3.7.25. The constructions of 3.7.24, 3.7.20 easily combine. Namely, assume that we have A, L as in 3.7.24 and $L^\flat$ as in 3.7.20. Look at pairs $i: A \to B$, $\psi^\flat: L^\flat \to B$ that satisfy same conditions as i, ψ from 3.7.24 plus we assume that $\psi^\flat(1^\flat) = 1_B$. As in 3.7.24 there exists a universal pair $(i, \psi^\flat)$, and we denote the corresponding B by $A \circledast U(L)^\flat$. One has $A \circledast U(L)^\flat = A \circledast U(L^\flat)/(1-1^\flat)A \circledast U(L^\flat)$. There is a canonical identification $A \circledast U(L)^\flat = \mathrm{Ind}_L^\flat A$. The canonical filtration on $\mathrm{Ind}_L^\flat A$ is a chiral algebra filtration, and the canonical morphism $A \otimes \mathrm{Sym}^\cdot L^\ell \to \mathrm{gr.}(A \circledast U(L)^\flat)$ is a morphism of chiral algebras. If L is $\mathcal{O}_X$-flat then this is an isomorphism (see 3.7.22). This version of the PBW theorem will be of use in Section 3.8.

3.7.26. In the rest of this section we give a different prove of the PBW theorem 3.7.14 which does not use the factorization algebra picture but followss the approach of 3.7.2–3.7.4. It will not be used in the rest of this work.

We get control on $U(L)$ by means of a canonical resolution $\mathcal{C}(L)$ defined in 3.7.27. Its definition uses Quillen's adjoint functors identifying the derived categories of DG Lie algebras and DG cocommutative coalgebras [QR]. Recall the definitions.

Let $\mathcal{A}$ be a tensor $\mathbb{Q}$=category (we do not assume it to be unital) which admits countable direct sums, so we have the tensor category $C\mathcal{A}$ of complexes in $\mathcal{A}$. Consider the categories $\mathcal{L}ie(C\mathcal{A})$, $\mathcal{C}oc(C\mathcal{A})$ of Lie DG algebras, resp. cocommutative and coassociative DG coalgebras in $\mathcal{A}$.

For $L \in \mathcal{L}ie(C\mathcal{A})$, $B \in \mathcal{C}oc(C\mathcal{A})$ a *Maurer-Cartan map* $\varphi : B \to L$ is a morphism of graded objects $B \to L[1]$ such that $d\varphi + \varphi d = \frac{1}{2}[\varphi, \varphi]$. Here $[\varphi, \varphi]$ is the composition $B \to B \otimes B \xrightarrow{\varphi \otimes \varphi} L \otimes L \to L$, the first arrow is the coproduct, the last one is the bracket. Denote by $\mathrm{MC}(B, L)$ the set of Maurer-Cartan maps; it depends on B, L in functorial way.

Both functors $\mathrm{MC}(B, \cdot)$, $\mathrm{MC}(\cdot, L)$ are representable. Denote the corresponding objects by $F(B) \in \mathcal{L}ie(C\mathcal{A})$, $G(L) \in \mathcal{C}oc(C\mathcal{A})$. They can be described explicitely as follows. Consider the universal Maurer-Cartan maps as morphisms of graded objects $B[-1] \to F(B)$, $G(L) \to L[1]$. They identify $F(B)$, $G(L)$ as plain graded objects with, respectively, the free Lie graded algebra generated by $B[-1]$ and the cofree cocommutative graded

algebra cogenerated by $L[1]$. The differentials are determined on generators by the Maurer-Cartan property.

The functor $F : \mathcal{C}oc(C\mathcal{A}) \to \mathcal{L}ie(C\mathcal{A})$ is left adjoint to $G : \mathcal{L}ie(C\mathcal{A}) \to \mathcal{C}oc(C\mathcal{A})$ in the obvious way.

Remark. $G(L)$ is the truncated Chevalley complex of L (if $L \in \mathcal{L}ie(\mathcal{A})$ then $G(L)^i = \Lambda^{-i}L$ if $i < 0$ and 0 otherwise).

3.7.27. Let us return to our situation, so we have a Lie* algebra L for which we want to check the PBW property. Consider L as a Lie* algebra on X^{S} (see 3.7.3). Denote by $\mathcal{C}(L)$ the Lie^{ch} algebra on X^{S} defined as the image of L by the composition of functors

$$\mathcal{L}ie^*(X^{\mathrm{S}}) \xrightarrow{G^*} \mathcal{C}oc^*(X^{\mathrm{S}}) \hookrightarrow \mathcal{C}oc^{ch}(X^{\mathrm{S}}) \xrightarrow{F^{ch}} \mathcal{L}ie^{ch}(X^{\mathrm{S}}). \tag{3.7.27.1}$$

Here the indices $*$, ch show that we deal with either of the two tensor category structures on $\mathcal{M}(X^{\mathrm{S}})$ (see 3.7.3); the middle functor is the obvious embedding (any $*$ coalgebra on X^{S} is automatically a ch coalgebra by (3.7.3.3)).

