

# COUNTING COLORINGS ON VARIETIES

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1.

**1.1. Introduction.** The goal of this note is to present a combinatorial mechanism for counting certain objects associated to a variety  $X$  defined over a finite field. The basic example, discussed in §2.2, is that of counting conjugacy classes in  $GL_n(\mathbb{F}_q)$ , where  $X = \mathbb{G}_m$  (the multiplicative group).

We give four different forms of the main formula (which is somewhat reminiscent of Polya's theory of counting). The principle that emerges is that in a given setup the counting generating functions for  $X = \bullet$  (a point),  $X = \mathbb{G}_a$  (the additive group) and  $X = \mathbb{G}_m$  are related to one another in a simple way. Often one of the cases will be significantly easier to compute than the others yielding a closed formula for all three generating functions. For example, in §3 we describe how one can go from counting all matrices in  $M_n(\mathbb{F}_q)$ , corresponding to  $X = \mathbb{G}_a$ , to counting unipotent matrices in  $M_n(\mathbb{F}_q)$ , corresponding to  $X = \bullet$ .

None of the special cases considered here are really new; the point is, instead, to stress the main combinatorial principle. For more general applications (to quiver and character varieties) we refer the reader to [1] and [2].

**1.2. The Zeta Function of  $X$ .** Let  $\mathbb{F}_q$  be a finite field with  $q$  elements. Fix an algebraic closure  $\overline{\mathbb{F}}_q$  of  $\mathbb{F}_q$ . For each  $r \in \mathbb{N}$  let  $\mathbb{F}_{q^r}$  be the unique subfield of  $\overline{\mathbb{F}}_q$  of cardinality  $q^r$ . Let  $\text{Frob}_q \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$  be the Frobenius automorphism  $x \mapsto x^q$ . Then  $\mathbb{F}_{q^r}$  is the fixed field of  $\text{Frob}_q^r$ .

Let  $X$  be an algebraic variety defined over  $\mathbb{F}_q$ . For each  $r \in \mathbb{N}$  let  $N_r(X) := \#X(\mathbb{F}_{q^r})$ . The *zeta function* of  $X$  is defined as

$$(1) \quad Z(X, T) := \exp \left( \sum_{r \geq 1} N_r(X) \frac{T^r}{r} \right).$$

Let  $\tilde{N}_d(X)$  be the number of Frobenius orbits in  $X(\overline{\mathbb{F}}_q)$  of size  $d$ . Then

$$(2) \quad N_r(X) = \sum_{d|r} d \tilde{N}_d(X).$$

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We can write the zeta function as an Euler product

$$(3) \quad Z(X, T) = \prod_{d \geq 1} (1 - t^d)^{-\tilde{N}_d(X)}.$$

**1.3. Colorings on  $X$ .** We consider the following general setup. Let  $\mathcal{C}$  be a set, whose members we call *colors*, and

$$(4) \quad |\cdot| : \mathcal{C} \longrightarrow \mathbb{Z}_{\geq 0}$$

a function called *degree* such that

- (1) there are finitely many colors of a given degree;
- (2) there is a unique color  $0 \in \mathcal{C}$  of degree 0.

A *coloring* on  $X$  is a map

$$(5) \quad \Lambda : X(\overline{\mathbb{F}}_q) \longrightarrow \mathcal{C}.$$

The *degree* of  $\Lambda$  is defined as

$$(6) \quad |\Lambda| := \sum_{x \in X(\overline{\mathbb{F}}_q)} |\Lambda(x)|.$$

We will only consider colorings of finite degree, so that  $\Lambda(x) = 0$  for all but finitely many  $x$ . We let the Frobenius automorphism act on colorings via

$$(7) \quad \Lambda^{\text{Frob}_q}(x) := \Lambda(\text{Frob}_q(x))$$

and say  $\Lambda$  is defined over  $\mathbb{F}_{q^r}$  if  $\Lambda$  is fixed by  $\text{Frob}_q^r$ . In this case we will write:  $\Lambda$  is a coloring of  $X/\mathbb{F}_{q^r}$ .

