HANDOUT 6

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1. Group generators

Given a finite group G and a set $U \subset G$ we denote by $\langle U \rangle$ the subgroup of G *generated* by U. This subgroup can be defined as the smallest subgroup containing U, or, what is the same, the interesection of all the subgroups containing U); alternatively, $\langle U \rangle$ consists of all the elements of G that can be expressed as a word in the elements of U or their inverses.

For example, in the case of the dihedral group $G = D_n$ of symmetries of the regular n-gon, the rotation $U = \{\tau\}$ generates the subgroup of all rotations, whereas $U = \{\sigma, \tau\}$ generates the whole group. Concretely, all rotations are of the form τ^k and all reflections are of the form $\tau^k\sigma$ for some $k = 0, 1, \cdots, n-1$. If $U = \{u_1, u_2, \cdots, u_m\}$ we often write $\langle u_1, u_2, \cdots u_m \rangle$ for $\langle U \rangle$.

2. Permutation Puzzles

A number of puzzles, including the fifteen-puzzle, Rubik's cube and its variants, etc., can be phrased in the language of group actions. We have a finite group G generated by a set U acting on a finite set X. The moves of the puzzle correspond to the generators in U and their inverses. Typically, we start with a random "scrambled" arrangement of X and want to "unscramble" taking it to a prefixed final state. This amounts to being able to figure out given $g \in G$ how to write it explicitly as a word in the generators in U. This word will describe one, of possibly many, sequences of moves that would solve the puzzle.

This description also applies to the linear algebra puzzles of Handout 4. Here the set X consists of all the state vectors s and the move in the puzzle corresponding to a clicking a button b gives the action

$$s \mapsto s + u_b$$

where u_b is a vector describing the neigbhors of b. Hence $U = \{u_b\}$, with b running through all buttons of the puzzle, and G is the group they generate.

In this puzzle we start with a scrambled state s_I and we want to take it to an unscrambled state s_F (for example, all buttons colored white). Doing this means writing $s_F - s_I$ in terms of the generators u_b.

This puzzle is simpler than others of this general type because the group in question is abelian, which has the effect that the order in which we do the moves is irrelevant, and, also because we can use the methods of linear algebra to solve it. For a general permutation puzzle none of these facts is valid and the puzzle could be much harder to solve.

3. A simpler version of the 15 puzzle

To illustrate the above notions consider the following simpler version of the 15 puzzle. The numbers $1, 2, 3, 4$ are arranged in four of the five points consisting of the vertices of a square and its center; the remaining point is the "blank" piece B. The puzzle consists of rearranging the numbers in the form

$$\begin{array}{ccc} 1 & & 2 \\ & B & \\ 4 & & 3 \end{array}$$

by succesively swapping B with any of its neighboring numbers.

Let us label the points as in the final configuration. To simplify the discussion somewhat we will consider only moves that take the blank back to the center of the square. What subgroup H of permutations of the numbers $1, 2, 3, 4$ do we obtain in this way? Do we get all of S_4? Notice that the puzzle will be solvable for any initial configuration only if indeed $H = S_4$.

The group H is generated by the permutations obtained by moving the blank along the four triangles with vertices the center and two of the vertices of the square, say, counterclockwise. If we label these triangles N, E, S, W they correspond to the following permutations in H

$$\begin{aligned} N = &\quad (12) \\ E = &\quad (23) \\ S = &\quad (34) \\ W = &\quad (14) \end{aligned}$$

I claim that these four permutation generate S_4 and hence $H = S_4$ and the puzzle is always solvable. To show this note that if $\sigma \in S_4$ and $\tau = (ij)$ is a transposition then

$$\sigma\tau\sigma^{-1} = (\sigma(i)\sigma(j)). \tag{1}$$

Hence, for example,

$$ENE^{-1} = (13), \qquad (SE)N(SE)^{-1} = (14).$$

In a similar way we may obtain all transpositions with which in turn we get all permutations of S_4.

Remarks In the original 15-puzzle of Sam Lloyd the subgroup of S_{15} of all sequence of moves that take the blank from a given position back to it is *not* all of S_{15} but rather A_{15}. This means that only half of all possible arrangements of the 15 numbers in the 4×4 array can be solved. For example, as Lloyd discovered, swapping, say, the numbers 14 and 15 in the solved puzzle gives an unsolvable configuration. In general, there is a parity condition that determines if a given configuration is solvable or not.

One can define a 15-type puzzle on any simple graph Γ (a graph with no loops or multiple edges). It was proved by Richard Wilson in 1974 that except when Γ is a regular n-gon or the very special graph Γ_0 given below the group of moves of taking the blank back to its initial position is either A_n or S_n.

For the special graph Γ_0 the configurations divide into six different equivalence classes; the group of permutations fixing the blank is of order 120.

Homework (Due Tue May 25)

(1) What is the order of the subgroup of S_4 generated by $(13), (1234)$? Do you recognize this group?
(2) What is the subgroup of S_4 generated by the 3-cycles (i.e. permutations of the form (ijk))?
(3) What is the subgroup of S_n, for $n > 2$, generated by the 3-cycles?
(4) Give the details showing why (1) is valid.
(5) Show that S_n is generated by (12) and $(12\cdots n)$ (Hint: show that you can get all transpositions of the form $(12), (23), (34), \ldots (n1)$ and argue like in the notes above.)
(6) Give a sequence of moves that solves the square puzzle described above if the original configuration is

$$\begin{array}{ccc} 3 & & 1 \\ & B & \\ 2 & & 4 \end{array}.$$

Can you find more than one sequence of moves that solve this case of the puzzle?

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