One has $\mathcal{C}(L)^{>0} = 0$ and $\mathcal{C}(L)^0$ is the free Lie algebra in $\mathcal{M}(X^{\mathrm{S}})^{ch}$ generated by $G^*(L)^{-1} = L$. An immediate calculation shows that the image of $\mathcal{C}(L)^{-1} \xrightarrow{d} \mathcal{C}(L)^0$ equals the ideal $\mathcal{I}$ from the end of 3.7.3, therefore one has (here $U^{\mathrm{S}} := U^{\mathrm{S}}_{\mathcal{L}ie}$)

$$H^0\mathcal{C}(L) = U^{\mathrm{S}}(L). \tag{3.7.27.2}$$

3.7.28. Let us compute $\mathcal{C}(L)$ explicitly. At the moment we forget about the differentials. One has $G^*(L) = \oplus (L[1])^{\overset{*}{\otimes}{}^n_{\Sigma_n}}$

§8 BRST, ALIAS SEMI-INFINITE, HOMOLOGY

Below "$\mathcal{D}$-module", etc., always means "super $\mathcal{D}$-module" (see 1.1.16 for "super" conventions). As in 1.4.20 *for a graded $\mathcal{D}$-module M we set $\hbar(M) := h(M^{0even})$.*

We discuss BRST homology in the setting of chiral algebras (3.8.1–3.8.16) and in the more traditional setting of Tate's linear algebra (3.8.17–3.8.22). The two pictures are compared in 3.8.23–3.8.25.

The BRST homology construction is a neat quantum version of the classical Hamiltonian reduction. It originated in mathematical physics (BRST is for physicists Becchi-Rouet-Stora-Tyupin), came to representation theory in a pioneering work of B.Feigin [F] (his term "semi-infinite" has to do with decomposition of a Tate vector space in a sum of compact and discrete "halves"), and was further developed in [FHZ]. Our exposition of the linear algebra picture follows these references except that we tried to get rid of many redundant structures (gradings, normal orderings, explicit cocycles, etc.)

3.8.1 Chiral Weyl algebras. Let T be a $\mathcal{D}_X$-module; we assume that it has no $\mathcal{O}$-torsion. Let $<>_T \in P_2^*(\{T,T\},\omega_X)$ be a skew-symmetric[168] $*$ pairing. Set $T^\flat := T \oplus \omega_X$. This is a Lie* algebra with commutator equal to $<,>_T$ (plus zero components). So $T^\flat$ is an extension of a commutative Lie* algebra T by ω_X. Let $\mathcal{W} = \mathcal{W}(T, <>_T)$ be the corresponding twisted enveloping chiral algebra $U(T)^\flat$ (see 3.7.20). This is the *Weyl algebra* of $(T, <>_T)$. It carries the standard filtration so that $T^\flat = \mathcal{W}_1$, and gr.$\mathcal{W} = \mathrm{Sym}^\cdot T$ (see 3.7.21).

The *classical*, or *coisson*, Weyl algebra $\mathcal{W}^c$ is Sym$T$ equipped with coisson bracket that equals to $<>_T$ on $T = \mathrm{Sym}^1 T$. It differs from gr.$\mathcal{W}$ as coisson algebra: the coisson bracket on gr.$\mathcal{W}$ is trivial.

3.8.2. Assume our T is a $\mathcal{D}_X$-vector bundle and $<>_T$ is non-degenerate, i.e., the corresponding morphism $T \to T'$ (see 2.2.13) is an isomorphism. Then the center of $\mathcal{W}(T)$ equals to ω_X. Let A be any chiral algebra. The tensor product $A \otimes \mathcal{W}(T)$ (see 3.5.15) contains A and $\mathcal{W}(T)$ as subalgebras, and A coincides with the centralizer of $T^\flat \subset \mathcal{W}(T) \subset A \otimes \mathcal{W}(T)$. We have a pair of adjoint functors

$$\mathcal{M}(X,A) \to \mathcal{M}(X, A \otimes \mathcal{W}(T)), \quad \mathcal{M}(X, A \otimes \mathcal{W}(T)) \to \mathcal{M}(X,A), \tag{3.8.2.1}$$

where the left functor is $M \mapsto M \otimes \mathcal{W}(T)$, the right one sends $N \in \mathcal{M}(X, A \otimes \mathcal{W}(T))$ to the centralizer of T.

[168] As always, "skew-symmetric" refers to the "super" tensor structure: if T is purely odd this amounts to "naive symmetric".

3.8.3 Lemma. The functor $\mathcal{M}(X,A) \to \mathcal{M}(X, A \otimes \mathcal{W}(T))$ is fully faithful. Its image consists of those $A \otimes \mathcal{W}(T)$-modules on which the Lie* algebra $T^\flat$ acts in a locally nilpotent way. In particular, if T is purely odd, then 3.8.2.1 are mutually inverse equivalences of categories. □

3.8.4. Let us return to situation 3.8.1 (so we drop the non-degeneracy condition). Assume that we have a direct sum decomposition $T = V \oplus V'$ such that $<>_T$ vanishes on both V and V'. Let $<> \in P_2^*(\{V', V\}, \omega_X)$ be the restriction of $<>_T$ (notice that the sign of $<>$ depends on the order of V, V'). One recovers $(T, <>_T)$ from $(V, V', <>)$ in the obvious way, so we will write $\mathcal{W} = \mathcal{W}(V, V', <>)$. One has the obvious embeddings of chiral algebras $\mathrm{Sym}V, \mathrm{Sym}V' \hookrightarrow \mathcal{W}(V, V')$.

Define a $\mathbb{Z}$-grading on $T^\flat$ setting $T^{\flat -1} := V$, $T^{\flat 1} := V'$, $T^{\flat 0} := \omega_X$. Therefore $\mathcal{W}$ acquires a $\mathbb{Z}$-grading.[169] One has $\mathcal{W}_1^{-1} = V$, $\mathcal{W}_1^1 = V'$, and $\mathrm{gr}_a \mathcal{W}^n = \mathrm{Sym}^{\frac{a-n}{2}} V \otimes \mathrm{Sym}^{\frac{a+n}{2}} V'$.

