HANDOUT 5

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1. Nim

In the game of Nim one starts with a certain number of chips distributed into piles. Each of two players alternatively remove chips from some pile and the person to remove the last wins. (Players must remove at least one chip from only one pile.)

This game is the prototype for what are called impartial games, in which players take turns playing and

1. there are no chance moves
2. moves are the same for both players
3. players have complete information on the status of the game
4. a player unable to move loses (normal play)
5. there are no draws and the game ends in a finite number of moves.

Nim was analyzed completely by Ch. Bouton in 1901; this beautiful theory is as follows. Let $n_1, \cdots, n_k$ be the number of chips in the different piles at some point in the game. Label this position $v = (n_1, \cdots, n_k)$. We will call v a *P-position* (P for previous player wins) if the Nim sum

$$n_1 * n_2 * \cdots * n_k = 0,$$

where $c = a * b$, the *Nim sum* of the numbers a and b, is defined as follows. If we write the numbers in binary,

$$a = \cdots a_2 a_1 a_0, \qquad b = \cdots b_2 b_1 b_0,$$

then c has binary expansion

$$c = \cdots c_2 c_1 c_0$$

where

$$c_j \equiv a_j + b_j \bmod 2, \qquad j = 0, 1, 2, \cdots$$

So, for example, the position $v = (5, 7, 2)$ is a P-position since $5 = 101_2, 7 = 111_2, 2 = 10_2$, with Nim sum 0. It helps do the calculation to organize the numbers in a table like this

$$\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

A position that is not a P-position is called an *N-position* (N for next player wins); hence, every position in the game is either P or N.

What we will show is that these positions have the property that

1. Any move from a P-position ends in an N-position.
2. From an N-position there is at least one move that ends in a P-position.

Granted this, the strategy to win the game is to always move to a P-position (whether you can get this started or not depends on the original position of the game if you are the first player and otherwise on how your opponent plays!). The reason is that by moving to a P-position you force your opponent to move to an N-position, from which you can then again move to a P-position and so on until the game ends. (The game ends in finitely many steps, as the total number of chips decreases at every move.) The end position, when no chips are left, is a P-position and hence it must be you who got there, winning the game.

In order to prove the two crucial statements about the N and P-positions we need to look at Nim addition a bit more closely. First, it is not hard to verify that $*$ is a commutative and associative operation on the numbers $0, 1, 2 \cdots$ and hence behaves like ordinary sum except for the unusual property that for any n

$$n * n = 0$$

or *for Nim addition, a number is its own negative.*

To prove property (1) of a P-position note that if

$$n_1 * n_2 * \cdots * n_k = 0$$

then each n_j is *uniquely* determined by the rest. Indeed,

$$\begin{aligned} n_1 &= n_2 * n_3 * \cdots * n_k \\ n_2 &= n_1 * n_3 * \cdots * n_k \\ &\cdots \end{aligned}$$

etc. Since Nim moves change *only one* of the numbers n_j, faced with a P-position, your opponent destroys the property that their Nim sum is zero.

To show property (2) is best to give an example. Say our position is $v = (5, 3, 7, 3, 1)$. We check that $5 * 3 * 7 * 3 * 1 = 3$ so v is an N-position. Writing the numbers in binary in a table we get

$$\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 3 \end{array}$$

Looking at the columns of this table from left to right we find the first for which the sum has a 1. In our case that is the second column from the left. We now pick one of the three numbers $3, 7$ or 2 which also has a 1 in that column. Say we pick 7. Our move will then consist in change this 7 into the unique number which will make the total Nim sum to be zero. In this case, this number is 4. Hence our move will be to take 3 chips from the pile that has 7. Note that in this case we had more than one choice of move to get a P-position; the important point is that there was at least one.