Given a pair  $(d, \lambda)$ , with  $d \in \mathbb{N}$  and  $\lambda \in \mathcal{C}$  a non-zero color, we define its *multiplicity*  $m_{d,\lambda}$  in a coloring  $\Lambda$  of  $X/\mathbb{F}_q$  to be the number of Frobenius orbits  $\{x\}$  in  $X(\overline{\mathbb{F}}_q)$  of degree  $d$  with  $\Lambda(x) = \lambda$ . Note that

$$(8) \quad |\Lambda| = \sum_{d \geq 1, \lambda \neq 0} m_{d,\lambda} d |\lambda|.$$

We call the combinatorial data  $\{m_{d,\lambda}\}$  of multiplicities the *type* of  $\Lambda$  and denote it  $\tau(\Lambda)$ .

*Example 1.* Let  $\mathcal{C} = \mathbb{Z}_{\geq 0}$  with degree function  $|n| = n$ . Then a coloring is an effective 0-cycle on  $X$ . The actions of Frobenius are compatible hence  $\Lambda$  is defined over  $\mathbb{F}_{q^r}$  if and only if the corresponding 0-cycle is.

*Example 2.* If  $X = \mathbb{G}_m$  and  $\mathcal{C} = \mathcal{P}$  is the set of all partitions of non-negative integers with  $|\lambda| = \lambda_1 + \lambda_2 + \dots$  if  $\lambda = \lambda_1 \geq \lambda_2 \geq \dots$  then colorings of degree  $n$  are in one-to-one correspondence with conjugacy classes in  $\text{GL}_n(\overline{\mathbb{F}}_q)$  by the Jordan decomposition theorem. Indeed, to each coloring  $\Lambda$  we may associate the direct sum of Jordan blocks with eigenvalue  $x \in \overline{\mathbb{F}}_q^*$  and size  $\lambda_i$ , where  $\Lambda(x) = \lambda_1 \geq \lambda_2 \geq \dots$ . This correspondence preserves the action of Frobenius and therefore colorings defined over  $\mathbb{F}_{q^r}$  are in bijection to conjugacy classes of  $\text{GL}_n(\mathbb{F}_{q^r})$ .

Similar statements hold for  $X = \mathbb{G}_a$  with colorings corresponding to conjugacy classes in  $M_n(\overline{\mathbb{F}}_q)$  instead.

We need one more ingredient. Let  $R = \mathbb{Z}[[t_1, \dots, t_N]][t_1^{-1}, \dots, t_N^{-1}]$  be the ring of Laurent series with integer coefficients in the variables  $t_1, \dots, t_N$ . We let

$$(9) \quad W : \mathcal{C} \longrightarrow R$$

be a function called *weight* satisfying  $W(0) = 1$ .

We define the weight of a coloring  $\Lambda$  on  $X/\mathbb{F}_q$  as

$$(10) \quad W(\Lambda) := \prod_{\{x\}} W(\Lambda(x))(t^{d(x)}),$$

where  $\{x\}$  runs through the Frobenius orbits in  $X(\overline{\mathbb{F}}_q)$ ,  $d(x) = \#\{x\}$  is the degree of  $x$  (the size of its Frobenius orbit) and  $t^d := t_1^d \cdots t_N^d$ . Note that  $W(\Lambda)$  only depends on the type  $\tau(\Lambda)$ :

$$(11) \quad W(\Lambda) = \prod_{d \geq 1, \lambda \neq 0} W(\lambda)(t^d)^{m_{d,\lambda}}.$$

We say that  $W$  is *homogeneous* if for each  $\lambda \in \mathcal{C}$  we have that  $W(\lambda) \in R$  is homogeneous of degree  $|\lambda|$ . In this case  $W(\Lambda)$  is also homogeneous of degree  $|\Lambda|$ .

**1.4. Coloring Zeta Function of  $X$ .** Given the coloring data  $C = (\mathcal{C}, |\cdot|, W(\cdot))$  we define the *coloring zeta function* of  $X$  as the formal power series in  $R[[T]]$

$$(12) \quad Z_C(X, t, T) := \sum_{\Lambda} W(\Lambda) T^{|\Lambda|},$$

where the sum runs over all colorings of  $X/\mathbb{F}_q$ . If  $X = \bullet$  (a point) then the coloring zeta function simply reduces to

$$(13) \quad Z_C(\bullet, t, T) := \sum_{\lambda \in \mathcal{C}} W(\lambda) T^{|\lambda|}.$$

