Representation Theory (Fall 2004)
Lecture 8

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Representations over $\mathbb{R}$ (cont.)

Let $G = H_8/\{\pm 1\}$. Then G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and therefore its irreducible representations are one dimensional. Now all of the elements of G have order at most two. This means that the characters of G can only take the values ± 1. By projecting H_8 onto G we obtain four different representations of H_8. Since H_8 has only five conjugacy classes, we can obtain the fifth character from the first four. Here is the character table for H_8:

H_8	1	-1	$\pm i$	$\pm j$	$\pm k$
	1	1	1	1	1
	1	1	1	-1	-1
	1	1	-1	1	-1
	1	1	-1	-1	1
R	2	-2	0	0	0

In the previous section we found a representation of H_8 as a subgroup of $SU(2)$. We could find the 2-dimensional representation R directly from taken traces of the following matrices:

$$
\begin{aligned}
1 &\overset{R}{\mapsto} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
-1 &\overset{R}{\mapsto} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
i &\overset{R}{\mapsto} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
j &\overset{R}{\mapsto} \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \\
k &\overset{R}{\mapsto} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\end{aligned}
$$

D_4 also has a subgroup of order 2 such that $D_4/\{1, \sigma\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then D_4 and H_8 have the same character table, even though they are not isomorphic. D_4 acts on $\mathbb{R}^2$ as the symmetries of the square and therefore is a real representation. We have then shown that two groups can have

Notes by Miguel Rodrguez; edited by Adriana Salerno the same character table, yet one of them may have real representations while the other does not. We would like to find a way of determining when a representation with real character is real.

Let G be some finite group with a representation V. If $\chi_V(g) \in \mathbb{R}$ for all $g \in G$ then $\overline{\chi_V(g)} = \chi_V(g)$. This means that $\chi_{V^*}(g) = \chi_V(g)$, which is true if, and only if, V is isomorphic to V^* as a representation of G. Let ϕ be a G-linear isomorphism of V and V^*. We can construct a bilinear pairing $B : V \times V \to V$, $(u, v) \mapsto \phi(u)(v) = B(u, v)$. This B is G-stable since $B(gu, gv) = \phi(gu)(gv) = g\phi(u)(gv) = \phi(u)(g^{-1}gv) = \phi(u)(v) = B(u, v)$. B is also non-degenerate.

Now suppose V is irreducible and suppose $\psi : V \to V^*$ is some G-linear isomorphism. If ϕ is as in the previous paragraph, then $\phi^{-1} \circ \psi : V \to V$ is an isomorphism and, by Schur's Lemma, a scalar. Then, if V is an irreducible representation of G with real character χ_V, V has a unique bilinear ,non-degenerate and G-stable pairing, up to scalars.

Suppose V is an irreducible representation of G. Then V has a G-stable Hermitian pairing, H, which is positive definite.

Exercise. *Prove H is unique up to a scalar, if V is an irreducible representation.*

Now we have a conjugate linear map from V to V^*, $u \mapsto H(u, \cdot)$. We can produce a conjugate linear automorphism of V, ψ, which is G-stable, as follows: given $u \in V$ there exists $\psi(u) \in V$ such that $B(u, \cdot) = H(\psi(u), \cdot)$.

Claim. *$B(u, v) = \epsilon B(v, u)$, where $\epsilon^2 = 1$.*

Proof. Define $B'(u, v) = B(v, u)$. By the uniqueness up to scalar of B, $B' = \lambda B$, $\lambda \in \mathbb{C}$. Then $\lambda^2 = 1$. □

$\psi \circ \psi = \psi^2 : V \to V$ is $\mathbb{C}$-linear. Then, by Schur's lemma, $\psi^2 = \lambda id_V$. By definition, $H(\psi(u), v) = B(u, v) = \epsilon B(v, u) = \epsilon \overline{H(u, \psi(v))}$. Applying this step twice we can show that $H(\psi^2(u), v) = H(u, \psi^2(v))$. This implies $\overline{\lambda} H(u, v) = \lambda H(u, v)$, and so $\lambda = \overline{\lambda}$. Then $\lambda \in \mathbb{R}$.

Take $v = \psi(u)$. Then, $H(\psi(u), \psi(u)) = \epsilon \overline{H(u, \psi^2(u))} = \epsilon\lambda H(u, u)$. If $u \neq 0$ then $\psi(u) \neq 0$. $H(u, u) > 0$ and so $H(\psi(u), \psi(u)) > 0$ and $sgn(\lambda) = \epsilon$. Scale by $|\lambda|$ so that $\lambda = \epsilon$. Then, $\psi : V \to V$ can be taken such that it is conjugate linear, G-linear and $\psi^2 = \epsilon id_V$.

Suppose $\epsilon = 1$, that is, B is symmetric. Since $\psi^2 = 1$, $V = V_+ \oplus V_-$, where V_+ and V_- are the eigenspaces of 1 and -1 respectively (regarding V as a vector space over $\mathbb{R}$). $\psi(iu) = -i\psi(u)$. Multiplication by i is an $\mathbb{R}$-linear homomorphism from V to itself. If $u \in V_+$, $\psi(u) = u$, then, $\psi(iu) = -i\psi(u) = -iu$ and $-iu \in V_-$. $i : V_+ \to V_-$, $i : V_- \to V_+$, are $\mathbb{R}$-linear isometries. Then $V = V_+ \oplus iV_+ = V_+ \otimes_{\mathbb{R}} \mathbb{C}$, as a complex vector space, i.e. V is real.

Conversely, if V is a real irreducible representation, then there exists a non-degenerate bilinear G-stable pairing, symmetric on V: $(u, v) = \sum_{g \in G} \langle gu, gv \rangle$, $\langle , \rangle = B$.

Now suppose $\epsilon = -1$, i.e. B is skew-symmetric. Then $\psi^2 = -1 id_V$ and V is not real since it does not have a bilinear symmetric pairing.

Claim. *V is a module over $\mathbb{H}$.*

Proof. $j \mapsto \psi$ gives an action of $\mathbb{H}$ on V. Then V is an $\mathbb{H}[G]$-module. □

An $\mathbb{H}$-module is a $\mathbb{C}$-module ($\mathbb{C} = \mathbb{R} + i\mathbb{R} \subset \mathbb{H}$). Multiplication by j gives us a homomorphism, $j : V \to V$, where $j(iv) = jiv = -ij(v) = -i(jv)$; i.e. multiplication by j is conjugate linear.

Summarizing: If V is an irreducible representation, then

$$V = \begin{cases} \text{complex} & \chi_V \notin \mathbb{R} \\ \text{real} & \chi_V \in \mathbb{R} \text{ and } V \text{ real} \\ \text{quaternionic} & \chi_V \in \mathbb{R} \text{ and } V \text{ an } \mathbb{H}-\text{module} \end{cases}$$

Schur Indicator:

Let $\chi_V \in \mathbb{R}$. We know

$$(\chi_{\wedge^2 V^*}, 1) = \begin{cases} 0 & B \text{ symmetric} \\ 1 & B \text{ skew symmetric} \end{cases}$$

We also know

$$(\chi_{\wedge^2 V^*}, 1) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)^2 - \chi_V(g^2)$$

And since the character is real

$$(\chi_V, \chi_V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)^2 = 1$$

Therefore, what will determine whether the representation is real or not, called the Schur indicator, is the following:

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) = \begin{cases} 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic} \\ 0 & \text{if } V \text{ is complex} \end{cases}$$