Remark. $\mathcal{W}$ carries another filtration – the one generated by $(\mathrm{Sym}V', V + \mathrm{Sym}V')$ (see 3.7.13). This filtration is commutative, and the associated graded algebra $\mathrm{gr}'\mathcal{W}$ equals $\mathrm{Sym}V' \otimes \mathrm{Sym}V$. So we have a canonical identification of graded algebras[170] $\mathrm{gr}'\mathcal{W} = \mathrm{Sym}(V' \oplus V) = \mathcal{W}^c$ which is compatible with coisson brackets. So if V, V' are mutually dual $\mathcal{D}_X$-vector bundles then $\mathcal{W}$ is a chiral $\mathrm{Sym}V'$-tdo (see 3.8.5 for terminology).

3.8.5 The Tate extension. Consider the $\mathcal{D}_X$-module $\mathcal{W}_2^0$. It contains $\mathcal{W}_0^0 = \mathcal{W}_0 = \omega_X$ and $\mathcal{W}_2^0/\omega_X = \mathrm{gr}_2 \mathcal{W}^0 = V \otimes V'$. The adjoint action of $\mathcal{W}_2^0$ preserves $\mathcal{W}_1^\pm$. Since $\omega_X \subset \mathcal{W}_2^0$ acts trivially, we get $*$ actions of $V \otimes V'$ on V, V'. Since $\mathcal{W}_2^0 = \mu(V, V') + \omega_X$ we see that $\mathcal{W}_2^0$ is a Lie* subalgebra of $\mathcal{W}$.

Recall that in 1.4.2 we defined the associative* algebra structure on $V \otimes V'$ together with its left action on V and right action on V'. We leave it to the reader to check that the above action of $V \otimes V'$ on V coincides with that of 1.4.2, the one on V' is opposite

[169] Notice that, unless V, V' are purely odd, the $\mathbb{Z}$-grading mod 2 does *not* coincide with the structure "super" $\mathbb{Z}/2$-grading.

[170] The grading we consider comes from the grading on $\mathcal{W}$; we forget about the grading coming from the filtration.

to that of 1.4.2, and the Lie* algebra structure on $V \otimes V' = \mathcal{W}_2^0/\omega_X$ coincides with the Lie* algebra structure associated with the associative* algebra structure from 1.4.2. So the quotient $V \otimes V' = \mathcal{W}_2^0/\omega_X$ is the Lie* algebra $\mathcal{G}^{Lie}$ from 2.6.2, and we have an extension of Lie* algebras

$$0 \to \omega_X \to \mathcal{W}_2^0 \to \mathcal{G} \to 0. \tag{3.8.5.1}$$

This extension canonically identifies with the extension $\mathcal{G}^{-\flat}$ opposite to the Tate extension $\mathcal{G}^{\flat}$ from 2.6.2. Namely, consider the chiral product $j_*j^*V \boxtimes V' \to \Delta_*\mathcal{W}_2^0$. Its restriction to $V \boxtimes V'$ equals to *minus* $<>$: $V \boxtimes V' \to \Delta_*\omega_X$, and its composition with the projection $\mathcal{W}_2^0 \to V \otimes V'$ is the standard projection $j_*j^*V \boxtimes V' \to \Delta_*(V \otimes V')$. Therefore it yields a canonical identification of extensions (see the construction of $\mathcal{G}^{\flat}$ in 2.6.2)

$$\mathcal{W}_2^0 \xrightarrow{\sim} \mathcal{G}^{-\flat}. \tag{3.8.5.2}$$

This identification is obviously compatible with the Lie* brackets.

3.8.6 Chiral Clifford algebras. For a vector space or a $\mathcal{D}$-module M set $M^{\Pi} := M \otimes k^{odd}$; this is a copy of M with reversed parity. Below we always use a canonical isomorphism $M^{\Pi} \otimes N^{\Pi} = M \otimes N$ defined as composition $(M \otimes k^{odd}) \otimes (N \otimes k^{odd}) = (M \otimes N) \otimes (k^{odd} \otimes k^{odd}) = M \otimes N$. We identify chiral or $*$ operations of arguments $\{M, N\}$ and $\{M^{\Pi}, N^{\Pi}\}$ in the similar way.

Let now Q be a $\mathcal{D}_X$-module without $\mathcal{O}$-torsion equipped with a *symmetric* $*$ pairing $(\,)_Q \in P_2^*(\{Q,Q\}, \omega_X)$. It amounts to a skew-symmetric $*$ pairing[171] $<>_{Q^{\Pi}}$ on Q^{Π}. Set $\mathcal{C}l = \mathcal{C}l(Q, (\,)_Q) := \mathcal{W}(Q^{\Pi}, <>_{Q^{\Pi}})$; this is the *Clifford* chiral algebra of $(Q, (\,)_Q)$.

Everything said in 3.8.4 has an obvious Clifford version, so we have a $\mathbb{Z}$-graded Clifford algebra $\mathcal{C}l^{\cdot}$ associated with a $*$ pairing $(\,) \in P_2^*(\{L', L\}, \omega_X) = P_2^*(\{L'[-1], L[1]\}, \omega_X)$ which contains commutative subalgebras $\mathrm{Sym}(L'[-1])$, $\mathrm{Sym}(L[1])$. The $*$ bracket of $\mathcal{C}l$ restricted to $L'[-1], L[1]$ equals to $(\,)$. On $\mathcal{C}l$ there is the standard filtration $\mathcal{C}l_{\cdot}$, and $\mathrm{gr}\mathcal{C}l = \mathrm{Sym}(L'[-1] \oplus L[1])$.[172]

[171] So $<>_{Q^{\Pi}}$ equals to $(\,)_Q$ on $Q^{ev} = Q^{\Pi odd}$ and to $-(\,)_Q$ on $Q^{odd} = Q^{\Pi even}$.

[172] See 1.1.16 for notation.