*Example 3.* In the *standard setup*  $C = (\mathbb{Z}_{\geq 0}, |\cdot|, 1)$ ,  $Z_C(X, T)$  is just the usual zeta function  $Z(X, T)$ . In particular, if  $X = \bullet$  then

$$Z_C(\bullet, T) = \sum_{n \geq 0} T^n = (1 - T)^{-1}.$$

*Example 4.* In the *partition setup*  $C = (\mathcal{P}, |\cdot|, 1)$  and if  $X = \bullet$  then

$$Z_C(\bullet, T) = \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} = \prod_{d \geq 1} (1 - T^d)^{-1}.$$

## 2.

**2.1. First Form.** This form of the main formula is similar to the Euler product (3) for the usual zeta function (to which it reduces to in the standard setup).

**Theorem 1.** *The following identity of generating functions holds*

$$(14) \quad Z_C(X, t, T) = \prod_{d \geq 1} Z_C(\bullet, t^d, T^d)^{\tilde{N}_d}$$

*Proof.* Write  $Z_C(\bullet, t, T) = 1 + z(T)$ . For  $N \in \mathbb{N}$  we have

$$Z_C(\bullet, t, T)^N = 1 + \sum_{m \geq 1} N(N-1) \cdots (N-m+1) \frac{z(T)^m}{m!}$$

by the binomial theorem. On the other hand by the multinomial theorem

$$\frac{z(T)^m}{m!} = \sum_{m_\lambda} \prod_{\lambda \neq 0} \frac{W(\lambda)^{m_\lambda}}{m_\lambda!} T^{m_\lambda |\lambda|}$$

summed over all sequences of non-negative integers  $m_\lambda$  with  $\sum_\lambda m_\lambda = m$ . Putting these two identities together we get that the coefficient of  $T^n$  on the right hand side of (14) equals

$$\sum_{m_{d,\lambda}} \prod_{d \geq 1, \lambda \neq 0} \tilde{N}_d (\tilde{N}_d - 1) \cdots (\tilde{N}_d - m_d + 1) \frac{W(\lambda) (t^d)^{m_{d,\lambda}}}{m_{d,\lambda}!}$$

summed over all  $m_{d,\lambda}$  sequences of non-zero integers satisfying

$$n = \sum_{d \geq 1, \lambda \neq 0} m_{d,\lambda} d |\lambda|,$$

where  $m_d := \sum_{\lambda \neq 0} m_{\lambda,d}$ .

On the other hand to give a coloring of  $X/\mathbb{F}_q$  with multiplicities  $m_{d,\lambda}$  we need to pick  $m_d = \sum_{\lambda \neq 0} m_{\lambda,d}$  Frobenius orbits of size  $d$  and color  $m_{d,\lambda}$  of them with color  $\lambda \neq 0$ . There are  $\binom{\tilde{N}_d}{m_d}$  ways of picking the orbits and  $m_d! / \prod_{\lambda \neq 0} m_{d,\lambda}!$  ways to color them in this way and the weight of  $\Lambda$  is  $W(\Lambda) = \prod_{d \geq 1, \lambda \neq 0} W(\lambda) (t^d)^{m_{d,\lambda}}$ . It follows that the coefficients of  $T^n$  on both sides of (14) agree.  $\square$

## 2.2. Second Form.

**Theorem 2.** *The following identity of generating functions holds*

$$(15) \quad Z_C(X, t, T) = \prod_{m \in \mathbb{Z}^N, d \geq 1} Z(X, t^m T^d)^{v_{d,m}}$$

where the exponents  $v_{d,m}$  are defined by the formal identity

$$(16) \quad Z_C(\bullet, t, T) = \prod_{m \in \mathbb{Z}^N, d \geq 1} (1 - t^m T^d)^{-v_{d,m}}.$$

*Proof.* Taking logarithms of both sides of (14) we get

$$\log Z_C(X, t, T) = \sum_{d \geq 1} \tilde{N}_d \log Z_C(\bullet, t^d, T^d).$$

By Möbius inversion of (2)

$$\tilde{N}_d = \frac{1}{d} \sum_{e|d} \mu(e) N_{d/e}.$$

Taking logarithm of both sides of (16) we get

$$\log Z_C(\bullet, t, T) = - \sum_{m \in \mathbb{Z}^n, d \geq 1} v_{d,m} \log(1 - t^m T^d).$$

