Representation Theory (Fall 2004)
Lecture 9, 10

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Summary of Real Representations

Let V be an irreducible representation of a group G. We have the following results:
1) V admits a Hermitian, positive definite, G-stable form H (i.e. $H(v,v) > 0$ for any $0 \neq v \in V$, $H(\lambda u, v) = H(u, \bar{\lambda} v) = \bar{\lambda} H(u,v)$ and $H(g \cdot u, g \cdot v) = H(u,v)$ for $g \in G$) unique up to scalars.
2) If χ_V is real, then V admits a bilinear, non-degenerate, G-stable form B which is unique up to scalars, such that $B(u,v) = \varepsilon B(v,u)$ where $\varepsilon = \pm 1$.
3)If $\varepsilon = +1$, then V is real, that is, $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, where V_0 is a real vector space and G acts on it. Moreover, we can scale B to make it positive definite. If $\varepsilon = -1$, then V is quaternionic, i.e. V is an $\mathbb{H}[G]$-module.

Schur Indicator

Theorem. *Let V be an irreducible representation of G, then*

$$\frac{1}{|G|}\sum_{g \in G} \chi_V(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex} \\ 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic} \end{cases}.$$

For example, let $G = H_8$, the quaternion group which has a 2-dimensional irreducible representation since it can be embedded as a subgroup of $SU(2)$. We worked out the characters of this representation before.

conjugacy classes	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
characters	2	-2	0	0	0

By simple calculation, the Schur indicator is -1 which means this representation is complex, as we have already seen.

For another example, let $G = D_8$, the dihedral group of order 8, which has a 2-dimensional representation with exactly the same characters as above. Assume $D_8 = \{\sigma, \tau | \sigma^2 = \tau^4 = 1, \tau\sigma = \sigma\tau^{-1}\}$.

conjugacy classes	$\{1\}$	$\{\tau^2\}$	$\{\tau, \tau^3\}$	$\{\sigma, \sigma\tau^2\}$	$\{\sigma\tau, \sigma\tau^3\}$
characters	2	-2	0	0	0

And the Schur indicator for this representation is 1. We have already realized this is a real representation as rotations and flips of a square in a real 2-space.

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Proof. It is well known that

$$V^* \otimes V^* = \mathrm{Sym}^2 V^* \oplus \wedge^2 V^*,$$

thus all G-stable bilinear forms are

$$(V^* \otimes V^*)^G = (\mathrm{Sym}^2 V^*)^G \oplus (\wedge^2 V^*)^G.$$

Three cases of V correspond to the differences of dimensions. See the table below.

	$\dim(\mathrm{Sym}^2 V^*)^G$	$\dim(\wedge^2 V^*)^G$	$\dim(V^* \otimes V^*)^G$
V complex	0	0	0
V real	1	0	1
V quaternionic	0	1	1

If V is real, then $\mathrm{Sym}^2 V^*$ contains a copy of trivial representation. Taking the inner product, we get

$$1 = (\chi_{\mathrm{Sym}^2 V^*}, \chi_{\text{trivial}}) = \frac{1}{|G|}\sum_{g\in G} \chi_{\mathrm{Sym}^2 V^*}(g) = \frac{1}{|G|}\sum_{g\in G} \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2)).$$

Note that $\chi_{V^*} = \overline{\chi}_V = \chi_V$. For an irreducible representation V,

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g)^2 = 1,$$

hence, we obtain

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g^2) = 1.$$

Similarly, if V is quaternionic, we take the inner product of $\chi_{\wedge^2 V^*}$ and χ_{trivial}, then

$$1 = (\chi_{\wedge^2 V^*}, \chi_{\text{trivial}}) = \frac{1}{|G|}\sum_{g\in G} \chi_{\wedge^2 V^*}(g) = \frac{1}{|G|}\sum_{g\in G} \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)),$$

and therefore

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g^2) = -1.$$

If V is complex, then $(\chi_{\mathrm{Sym}^2 V^*}, \chi_{\text{trivial}}) = 0$, and $(\chi_{\wedge^2 V^*}, \chi_{\text{trivial}}) = 0$. Taking their difference, we have

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g^2) = 0.$$

□

Examples and Exercises

Example. χ_V are real for all V if and only if g is conjugate to g^{-1} for every $g \in G$.

Proof. "$\Leftarrow$": Since $\chi(g) = \chi(g^{-1}) = \bar{\chi}(g)$, $\chi(g)$ is real. The second equality is because $\chi(g)$ is the sum of g's eigenvalues as a linear transformation of V, which are roots of unity since g is of finite order, say $\chi(g) = \sum \lambda_i$, then $\chi(g^{-1}) = \sum \lambda^{-1} = \sum \bar{\lambda}_i = \bar{\chi}(g)$.