Let us pass to 3.8.5. Consider the Lie* algebra $\mathcal{C}l_2^0$. The chiral product $j_*j^*(L' \boxtimes L) = j_*j^*(L'[-1] \boxtimes L[1]) = j_*j^*(\mathcal{C}l_1^1 \boxtimes \mathcal{C}l_1^{-1}) \to \mathcal{C}l_2^0$ yields the $*$ pairing equal to $(\ )$, so it defines, as in 3.8.5, an isomorphism of ω_X-extensions of $\mathcal{G} = L \otimes L'$ (cf. 3.8.5.2)

$$\mathcal{C}l_2^0 \xrightarrow{\sim} \mathcal{G}^\flat \tag{3.8.6.1}$$

where $\mathcal{G}^\flat$ is the Tate extension defined by $L, L', (\ )$. This is an isomorphism of Lie* algebras, and it identifies the adjoint action of $\mathcal{C}l_2^0/\omega_X$ on $\mathcal{C}l_1^{\pm 1}$ with the usual action of $\mathcal{G}$ on $L'[-1]$, $L[1]$. The proof coincides with that of similar statement for Weyl algebras plus a sign exercise.

3.8.7. Let now L be a Lie* algebra on X which is a $\mathcal{D}_X$-vector bundle.[173] Let L' be the dual $\mathcal{D}_X$-vector bundle and $(\ ) \in P_2^*(\{L', L\}, \omega_X)$ the canonical non-degenerate pairing (see 2.2.13). As in 3.8.6 we have $T := L \oplus L'$ with the symmetric $*$ pairing $(\ )_T$, the $\mathbb{Z}$-graded Clifford algebra $\mathcal{C}l = \mathcal{C}l(T)$ and the Tate extension $\mathfrak{g}l(L)^\flat$ of Lie* algebra $\mathfrak{g}l(L) = L \otimes L'$ identified with $\mathcal{C}l(T)_2^0$. The adjoint action of L yields a morphism of Lie* algebras $L \to \mathfrak{g}l(L)$. Let $L^\flat$ be the pull-back to L of the extension $\mathfrak{g}l(L)^\flat$; this is *the Tate extension* of L. So we have a canonical morphism of Lie* algebras $\beta : L^\flat \to \mathfrak{g}l(L)^\flat \subset \mathcal{C}l$ which sends $1^\flat \in L^{\flat\ell}$ to $1 \in \mathcal{C}l^\ell$.

3.8.8. Let A be a chiral algebra and $\alpha : L^\flat \to A$ a morphism of Lie* algebras such that $\alpha(1^\flat) = -1_A$; such pair (A, α) is called *a BRST datum.* Since α and β take opposite values on $1^\flat \in L^\flat$ we have a morphism of Lie* algebras

$$\ell^0 := \alpha + \beta : L \to A \otimes \mathcal{C}l^0. \tag{3.8.8.1}$$

Let $L_\dagger$ be the DG Lie* algebra introduced in 1.4.5 (as a plain complex it equals $\mathcal{C}one(id_L)$). The above ℓ^0 extends to a morphism of graded Lie* algebras

$$\ell : L_\dagger \to A \otimes \mathcal{C}l, \tag{3.8.8.2}$$

where ℓ^{-1} is the composition of the embeddings $L[1] \to \mathcal{C}l^{-1} \to A \otimes \mathcal{C}l$.

[173] i.e., X-locally L is a projective $\mathcal{D}_X$-module of finite rank.

To define the BRST current we need a simple lemma. Let $\delta : L' \to \mathrm{Sym}^2(L'[-1])[2]$ be the Chevalley differential (see 1.4.9) and $i : \mathrm{Sym}(L'[-1]) \hookrightarrow \mathcal{C}l \hookrightarrow A \otimes \mathcal{C}l$ the usual embedding. As usual, $i^a : \mathrm{Sym}^n(L'[-1]) \hookrightarrow A \otimes \mathcal{C}l^n$ are the components of i.

3.8.9 Lemma. The $*$ operation $[\ell^0, i^1] + [\ell^{-1}, i^2\delta] \in P_2^*(\{L, L'\}, A \otimes \mathcal{C}l[1])$ vanishes. Here $[\ \ ]$ is the bracket on $A \otimes \mathcal{C}l$.

Proof. This statement has nothing to do with A. It follows immediately from the coincidence of $[\beta, i^1], -[\ell^{-1}, i^2\delta] \in P_2^*(\{L, L'\}, \mathcal{C}l[1])$ (now $[\ \ ]$ is the bracket in $\mathcal{C}l$): both operations equal to the coadjoint action of L composed with embedding i^1. □

Consider a chiral operation

$$\tilde{\chi} := \mu(\ell^0, i^1) + \mu(\ell^{-1}, i^2\delta) \in P_2^{ch}(\{L, L'\}, A \otimes \mathcal{C}l[1]) \tag{3.8.9.1}$$

where μ is the chiral product on $A \otimes \mathcal{C}l$. The corresponding $*$ operation vanishes by 3.8.9, so it amounts to a morphism of graded $\mathcal{D}_X$-modules

$$\chi : L \otimes L' \to A \otimes \mathcal{C}l[1]. \tag{3.8.9.2}$$

Since L is a $\mathcal{D}_X$-bundle we have $h(L \otimes L') \xrightarrow{\sim} \mathrm{End}(L)$. Set

$$\mathfrak{d} = \mathfrak{d}_{A,\alpha} := -\chi(id_L) \in \hbar(A \otimes \mathcal{C}l[1]) = h(A \otimes \mathcal{C}l^1)^{odd}. \tag{3.8.9.3}$$

This is *the BRST current* for A with respect to α. Its adjoint action is an odd derivation d of $A \otimes \mathcal{C}l$ of degree 1 is called *the BRST differential.*

3.8.10 Theorem. (i) One has $[\mathfrak{d}, \mathfrak{d}] = 0$, hence $d^2 = 0$.

The DG chiral algbebra $(A \otimes \mathcal{C}l, d)$ is denoted by $\mathcal{C}_{\mathrm{BRST}}(L, A)$.

