $\varphi = \sum x, y[x,y]$, and showing that φ is G-linear, using Schur's lemma to show it is a scalar, and then taking the trace of φt, we could similarly show that

$$\frac{1}{|G|^2}\sum_{x,y\in G}\chi([x,y]t) = \frac{\chi(t)}{\chi(1)^2}$$

By induction, we get

$$\frac{1}{|G|^{2k}}\sum_{x_1,\ldots,x_k,y_1,\ldots,y_k}\chi([x_1,y_1]\ldots[x_k,y_k]t) = \frac{\chi(t)}{\chi(1)^{2k}}$$

Recall,

$$\delta(g) = \begin{cases} 1 & g = 1 \\ 0 & g \neq 1 \end{cases}$$

We already have shown

$$\delta(g) = \frac{1}{|G|}\sum_{\chi}\chi(1)\chi(g)$$

More generally,

$$\frac{N_k(t)}{|G|^{2k}} = \sum\chi([x_1,y_1]\ldots[x_k,y_k]t) = \frac{1}{|G|}\sum_{\chi}\chi(1)\frac{\chi(t)}{\chi(1)^{2k}}$$

So,

$$N_k(t) = \sum_{\chi}(\frac{|G|^{2k-1}}{\chi(1)})\chi(t)$$

If the size of U_t is uniformly distributed over $t\in G$, then $N_k(t) = |G|^{2k-1}$.