On the other hand,

$$T = - \sum_{r \geq 1} \mu(r) \log(1 - t^r)$$

and hence

$$\begin{aligned} \log Z_C(X, t, T) &= - \sum_{r, s \geq 1} \frac{1}{rs} \mu(r) N_s \sum_{m \in \mathbb{Z}^n, d \geq 1} v_{d,m} \log(1 - t^{msr} T^{dsr}) \\ &= \sum_{s \geq 1} \frac{1}{s} N_s \sum_{m \in \mathbb{Z}^n, d \geq 1} v_{d,m} t^{sm} T^{sd} \\ &= \sum_{m \in \mathbb{Z}^n, d \geq 1} v_{d,m} \log Z(X, t^m T^d) \end{aligned}$$

proving our claim.  $\square$

*Remark.* It is easy to see by induction that the  $v_{d,m}$  in (16) are integers uniquely determined by  $Z_C(\bullet, t, T)$ .

*Example 5.* In the *partition setup*  $C = (\mathcal{P}, |\cdot|, 1)$  with  $X = \mathbb{G}_m$  we have

$$Z_C(\mathbb{G}_m, T) = \sum_{n \geq 0} C_n T^n,$$

where  $C_n$  is the number of conjugacy classes in  $\mathrm{GL}_n(\mathbb{F}_q)$  (see example 5), by (15) this equals

$$(17) \quad \prod_{n \geq 1} \left( \frac{1 - T^n}{1 - qT^n} \right)$$

as  $Z(\mathbb{G}_m, T) = (1 - T)/(1 - qT)$ . See [3], [5], [6].

Similarly, if we take  $X = \mathbb{G}_a$  then we get the expression

$$(18) \quad \prod_{n \geq 1} (1 - qT^n)^{-1}$$

for the generating function for the conjugacy classes in  $M_n(\mathbb{F}_q)$  instead.

**2.3. Third Form.** This form of the expression for the coloring zeta function is a simple variant of the second form (15) but it is convenient to state it separately.

Let

$$(19) \quad Z_C(u, t, T) := \prod_{m \in \mathbb{Z}^n, d \geq 1} (1 - u t^m T^d)^{-v_{d,m}}$$

where  $u$  is another formal variable and  $v_{d,m}$  is as in (16).

**Theorem 3.** *Let*

$$(20) \quad Z(X, T) = \prod_i (1 - x_i T)^{-n_i},$$

for some  $x_i \in \mathbb{C}$  and  $n_i \in \mathbb{Z}$ . Then with the above notation we have

$$(21) \quad Z_C(X, t, T) = \prod_i Z_C(x_i, t, T)^{n_i}.$$

*Example 6.* In the standard setup  $Z_C(\bullet, T) = (1 - T)^{-1}$  so that  $Z_C(u, T) = (1 - uT)^{-1}$  and (21) is simply a restatement of (20).

*Remark.* It is known by the work of Dwork that  $Z(X, T)$  is a rational function of  $T$  of the form (20).

**2.4. Fourth Form.** Recall that  $R = \mathbb{Z}[[t_1, \dots, t_N]][t_1^{-1}, \dots, t_N^{-1}]$  is the ring of Laurent series in variables  $t_1, \dots, t_N$  with integer coefficients. Given  $Z \in 1 + TR[[T]]$  we define, following Getzler [4]

$$(22) \quad \text{Log}(Z) := \sum_{d \geq 1, m \in \mathbb{Z}^N} v_{d,m} t^m T^d \in R[[T]],$$

where

$$Z = \prod_{m \in \mathbb{Z}^N, d \geq 1} (1 - t^m T^d)^{-v_{d,m}},$$

as in (16).

In this section we assume that  $X$  is a polynomial count variety; i.e.

$$N_r(X) = N_X(q^r), \quad r \in \mathbb{N},$$

for some fixed polynomial  $N_X \in \mathbb{Z}[q]$ . We also assume that one of the variables in  $R$  is  $q$ . To simplify the notation we relabel the variables as  $q, t_1, \dots, t_N$  and the exponents as  $i \in \mathbb{Z}$  for  $q$  and  $m \in \mathbb{Z}^N$  for  $t_1, \dots, t_N$ .