(ii) ℓ commutes with differentials, so we have a morphism of DG Lie* algebras

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}(L, A). \tag{3.8.10.1}$$

(iii) $\mathfrak{d}$ is a unique element of $\hbar(A \otimes \mathcal{C}l)$ such that $ad_{\mathfrak{d}}\ell^{-1} = \ell^0$. Similarly, d is a unique derivation of $A \otimes \mathcal{C}l$ of degree 1 such that $d\ell^{-1} = \ell^0$.

Remarks. (i) Property (ii) says that ℓ^{-1} is a d-homotopy between $\ell^0 = \alpha + \beta$ and 0; we refer to it as *the BRST property* of $\mathfrak{d}$ or d. According to (iii), the BRST property determines $\mathfrak{d}$ and d uniquely.[174]

(ii) The construction of the BRST element and the above theorem remain literally valid in the setting of chiral algebras on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ (see 3.3.9). We assume that L is a Lie* algebra on $\mathcal{Y}$ which is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$, etc.

Proof. It occupies 3.8.11–3.8.13.

3.8.11. First we show that $d\ell^{-1} = \ell^0 \in \mathrm{Hom}(L, A \otimes \mathcal{C}l^0)$, i.e., $\mathfrak{d}$ and d satisfy (iii).

Consider the operation $\mu(\tilde{\chi}, \ell^{-1}) \in P_3^{ch}(\{L, L', L\}A \otimes \mathcal{C}l)$. One has (in formulas below lower indices 1,2,3 label the variables) $\mu(\tilde{\chi}, \ell^{-1}) = \mu(\mu(\ell_1^0, i_2^1), \ell_3^{-1}) + \mu(\mu(\ell_1^{-1}, i^2\delta_2), \ell_3^{-1}) = \mu(\mu(\ell_1^0, \ell_3^{-1}), i_2^1) + \mu(\ell_1^0, \mu(i_2^1, \ell_3^{-1})) + \mu(i^2\delta_2, \mu(\ell_1^{-1}, \ell_3^{-1})) + \mu(\ell_1^{-1}, \mu(\ell_3^{-1}, i^2\delta_2))$. Let $[\tilde{\chi}, \ell^{-1}]$ be the restriction of this operation to $j_*j^*(L \boxtimes L') \boxtimes L \subset j_*^{(3)}j^{(3)*}(L \boxtimes L' \boxtimes L)$. One has $[\tilde{\chi}, \ell^{-1}] = \mu(\ell^{-1}[\ \ ]_{13}, i_2^1) + \mu(\ell_1^0, <>_{23}) + \mu(\ell_1^{-1}, [\ell_3^{-1}, i^2\delta_2])$. Notice that $[\ell^{-1}, i^2\delta] = -[\ell^0, i^1]$ since the ℓ-adjoint action of the graded Lie* algebra $L_\dagger$ on $\mathrm{Sym}(L'[-1]) \subset A \otimes \mathcal{C}l$ coincides with the canonical action from 1.4.9 which is compatible with the differentials. Thus $[\tilde{\chi}, \ell^{-1}] = -\mu(<>_{23}, \ell_1^0) - \mu([\ell_3^0, \ell_1^{-1}], i_2^1) - \mu(\ell_1^{-1}, [\ell_3^0, i_2^1]) = -\mu(<>_{23}, \ell_1^0) - [\mu(i_2^1, \ell_1^{-1}), \ell_3^0]$. The triple $*$ operations that correspond to both of the latter summands vanish, so we can rewrite them as binary $*$ operations $\gamma_1, \gamma_2 \in P_2^*(\{L \otimes L', L\}, A \otimes \mathcal{C}l)$, and our equality becomes $[\chi, \ell^{-1}] = -\gamma_1 - \gamma_2$. Thus $d\ell^{-1} = \gamma_1(id_L) + \gamma_2(id_L)$ where $\gamma_i(id_L) \in \mathrm{Hom}(L, A \otimes \mathcal{C}l)$ are morphisms obtained from γ_i by convolution with $id_L \in h(L \otimes L')$. One has $\gamma_1(id_L) = \ell^0$ and $\gamma_2(id_L) = 0$, and we are done.

3.8.12. Now let us prove the uniqueness statement of 3.8.10(iii).

(a) Let $F \subset h(A \otimes \mathcal{C}l)$ be the centralizer of ℓ^{-1}, so F contains $A \otimes \mathrm{Sym}(L[1])$. Let us show that actually $F = A \otimes \mathrm{Sym}(L[1])$. This statement follows from its "classical" version 1.4.21. Indeed, consider the ring filtration on $A \otimes \mathcal{C}l$ generated by $(A \otimes \mathrm{Sym}(L[1]), A \otimes (\mathrm{Sym}(L[1]) + L'))$ (see 3.7.13). The associated graded algebra equals $A \otimes \mathcal{C}l_c$ (see 1.4.20). Consider the corresponding filtration on $h(A \otimes \mathcal{C}l)$; we have $\mathrm{gr}h(A \otimes \mathcal{C}l) = h(A \otimes \mathcal{C}l_c)$. Take

[174] This is a chiral version of a theorem of Akman [Ak].