“$\Rightarrow$”: For any class function f, f is a $\mathbb{C}$-linear combination of irreducible characters since they are a basis of the class function space. Thus

$$f(g^{-1}) = \sum_{\chi \text{ irreducible}} C_\chi \chi(g^{-1}) = \sum C_\chi \chi(g) = f(g).$$

Therefore g and g^{-1} are in same conjugacy class. ☐

Exercise. Let V, W be two representations of G, then:
1) if V is irreducible, then $V \otimes V^*$ is real.
2) if V and W are both real or quaternionic, then $V \otimes W$ is real.
3) if V is real, then $\wedge^k V$ is real.
4) if V is quaternionic, then $\wedge^k V$ is real if k is even, or quaternionic if k is odd.

Example. If G has odd order, then every nontrivial irreducible representation is complex.

Proof. The map $g \mapsto g^2$ is a bijection, hence

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g) = \left\{\begin{array}{ll} 1 & \text{if } \chi = \chi_{\text{trivial}} \\ 0 & \text{otherwise} \end{array}\right. .$$

The result follows from the Schur indicator. ☐

Exercise. Suppose V is an irreducible real representation of G, i.e. $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ and G acts on V_0, then V_0 is necessarily an irreducible representation of G over $\mathbb{R}$. However, the converse is not true in general. For example, let $G = \mathbb{Z}/3\mathbb{Z}$ with generator τ acting on $V_0 = \mathbb{R}^2$ as counterclockwise rotation by $2\pi/3$. This is an irreducible representation since there is no line fixed. Nonetheless, G acting on $V_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ is reducible since there is no irreducible representation of degree higher than 2 for an abelian group. Let V_0 be any irreducible representation over $\mathbb{R}$, show that $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ is either irreducible or $V = W_1 \oplus W_2$ such that $\chi_{W_1} = \bar{\chi}_{W_2}$.

Irreducibility of $\wedge^k V$

Claim. *$\wedge^k V$ is irreducible for $k = 0, 1, \ldots, n$ where V is the standard representation of S_n.*

Proof. Let U be the trivial representation, so $U \oplus V$ is the permutation representation.

$$\wedge^k(U \oplus V) = (\wedge^0 U \otimes \wedge^k V) \oplus (\wedge^1 U \otimes \wedge^{k-1} V) = \wedge^k V \oplus \wedge^{k-1} V.$$

Let χ be the character of $\wedge^k(U \oplus V)$, then it follows from above that it suffices to show $(\chi, \chi) = 2$. Let $e_1, e_2, \ldots, e_n$ be a basis of $U \oplus V$ on which S_n acts by $g \cdot e_i = e_{g(i)}$. Naturally $e_{i_k} \wedge \ldots \wedge e_{i_k}$, $i_1 < \ldots < i_k$ forms a basis of $U \oplus V$ on which S_n acts by $g \cdot e_{i_k} \wedge \ldots \wedge e_{i_k} = e_{g(i_1)} \wedge \ldots \wedge e_{g(i_k)}$. Hence the matrix of g will have 1 or -1 on the diagonal if g maps $\{i_1, \ldots, i_k\}$ to itself. Hence

$$\chi(g) = \sum_B \operatorname{sgn} g|_B,$$

where the sum runs over all $B \subset \mathbb{N}_n = \{1, 2, \ldots, n\}$ such that $|B| = k$ and $g(B) = B$. Thus

$$(\chi, \chi) = \frac{1}{n!}\sum_{g\in S_n}\sum_B\sum_C \operatorname{sgn} g|_B \cdot \operatorname{sgn} g|_C.$$

Interchanging the summation notation,

$$(\chi, \chi) = \frac{1}{n!} \sum_{B,C} \sum_{g} \text{sgn } g|_B \cdot \text{sgn } g|_C.$$

For fixed B,$C \subset \mathbb{N}_n$, $|B|$,$|C| = k$, let's consider

$$\sum_{g} \text{sgn } g|_B \cdot \text{sgn } g|_C,$$

where the sum runs over all $g \in S_n$ which preserves B and C. Obviously, such g preserves $B - C = A_1$, $C - B = A_2$, $B \cap C = A_3$ and $\overline{B \cup C} = A_4$. Consequently, g can be decomposed into $g = g_1 g_2 g_3 g_4$ where $g_i \in S_{A_i}$. Suppose $|B \cap C| = l \leq k$, then

$$\begin{aligned} \textstyle\sum_g \text{sgn } g|_B \cdot \text{sgn } g|_C &= \textstyle\sum_{g_1 \in S_{A_1}} \sum_{g_2 \in S_{A_2}} \sum_{g_3 \in S_{A_3}} \sum_{g_4 \in S_{A_4}} \text{sgn } (g_1 g_3)(g_2 g_3) \\ &= \textstyle\sum_{g_3 \in S_{A_3}} \sum_{g_4 \in S_{A_4}} \left(\sum_{g_1 \in S_{A_1}} \text{sgn } g_1 \right) \left(\sum_{g_2 \in S_{A_2}} \text{sgn } g_2 \right) \\ &= \begin{cases} l!(n + l - 2k)! & \text{if } k - l = 1 \text{ or } 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