**Theorem 4.** *The following identity holds*

$$(23) \quad \text{Log}(Z_C(X, t, T)) = N_X(q) \text{Log}(Z_C(\bullet, t, T)).$$

*Proof.* The claim is a simple consequence of the third form (21) of our main formula. If  $N_X(q) = \sum_j n_j q^j$  then

$$Z(X, T) = \prod_j (1 - q^j T)^{-n_j}.$$

Hence by (21)

$$\begin{aligned} Z_C(X, t, T) &= \prod_{j \in \mathbb{Z}} Z_C(q^j, t, T)^{n_j} \\ &= \prod_{i, j \in \mathbb{Z}, m \in \mathbb{Z}^N, d \geq 1} (1 - q^{i+j} t^m T^d)^{-n_j v_{d,i,m}} \\ &= \prod_{k \in \mathbb{Z}, m \in \mathbb{Z}^N, d \geq 1} (1 - q^k t^m T^d)^{-\sum_{i+j=k} n_j v_{d,i,m}}. \end{aligned}$$

Hence

$$\text{Log}(Z_C(X, t, T)) = \sum_{k \in \mathbb{Z}, m \in \mathbb{Z}^N, d \geq 1} \sum_{i+j=k} n_j v_{d,i,m} q^k t^m T^d$$

which equals the right hand side of (23)  $\square$

3.

**3.1. Unipotent matrices.** We consider the coloring data  $\mathcal{C} = \mathcal{P}$  with the usual degree function  $|\cdot|$  but with a non-trivial weight function. For all results and concepts related to partitions our reference will be [7], whose notation we will follow.

Let  $X = \mathbb{G}_a$  and  $\Lambda$  a coloring of  $X/\mathbb{F}_q$  corresponding to a conjugacy class  $c$  in  $M_n(\mathbb{F}_q)$ . The centralizer  $z_c$  of  $c$  in  $G_n := GL_n(\mathbb{F}_q)$  has order [7]

$$\prod_{\{x\}} a_{\Lambda(x)}(q^d(x)),$$

where, as before,  $\{x\}$  runs through the Frobenius orbits,  $d(x)$  is the size of  $\{x\}$  and where for  $\lambda \in \mathcal{P}$

$$(24) \quad a_\lambda(q) := q^{|\lambda|+2n(\lambda)} b_\lambda(q^{-1}),$$

with

$$(25) \quad n(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i$$

$$(26) \quad b_\lambda(q) := \prod_{i \geq 1} \phi_{m_i(\lambda)}(q)$$

$$(27) \quad \phi_m(q) := (1-q)(1-q^2) \cdots (1-q^m)$$

and, finally,  $m_i(\lambda)$  is the multiplicity of  $i$  in  $\lambda$ .

It follows that if we define our weight function as

$$W(\lambda) := a_\lambda(q)^{-1} \in R$$

where  $R = \mathbb{Z}[[q]][[q^{-1}]$  then

$$(28) \quad W(\Lambda) = \frac{1}{|z_c|}.$$

Consequently, if we now take  $X = \mathbb{G}_m$  then

$$\sum_{\Lambda/\mathbb{F}_q, |\Lambda|=n} W(\Lambda) = 1$$

and therefore

$$(29) \quad Z_C(\mathbb{G}_m, q, T) = \sum_{n \geq 0} T^n = (1-T)^{-1}.$$

Applying (23) to this situation we find that

$$(30) \quad Z_C(\bullet, q, T) = \prod_{n \geq 0} (1 - q^n T),$$

since  $N_{\mathbb{G}_m} = q - 1$  and  $(q - 1)^{-1} = -(1 + q + q^2 + \dots)$  and similarly

$$(31) \quad Z_C(\mathbb{G}_a, q, T) = \prod_{n \geq 1} (1 - q^n T).$$

On the other hand, for  $X = \mathbb{G}_a$  we have

$$\sum_{|\lambda|=n} \frac{1}{a_\lambda(q)} = \sum_{\Lambda/\mathbb{F}_q, |\Lambda|=n} W(\Lambda) = \frac{|M_n(\mathbb{F}_q)|}{|G_n|} = \frac{q^{\frac{1}{2}n(n+1)}}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}$$

and we have therefore proved the following identity of Euler

$$(32) \quad \sum_{n \geq 0} \frac{q^{\frac{1}{2}n(n+1)} T^n}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)} = \prod_{n \geq 1} (1 - q^n T).$$

On the other hand, for  $X = \bullet$  we have

$$\sum_{\Lambda/\mathbb{F}_q, |\Lambda|=n} W(\Lambda) = \frac{u_n}{|G_n|},$$

where  $u_n$  is the number of unipotent matrices in  $G_n$ . Using (30) we deduce the known result  $u_n = q^{n^2 - n}$ .