$v \in h(A \otimes \mathcal{C}l^a)$; let n be the smallest number such that it lies in nth term of the filtration, so the corresponding $\bar{v} \in \mathrm{gr}_n h(A \otimes \mathcal{C}l_c^a) = h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])$ is non-zero (cf. the proof of 1.4.21). Then the composition $ad_v \ell^{-1}$ takes values in the previous term, and the corresponding morphism from L^Π to $\mathrm{gr}_{n-1}(A \otimes \mathcal{C}l^a) \subset A \otimes \mathcal{C}l_c^{a-1}$ is the morphism $\psi(\bar{v})$ from 1.4.21. So 1.4.21 implies that for $v \in F$ one has $n = 0$, q.e.d.

(b) Consider the maps $h(A \otimes \mathcal{C}l^a) \to \mathrm{Der}(A \otimes \mathcal{C}l)^a \to \mathrm{Hom}(L^\Pi, A \otimes \mathcal{C}l^{a-1})$, the first arrow is the adjoint action, the second one is $\nu \mapsto \nu\ell^{-1}$. Now both of them are injective for $a \geq 1$; this obviously implies 3.8.10(iii). The proof is an obvious modification of that of its "classical" counterpart 1.4.23(iii) (replace 1.4.21 by (a) above and $\mathcal{C}l_c$ by $\mathcal{C}l$).

3.8.13. Let us show that $[\mathfrak{d}, \mathfrak{d}] = d(\mathfrak{d})$ vanishes. According to 3.8.12(a) it suffices to prove that $[\mathfrak{d}, \mathfrak{d}] \in F$, i.e., $d^2\ell^{-1} = 0$. One has $d\ell^{-1} = \ell^0$ (see 3.8.11), so we want to show that $d\ell^0 = 0$. Again by 3.8.12(a) it suffices to show that $d\ell^0$ takes values in F, i.e., $[d\ell^0, \ell^{-1}] = 0$. One has[175] $[d\ell^0, \ell^{-1}] = d[\ell^0, \ell^{-1}] - [\ell^0, d\ell^{-1}] = d\ell^{-1}[\quad] - [\ell^0, d\ell^{-1}] = \ell^0[\quad] - [\ell^0, \ell^0] = 0$, and we are done.

Finally 3.8.10(ii) follows from 3.8.11 and 3.8.13(a). □

3.8.14. The DG chiral algebra $\mathcal{C}_{\mathrm{BRST}}(L, A)$ considered as an object of an appropriate derived category is called *the BRST reduction* of A with respect to α. The reduction is *regular* if $H^a_{\mathrm{BRST}}(L, A) = 0$ for $a \neq 0$; then the BRST reduction is the plain chiral algebra $H^0_{\mathrm{BRST}}(L, A)$.

For any graded $A \otimes \mathcal{C}l$-module N the action of $\mathfrak{d}$ on N is a d-derivation of square 0, so N is automatically a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. In particular, for an A-module M the module $M \otimes \mathcal{C}l^\cdot$ (see 3.8.2) is a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. We denote it by $\mathcal{C}_{\mathrm{BRST}}(L, M)$ and call the BRST complex of L with coeficients in M. Its cohomology $H^\cdot_{\mathrm{BRST}}(L, M)$ is an $H^\cdot_{\mathrm{BRST}}(L, A)$-module; this is *BRST*, or *semi-infinite*, homology of L with coefficients in M. As follows from functoriality property (see 3.8.11(a)) $C_{\mathrm{BRST}}(L, M)$ depends, as a complex of plain $\mathcal{D}_X$-modules, only on the chiral L^b-module structure on M.

[175] The first equality comes since d is a derivation, the second and the fourth one follow since ℓ commutes with brackets, the third one is 3.8.11.

Variant. Let $x \in X$ be a point, $j_x : U_x \hookrightarrow X$ its complement. Assume that our BRST datum is given on U_x. According to 3.4.5, 3.5.17 the category of $j_{x*}A \otimes \mathcal{C}l$-modules supported at x coincides with the category of discrete $A_x^{as}\hat{\otimes}\mathcal{C}l_x^{as}$-modules. We can consider $\mathfrak{d}$ as an element of $h_x j_{x*}A \otimes \mathcal{C}l$, so its action yields a canonical differential on any discrete $A_x^{as}\hat{\otimes}\mathcal{C}l_x^{as}$-module. We will see in 3.8.25 that it can be described in terms of Tate linear algebra.

Remark. In this setting we have no canonical $j_{x*}\mathcal{C}l$-module supported at x, so to define the BRST homology of an A_x^{as}-module M we should choose some $\mathcal{C}l_x^{as}$-module C and then define the BRST homology of M as the homology of $M \otimes C$. If L is extended to X, i.e., we have a Lie* subalgebra $L_X \subset j_{x*}L$ then we can take for C the module corresponding to the fiber at x of the Clifford module for L_X.

3.8.15. Let us show that the BRST construction for coisson algebras from 1.4.20–1.4.24 can be considered as a "classical limit" of the chiral BRST.

Let A_t, $\alpha_t : L_t^\flat \to A_t$ be a $k[t]$-family of BRST data (see Remark (ii) in 3.8.10).[176] Assume that A_t is a flat $k[t]$-module, $A_0 := A/tA$ is a commutative chiral algebra, and the Lie* bracket $[\ \ ]_t$ on L_t is divisible by t.

The latter condition implies that the extension $L_0^\flat$ of $L_0 := L_t/tL_t$ splits canonically, so we have $\alpha_0 : L_0 \to A_0$. According to 3.2.6, A_0 is a coisson algebra. Set $[\ \ ]_t^- := t^{-1}[\ \ ]_t$; this is a Lie* bracket on L_t. Then α_0 is a morphism of Lie* algebras if we equip L_0 with bracket $[\ \ ]_0^- := [\ \ ]_t^- \mathrm{mod}t$. Thus we have the "classical" BRST complex $\mathcal{C}_{\mathrm{BRST}}(L_0, A_0)_c$ as defined in 1.4.23. We want to compare it with $\mathcal{C}_{\mathrm{BRST}}(L_t, A_t)$.