It is not hard to formalize what we did in this example into a mathematical proof. The one thing to be careful about is that though it is clear that with this procedure we will achieve Nim sum zero (and hence a P-position) we must be sure that our move is a legal one for the game. This means that the new number n_j' say of chips in the j-th pile that we chose to modify should be strictly smaller than the original number n_j. This works because the binary digits of n_j and n_j' are related as follows. To the left of j they are equal (why?); at j, n_j has a 1 (that is how we selected it) and n'_j has a zero (because we want the new Nim sum to be zero); to the right of j they are unrelated. This means that $n_j - n'_j = 2^j \pm m$ where m has binary expansion involving digits in the range $0, 1, \ldots, j-1$ only. Hence, $m < 2^j$ and therefore $n'_j < n_j$.

2. Directed graphs

We can represent an impartial game as a directed graph Γ whose vertices are the positions of the game and whose edges correspond to legal moves. We will write $v \mapsto w$ to mean that we can move from position v to position w by a legal move in the game. For example, here is the graph corresponding to the game of Nim with starting position $(2, 1, 1)$.

Note that Γ does not have any circuits as this would correspond to an infinite loop in the game. It can be proved that on any such Γ we can define the notion of N and P positions recursively as follows. 0) First label P all terminal positions, those with no edges coming out of them. 1) Then label N all positions that have an edge ending in a P-position. 2) Now look for any positions (if any) for which all edges stemming out of them end in an N-position; label these P. Repeat 1) and 2) until all positions have been labeled.

This labeling of positions will satisfy the same two main properties as those n in the case of Nim. By an identical reasoning we obtain a strategy to winning the game corresponding to Γ.

3. Subtraction Games

One simple source of impartial games are the subtraction games. These are played as follows. We start with a number n of chips in one pile. A move consist of removing a number s of chips where $s \in S$ for a fixed set S. For example, if $S = \{1, 2, 5\}$ and $n = 10$ a player can move by taking $1, 2$ or 5 chips leaving $9, 8$ or 5 for the other player. Here is the graph Γ for this game.

Following the recursive procedure of the previous section we can label all the positions of the subtraction game with $S+\{1, 2, 5\}$ and find they are $P, N, N, P, N, N, \ldots$ repeating indefinitely. In other words, n is a P position if and only if 3 divides n. So for example, if n was 26 and it was our turn to play we could take 2 chips leaving 24 or take 5 leaving 21 since both 21 and 24 are P-positions.

4. Sum of games

If Γ_1 and Γ_2 are two impartial games we can define their (disjunctive) sum Γ. To play in Γ a player chooses one of the two games Γ_1 or Γ_2 and makes a move in it. As usual a player unable to move loses.

Formally, the positions v of Γ are pairs of positions $v = (v_1, v_2)$ with v_i a position in Γ_i for $i = 1, 2$. We can move from v to v' in Γ if and only if $v' = (v_1', v_2)$ with $v_1 \mapsto v_1'$ or $v' = (v_1, v_2')$ with $v_2 \mapsto v_2'$.

As an illustration, consider the subtraction games Γ_i for $i = 1, 2$ with $S_1 = \{1, 2\}$ and $S_2 = \{1, 3\}$. We easily check that the positions in Γ_i are labeled $P, N, N, P, N, N, \ldots$ and $P, N, P, N, \ldots$ respectively. The positions in $\Gamma = \Gamma_1 * \Gamma_2$ can be put in an array with, say, positions in Γ_1 going horizontally and positions in Γ_2 vertically.

⋮	⋮	⋮	⋮	⋮	⋮	
N	P	N	N	P	N	...
P	N	N	P	N	N	...
N	P	N	N	P	N	...
P	N	N	P	N	N	...
N	P	N	N	P	N	...
P	N	N	P	N	N	...

Notice that the label of the positions in Γ is not uniquely determined by the labels of the corresponding positions in Γ_1 and Γ_2. For example, in Γ_1 1 and 2 are labeled N and in Γ_2 1 is labeled N. However, in Γ $(1,1)$ is P and $(2,1)$ is N.

In order to easily compute the labeling of positions in Γ we need more information on the positions of Γ_1 and Γ_2 than their labels. This is provided by the Grundy function described in the next section.