If $k - l = 0$, i.e. $B = C$, there are $\binom{n}{k}$ many choices for B and C. Summing over these B and C, we have

$$\frac{1}{n!} \sum_{B=C} \sum_{g} \text{sgn } g|_B \cdot \text{sgn } g|_C = \frac{l!(n + l - 2k)!}{n!} \binom{n}{k} = 1.$$

If $k - l = 1$, i.e. $|B \cap C| = k - 1$, there are $\binom{n}{k-1}(n - k + 1)(n - k)$ choices for B and C. Summing over these B and C, we get

$$\frac{1}{n!} \sum_{|B \cap C| = k-1} \sum_{g} \text{sgn } g|_B \cdot \text{sgn } g|_C = \frac{l!(n + l - 2k)!}{n!} \binom{n}{k-1} (n - k + 1)(n - k) = 1.$$

The result follows easily. □

Classifying Finite Subgroups of $SO(3)$

In this section, we will classify all isomorphism types of finite subgroups of $SO(3)$, the group 3×3 orthogonal matrices of determinant 1, which is also all the rotations in real 3-space. We will use the latter point of view in our approach. Let $G \leq SO(3)$ be a finite subgroup. For any nontrivial element $g \in G$, g fixes exactly 2 points on S^2, namely, the intersection points of the axis of rotation and S^2. Let $\wp$ be the set of points of S^2 fixed by some nontrivial $g \in G$. Then G acts on $\wp$ and so divides it into orbits, say $\wp = \cup_{i=1}^{N} \wp_i$. Let $n = |G|$, $G_i = \text{Stab}_G(\wp_i)$, $n_i = |G_i|$. Hence $|\wp_i| = \frac{|G|}{|G_i|} = \frac{n}{n_i}$. Let's compute $\sum_{P \in \wp} |\{1 \neq g \in G | g \cdot P = P\}|$. On one hand,

$$\sum_{P \in \wp} |\{1 \neq g \in G | g \cdot P = P\}| = \sum_{i=1}^{N} \sum_{P \in \wp_i} |\{1 \neq g \in G | g \cdot P = P\}| = \sum_{i=1}^{N} (n_i - 1) \frac{n}{n_i}.$$

On the other hand,

$$\sum_{P \in \wp} |\{1 \neq g \in G | g \cdot P = P\}| = \sum_{1 \neq g \in G} |\{P \in \wp | g \cdot P = P\}| = 2(n - 1).$$

Thus we establish the equation

$$\sum_{i=1}^{N}(n_i-1)\frac{n}{n_i}=2(n-1).$$

After dividing by $2n$, we get

$$1-\frac{1}{n}=\frac{1}{2}\sum_{i=1}^{N}\left(1-\frac{1}{n_i}\right).$$

Note that n_i divides n and also $n_i \geq 2$ since G does not act on $\wp_i$ freely (some $g \in G$ fixes some $P \in \wp_i$). Thus

$$1>1-\frac{1}{n}=\frac{1}{2}\sum_{i=1}^{N}\left(1-\frac{1}{n_i}\right)\geq\frac{N}{4}.$$

So N =1, 2, or 3. We will assume $n > 1$ since otherwise G is the trivial group and there is no $\wp$ by definition.

Case 1: $N = 1$

$$1-\frac{1}{n}=\frac{1}{2}\left(1-\frac{1}{n_1}\right)\Rightarrow n=2-\frac{n}{n_1}\leq 1\Rightarrow n=1,$$

giving a contradiction.

Case 2: $N = 2$

$$1-\frac{1}{n}=\frac{1}{2}\left(1-\frac{1}{n_1}\right)+\frac{1}{2}\left(1-\frac{1}{n_2}\right)\Rightarrow 2=\frac{n}{n_1}+\frac{n}{n_2}\Rightarrow n=n_1=n_2.$$

Hence $|\wp_1| = |\wp_2| = 1$ and every $g \in G$ fixes these two points and therefore $G \cong \mathbb{Z}/n\mathbb{Z}$.

Case 3: $N = 3$

$$1+\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}.$$

Suppose $2 \leq n_1 \leq n_2 \leq n_3$, since

$$1<1+\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}\leq\frac{3}{n_1},$$

we have $n_1 = 2$. Then the equation becomes

$$\frac{1}{2}+\frac{2}{n}=\frac{1}{n_2}+\frac{1}{n_3}.$$

By a similar estimation,

$$\frac{1}{2}<\frac{1}{2}+\frac{2}{n}=\frac{1}{n_2}+\frac{1}{n_3}\leq\frac{2}{n_2}$$

giving $n_2 = 2$ or 3. If $n_2 = 2$, then $n_3 = m \geq 2$ and $n = 2m$. If $n_2 = 3$, then we will have $3 \leq n_3 < 6$. Finally, we list our results by the following table.

n_1	n_2	n_3	n	G
2	2	$m \geq 2$	$2m$	D_{2m} as rigid motions of m-gon
2	3	3	12	A_4 as rigid motions of tetrahedron
2	3	4	24	S_4 as rigid motions of cube or octahedron
2	3	5	60	A_5 as rigid motions of icosahedron or dodecahedron