Let $\mathcal{C}l_t$ be the chiral Clifford $k[t]$-algebra for L_t, L_t', and $\mathcal{C}l_t^- \subset \mathcal{C}l_t$ the chiral subalgebra generated by $L[1]$ and $tL'[-1]$. Set $L_0 := L_t/tL_t$, $L_0' := L_t'/tL_t'$. We see that $\mathcal{C}l_0^- := \mathcal{C}l_t^-/t\mathcal{C}l_t^- = \mathrm{Sym}(L_0[1] \oplus L_0'[-1])$ is a commutative chiral algebra,[177] and the coisson bracket on $\mathcal{C}l_0^-$ that comes from the deformation $\mathcal{C}l_t^-$ is equal to the coisson Clifford bracket from 1.4.20. Therefore the coisson algebra $\mathcal{C}l_0^-$ coincides with the coisson Clifford algebra $\mathcal{C}l_c$ from 1.4.20.

[176] So L_t is a locally projective $\mathcal{D}_X[t]$-module, in particular, it is a $k[t]$-flat.

[177] Here the embedding $L_0'[-1] \hookrightarrow \mathcal{C}l_0^-$ comes from the the embedding $L_t'[-1] \hookrightarrow \mathcal{C}l_t$ multiplied by t.

Consider the graded chiral $k[t]$-subalgebra $A_t \otimes \mathcal{C}l_t^- \subset A_t \otimes \mathcal{C}l_t$. Its fiber over $t = 0$ equals $A_0 \otimes \mathcal{C}l_c$ (the tensor product of coisson algebras). The morphism $t\tilde{\chi}$ from (3.8.9.1) takes values in $A_t \otimes \mathcal{C}l_t^-$, so[178] we get a canonical element $\mathfrak{d}^- \in \hbar(A_t \otimes \mathcal{C}l_t^-[1])$ whose image in $\hbar(A_t \otimes \mathcal{C}l_t^-[1])$ equals $t\mathfrak{d}$. Set $\mathfrak{d}_c := \mathfrak{d}_0^- \in \hbar(A_0 \otimes \mathcal{C}l_c[1])$.

Since the algebra $A_0 \otimes \mathcal{C}l_0^-$ is commutative, the BRST differential $ad_{\mathfrak{d}}$ preserves the subalgebra $A_t \otimes \mathcal{C}l_t^- \subset A_t \otimes \mathcal{C}l$. We denote this DG $k[t]$-subalgebra of $\mathcal{C}(L_t, A_t)$ by $\mathcal{C}(L_t, A_t)^-$. As follows from the uniqueness statement 1.4.23(iii) its fiber at $t = 0$ coincides with the DG algebra $\mathcal{C}_{\mathrm{BRST}}(L_0, A_0)_c$ from 1.4.23. Notice that the differential on $\mathcal{C}(L_t, A_t)$ equals the adjoint action of $\mathfrak{d}^-$ with respect to the bracket $\{\quad\}_t$ from 3.2.6, so 1.4.23(iii) also shows that $\mathfrak{d}_c$ coincides with the "classical" BRST current of (1.4.22.2).

3.8.16 Example. Let A, $\alpha : L^\flat \to A$ be a BRST datum, and $A.$ be a commutative filtration on A (see 3.3.11) such that $\alpha(L^\flat) \subset A_1$. Define A_t as in Example from 3.2.6. Set $L_t := L \otimes k[t]$; let $\alpha_t : L_t \to A_t$ be the $k[t]$-linear morphism whose restriction to L equals the composition $L \to A_1 \subset A_t$. We equip L_t with a $k[t]$-linear Lie* bracket $[\quad]_t := t[\quad]_L$. Then (A_t, α_t) is a $k[t]$-family of BRST data that satisfies conditions of 3.8.15. So we have the corresponding flat $k[t]$-family $\mathcal{C}_{\mathrm{BRST}}(L_t, A_t)^-$ of DG chiral algebras whose fiber at $t = 1$ equals $\mathcal{C}_{\mathrm{BRST}}(L, A)$ and at $t = 0$ equals $\mathcal{C}_{\mathrm{BRST}}(L, \mathrm{gr}A)_c$.[179]

One can also look at the above situation as follows. Consider an increasing filtration on $A \otimes \mathcal{C}l$ with nth term equal to $\Sigma A_{n-i} \otimes \mathrm{Sym}(L'[-1]) \cdot \mathrm{Sym}^i(L[1])$. This is a commutative filtration; one has $\mathrm{gr}(A \otimes \mathcal{C}l) = \mathrm{gr}A \otimes \mathcal{C}l_c$ as coisson algebras. The BRST differential preserves our filtration. The classical BRST differential preserves the similar filtration[180] on $\mathrm{gr}A \otimes \mathcal{C}l_c$, and the above identification is compatible with the differentials: indeed, both associated graded algebras equal to the Chevalley complexes $\mathcal{C}(L, \mathrm{gr}A \otimes \mathrm{Sym}(L[1]))$. The above considerations show that the first terms of the corresponding spectral sequences also coincide. In particular, if the "classical" $E_1^{p,q}$ vanishes for $p + q \neq 0$ then both "classical" and "quantum" BRST reductions are regular.

[178] Replacing in (3.8.9.2), (3.8.9.3) χ by $t\chi$ and $\mathcal{C}l_t$ by $\mathcal{C}l_t^-$.
[179] With respect to the morphism $\alpha_0 : L \to \mathrm{gr}A$ equal to the composition $L \to A_1 \to \mathrm{gr}_1 A$.
[180] Which comes from the obvious grading.