5. Sprague-Grundy theory

We define the Grundy function G_Γ of an impartial game Γ recursively by the formula

$$G_\Gamma(v) = \operatorname{mex}\{G_\Gamma(w) \mid v \mapsto w\}.$$

Here $\operatorname{mex} S$, where S is a set of non-negative integers, is the smallest number *not* in S. (The name mex comes from the term *minimum-excludant.*)

For example, if v is a terminal position $G_\Gamma(v) = 0$, since by definition $\{w \mid v \mapsto w\}$ is the empty set. For the subtraction game with $S = \{1, 2\}$ the values of G_Γ are easily seen to be $0, 1, 2, 0, 1, 2, \dots$

The key property of mex that we need is that $\operatorname{mex} S = 0$ if and only if $0 \notin S$. This implies that: 1) $G_\Gamma(v) = 0$ if and only if $G_\Gamma(w) \neq 0$ for all w with $v \mapsto w$ and (equivalently) 2) $G_\Gamma(v) > 0$ if and only if $G_\Gamma(w) = 0$ for some w with $v \mapsto w$. We see that if we label P those positions v with $G_\Gamma(v) = 0$ and N those with $G_\Gamma(v) > 0$ this labeling is exactly the one we introduced before.

In other words, we have attached a numerical value to each position in the game from which, in particular, we can read off its label N or P. We can interpret the value of the Grundy function of a position as the size of a pile in a game of Nim (but with the possibility that the rules of the game may allow the pile to increase size).

The beauty of the extra information that the Grundy function carries is contained in the following theorem, which greatly extends the analysis of Nim by Bouton.

Theorem 1. *(Sprague-Grundy) Let $\Gamma_1, \dots, \Gamma_k$ be impartial games with Grundy functions $G_{\Gamma_1}, \dots, G_{\Gamma_k}$ respectively. If $\Gamma = \Gamma_1 * \cdots * \Gamma_k$ is the sum of these games then its Grundy function G_Γ is the Nim sum of the G_{Γ_i}; i.e.,*

$$G_\Gamma = G_{\Gamma_1} * \cdots * G_{\Gamma_k}$$

For example, applying the theorem to the two subtraction games of the previous section we find that, say,

$$G_\Gamma((4,3)) = G_{\Gamma_1}(4) * G_{\Gamma_2}(3) = 1 * 1 = 0$$

agreeing with our previous calculation that $(4, 3)$ is a P-position.

Homework (Due Thur April 8)

(1) Make a table of Nim addition for all numbers $n = 0, 1, \ldots, 10$. (Or, even better, write the code in your favorite programming language to compute the Nim sum of two numbers.)

(2) Compute the Grundy function for the subtraction game with $S = \{4, 10, 12\}$. (The function is periodic with what period?)

(3) Let Γ_1 be the substraction game with $S = \{1, 2, 5\}$ and Γ_2 the substraction game of the previous problem. Let Γ be the sum of Γ_1 and Γ_2. Draw the diagram of N and P positions (n, m) for Γ (like the one in page 5 of this handout) for say $1 \leq n, m \leq 6$. Verify by hand the statement of theorem 1 for this range.

(4) Consider the following impartial game. We start with an odd number of piles with n chips each. Players alternate removing 1 or 2 chips from one (and only one) pile. What would be your strategy if you are the first player? (Explain your reasoning.)

(5) Give all possible good moves in a game of Turning Turtles with position

$$T \quad T \quad H \quad H \quad T \quad H \quad H$$

(6) What would be your move in the following game of Hackenbush?

(7) Grundy's game is played starting with a pile with n chips. A move in the game is to take a pile and break it into two piles of *unequal* size. Draw the graph and compute the Grundy function for $n = 10$. (For this game the Grundy function does not have a known closed formula.)

(8) (Challenge) What is the value of

$$1 * 2 * \cdots * n$$

for $n = 1, 2, \ldots$?

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