# SOME CLASSICAL $p$-ADIC ANALYSIS

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## 1

In [1] D. Zagier solves the Monthly problem: prove that

$$v_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right)=v_3\left(n^2\binom{2n}{n}\right), \tag{1}$$

where $v_p$ denote the $p$-adic valuation. He does this by proving that there is a continuous function $f:\mathbb{Z}_3\longrightarrow -1+3\mathbb{Z}_3$ which interpolates the values

$$f(n)=\frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}, \qquad n\in\mathbb{N}. \tag{2}$$

The talk was a description of a general form of these facts.

Define formally

$$H(n,t):=\frac{1}{\binom{2n}{n}}\sum_{k=0}^{n-1}\binom{2k}{k}(t+4)^{n-1-k}, \qquad n\in\mathbb{N}, \tag{3}$$

where $t$ is a variable.

**Proposition 1.** *The following identity holds*

$$H(n,t)=\sum_{k=1}^{n}\frac{1}{\binom{2k}{k}}\binom{n}{k}t^{k-1}. \tag{4}$$

Fix once and for all a prime $p>2$. Let $t\in\mathbb{Z}_p$ with $|t|_p<1$. Then the Mahler series

$$H(x,t):=\sum_{k\geq 1}\frac{1}{\binom{2k}{k}}\binom{x}{k}t^{k-1}, \qquad x\in\mathbb{Z}_p \tag{5}$$

converges since the valuation $v_p(\binom{2k}{k})$ grows at most logarithmically with $k$. By proposition 1 the continuous function $H(\cdot,t)$ interpolates the values $H(n,t)$ of (3).

In fact, the function $H$ is analytic in the unit disk in $\mathbb{C}_p$. Expanding (5) formally as a power series we find

$$H(x,t)=\sum_{n\geq 0}b_n(t)x^n, \tag{6}$$

where

$$b_0=0, \qquad b_n(t)=\sum_{k\geq n}\frac{t^{k-1}}{k!\binom{2k}{k}}c_{n,k}, \qquad n\in\mathbb{N}, \tag{7}$$

for certain integers $c_{n,k}$ obtained from

$$\binom{x}{k} = \frac{1}{k!}\sum_{n\geq 0} c_{n,k}\, x^n.$$

We have the following analogue of (1)

**Theorem 1.** *For $u \in \mathbb{C}_p$ such that $v_p(u-4) \geq 2/(p-1)$ we have*

$$\left|\sum_{k=0}^{n-1}\binom{2k}{k}u^{n-1-k}\right|_p \leq \left|n\binom{2n}{n}\right|_p, \qquad n\in\mathbb{N}. \tag{8}$$

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Zagier proved that $H_3(x,1)$ is actually divisible by $x^2$ and hence gained an extra power of $n$ in the right hand side of the theorem.

To see how this comes about it will be convenient to give $t$ and $u$ in terms of another variable $w$ as follows:

$$t = (w-1/w)^2, \qquad u := t+4 = (w+1/w)^2. \tag{9}$$

For example, if $w = \zeta_3$, a primitive cubic root of unity, then $t = (w-w^{-1})^2 = -3$ and $u = (w+w^{-1})^2 = 1$.

We have

$$b_1(t) := \left.\frac{H(x,t)}{x}\right|_{x=0} = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k\binom{2k}{k}}t^{k-1}. \tag{10}$$

**Proposition 2.** *The following identity of formal power series in the variable $z$ holds*

$$\frac{1}{w^2-w^{-2}}\log w^2 = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k\binom{2k}{k}}(w-w^{-1})^{2(k-1)}, \qquad w = 1-z. \tag{11}$$

We now consider (11) $p$-adically for $|z|_p < 1$ (so that also $|w-w^{-1}|_p < 1$ with $w = 1-z$) to obtain

$$b_1(t) = \frac{1}{w^2-w^{-2}}\log w^2, \qquad |1-w|_p < 1, \qquad t = (w-w^{-1})^2.$$

As a special case we find

**Corollary 1.** *Let $\zeta_p \in \mathbb{C}_p$ be a primitive $p$-th root of unity. Then*

$$b_1((\zeta_p-\zeta_p^{-1})^2) = 0. \tag{12}$$

Combining Theorem 1 with the above corollary we obtain a closer generalization of the original Monthly problem (1).

**Theorem 2.** *For $p > 2$ and $\zeta_p \in \mathbb{C}_p$ a primitive $p$-th root of unity we have*

$$\left|\sum_{k=0}^{n-1}\binom{2k}{k}(\zeta_p+\zeta_p^{-1})^{2(n-1-k)}\right|_p \leq \left|n^2\binom{2n}{n}\right|_p, \qquad n\in\mathbb{N}. \tag{13}$$

The equality typically does not hold for all n but will, in fact, hold except for $n$'s in some excluded congruence classes modulo $p$.

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It is not hard to show that

$$b_n(t) = \frac{1}{2(n-1)!}\int_0^1 \frac{\log^{n-1}(1+tz(1-z))}{1+tz(1-z)}\,dz. \tag{14}$$

Manipulating the integral we find

$$\log(t+4)\log\left(\frac{\gamma^+}{\gamma^-}\right) + \sum_{k=0}^{n-1}\binom{n-1}{k}\int_{\gamma^-}^{\gamma^+} \log^k(1-v)\log^{n-k-1} v\,\frac{dv}{v}$$

where

$$\gamma^\pm := \tfrac{1}{2}(1\pm\gamma), \qquad \gamma := \sqrt{\frac{t}{t+4}}.$$

The individual integrals in this sum are essentially what are know as Nielsen polylogarithms and can be expressed in terms of multi-polylogarithms. We can express these multi-polylogarithms in terms of the usual polylogarithms for $n = 2, 3$. Here is the result for $n = 2$.

**Proposition 3.** *For $|t| < 1$ we have*

$$b_2(t) = \frac{\gamma^2-1}{2\gamma}\left[\tfrac{1}{2}\log^2(\gamma^+) - \tfrac{1}{2}\log^2(\gamma^-) + \mathrm{Li}_2(\gamma^+) - \mathrm{Li}_2(\gamma^-)\right]. \tag{15}$$

Alternatively, we also have for $n \in \mathbb{N}$

$$b_n(t) = \frac{1}{(t+4)}\sum_{0\le j_1<j_2<\cdots<j_n} \frac{(\frac{t}{t+4})^{j_n}}{(j_1+\frac{1}{2})(j_2+\frac{1}{2})\cdots(j_n+\frac{1}{2})}. \tag{16}$$

In Zagier's case $p = 3, t = -3$ and hence $\gamma^\pm$ are the primitive cubic roots of unity. The above expressions suggest a relation between $b2(-3)$ with $\zeta_3(2)$. Indeed, we find numerically that they are related up to a simple factor in $\mathbb{Q}$. Thanks to work done during the Oberwolfach workshop with H. Gangl and D. Zagier the proof of this fact seems reasonably close.

In general it seems that the higher coefficients $b_n(-3)$ are also related to values of 3-adic $L$-series at least up to $n = 6$; including the apparent equality $b_3(-3) = 0$, noticed by Zagier. It is not inconceivable that all of these could be proved in the near future. These identities should be part of the general picture of relations between periods and special values of $L$-functions in a $p$-adic context.

## References

[1] D. Zagier; J. Shallit; N. Strauss, Problems and Solutions: 6625, Amer. Math. Monthly **99** (1992), no. 1, 66–69

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