**3.2. Commuting pairs of matrices.** We now consider a weight function arising from the centralizer algebra  $\mathcal{Z}_A$  of a matrix  $A \in M_n(\mathbb{F}_q)$ . It is known that

$$\dim_{\mathbb{F}_q}(\mathcal{Z}_A) = \sum_{\{x\}} d(x) \langle \Lambda(x), \Lambda(x) \rangle$$

where for a partition  $\lambda$  we define  $\langle \lambda, \lambda \rangle := |\lambda| + 2n(\lambda)$ .

Since  $|\mathcal{Z}_A|$  only depends on the conjugacy class  $[A]$  of  $A$ , we can count pairs of commuting matrices in  $M_n(\mathbb{F}_q)$  as follows

$$\gamma_n := \#\{A, B \in M_n(\mathbb{F}_q) \mid AB = BA\} = \sum_{[A]} \#[A] |\mathcal{Z}_A|,$$

where  $[A]$  runs through the conjugacy classes in  $M_n(\mathbb{F}_q)$ . Hence if we define

$$W(\lambda) := \frac{q^{\langle \lambda, \lambda \rangle}}{a_\lambda(q)}$$

then

$$W(\Lambda) = \frac{\#[A]}{|G_n|} |\mathcal{Z}_A|,$$

where  $[A]$  corresponds to the coloring  $\Lambda$  on  $\mathbb{G}_a/\mathbb{F}_q$ , and therefore

$$\gamma_n = |G_n| \sum_{\Lambda} W(\Lambda).$$

Consequently,

$$(33) \quad Z_C(\mathbb{G}_a, q, T) = \sum_{n \geq 0} \frac{\gamma_n}{|G_n|} T^n.$$

On the other hand,

$$\begin{aligned}
Z_C(\bullet, q, T) &= \sum_{\lambda} \frac{q^{\langle \lambda, \lambda \rangle}}{a_{\lambda}(q)} T^{|\lambda|} \\
&= \sum_{\lambda} \frac{T^{|\lambda|}}{b_{\lambda}(q^{-1})} \\
&= \prod_{i \geq 1} \sum_{m_i \geq 0} \frac{T^{im_i}}{\phi_{m_i}(q^{-1})} \\
&= \prod_{i, n \geq 1} (1 - q^n T^i)
\end{aligned}$$

using Euler's identity (32). Applying (21) we recover (in an equivalent form) a result of Fine and Feit [3]

$$(34) \quad \sum_{n \geq 0} \frac{\gamma_n}{|G_n|} T^n = \prod_{i, n \geq 1} (1 - q^{n+1} T^i).$$

Similarly, we obtain

$$\sum_{n \geq 0} \frac{\gamma'_n}{|G_n|} T^n = Z_C(\mathbb{G}_m, q, T),$$

where

$$\gamma'_n := \#\{A \in GL_n(\mathbb{F}_q), B \in M_n(\mathbb{F}_q) \mid AB = BA\}.$$

Again by (21) we find

$$Z_C(\mathbb{G}_m, q, T) = \prod_{i, n \geq 1} \left( \frac{1 - q^{n+1} T^i}{1 - q^n T^i} \right) = \prod_{i \geq 1} (1 - q T^i)^{-1}.$$

We now recognize this generating series as (18) and conclude that  $\gamma'_n/|G_n|$  is the number of conjugacy classes in  $M_n(\mathbb{F}_q)$ . This, in fact, can be proved directly by a simple application of Burnside's lemma to  $GL_n(\mathbb{F}_q)$  acting on  $M_n(\mathbb{F}_q)$  by conjugation. By our main combinatorial principle, this means that we can run the argument backwards and prove (34) starting from (18).

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