In the rest of this section we will show that the associative topological DG algebras $\mathcal{C}_{\mathrm{BRST}}(L,A)^{as}_x$ can be computed purely in terms of Tate's linear algebra.

3.8.17 BRST in the Tate linear algebra setting. We assume that the reader knows the contents of 2.6.9–2.6.11. Below "topological vector space" means (unless stated explicitly otherwise) k-vector space equipped with a complete and separated linear topology. Topological vector spaces form a tensor category with respect to completed tensor product $\hat{\otimes}$.[181] If T is a Tate vector space, R any topological vector space then $R\hat{\otimes}T^*$ is equal to the space $\mathrm{Hom}(T,R)$ of continuous linear maps $T\to R$. More generally, $R\hat{\otimes}T^{\hat{\otimes}n}$ is equal to the space of continuous polylinear maps $T\times ..\times T\to R$. All topological associative algebras are assumed to be unital, complete and separated and the topology should be formed by open left ideals. The category of topological associative algebras is a tensor category with respect to $\hat{\otimes}$. For a topological associative algebra A we denote by $A\mathrm{mod}$ the category of left discrete A-modules. This is an abelian k-category.

For a topological Lie algebra L its *topological enveloping algebra* $U(L)$ is a topological associative algebra equipped with a continuous morphism of Lie algebras $L\to U(L)$ which is universal in the obvious sense. Thus $U(L)$ is a completion of the plain enveloping algebra with respect to topology made of left ideals generated by open subspaces $V\subset L$. If the topology on L is generated by open Lie subalgebras then $U(L)$ satisfies Poincaré-Birkhoff-Witt theorem: the associated graded algebra $\mathrm{gr}U(L)$ for the standard filtration[182] equals the completed symmetric algebra $\mathrm{Sym}L$. This condition holds automatically if L is a Tate vector space.[183]

Remark. If $L^\flat$ is a central extension of L by k then we have the corresponding twisted topological enveloping algebra $U(L)^\flat$; this is a topological associative algebra equipped with a morphism of topological Lie algebras $L^\flat\to U(L)^\flat$ which sends $1^\flat\in k\subset L^\flat$ to $1_{U(L)^\flat}$ and is universal morphism with these properties. The PBW theorem for $U(L)^\flat$

[181] $A\hat{\otimes}B$ is the completion of $A\otimes B$ with respect to topology formed by subspaces $U\otimes B+A\otimes V$ where U,V are open subspaces in A,B.

[182] Its terms are closures of the terms of the standard filtration on the plain enveloping algebra.

[183] Proof: for every c-lattice $V\subset L$ its normalizer in L intersected with V is an open Lie subalgebra of L contained in V.

holds if the topology on $L^\flat$ is generated by open Lie subalgebras (and we assume that the topology induced on $k \subset L^\flat$ is discrete).

Let T be a Tate (super) vector space, $<>=<>_T$ a skew-symmetric (continuous) bilinear form on T. Consider $T^\flat := T \oplus k$ as a Lie algebra with the non-zero component of the bracket equal to $<>$. This is a central extension of T (considered as a commutative Lie algebra) by k. The twisted topological enveloping algebra $W = U(T)^\flat$ is called *the topological Weyl algebra* of $(T, <>)$.

Assume that $<>$ is non-degenerate. Then the category Wmod can be described as follows. Let $V \subset T$ be a c-lattice; then its orthogonal complement $V^\perp$ is also a c-lattice. Assume that the restriction of $<>$ to V is trivial, i.e., $V^\perp \supset V$. The restriction of $<>$ to $V^\perp$ comes from the finite-dimensional vector space $V^\perp/V$. The Weyl algebra $W(V^\perp)$ is equal to the subalgebra of W generated by $V^\perp$ (and coincides with the centralizer of V in W). The projection $W(V^\perp) \to W(V^\perp/V)$ yields the functor $W(V^\perp/V)\text{mod} \to W\text{mod}$, $M \mapsto W \underset{W(V^\perp)}{\otimes} M$. This functor is fully faithful. According to a variant of Kashiwara's lemma it identifies $W(V^\perp/V)$mod with the subcategory of those W-modules on which every $v \in V \subset W$ acts in a locally nilpotent way. Every discrete W-module is a union of submodules which satisfy this property with respect to smaller and smaller V's.

As in 3.8.6 we can replace skew-symmetric pairing by a symmetric one to get a *topological Clifford algebra.* As above, we are interested in the graded version of this construction, so we start with a Tate vector space F and consider the topological graded Clifford algebra $\mathcal{C}l$ generated by the graded (super) Tate vector space $F^*[-1] \oplus F[1]$ such that the commutator pairing between $F^*[-1]$ and $F[1]$ equals the canonical pairing $F^*[-1] \otimes F[1] = F^* \otimes F \to k \subset \mathcal{C}l$. Our $\mathcal{C}l$ contains commutative graded subalgebras[184] $\text{Sym}(F^*[-1])$, $\text{Sym}(F[1])$. It carries the standard filtration $\mathcal{C}l.$ compatible with the grading; the algebra $\text{gr}\mathcal{C}l$ is equal to $\text{Sym}(F^*[-1] \oplus F[1])$.

We have $\mathcal{C}l_0^0 = k$, $\mathcal{C}l_1^1 = F^*[-1]$ and $\mathcal{C}l_1^{-1} = F[1]$. Notice that $\mathcal{C}l_2^0$ is equal to the normalizer of $F^*[-1] \oplus F[1]$ in $\mathcal{C}l$. The adjoint action of this Lie subalgebra preserves both

[184] Here Sym denotes the completed symmetric algebra.

$F^*[-1]$ and $F[1]$, and identifies $\mathcal{C}l_2^0/k$ with $\mathfrak{g}l(F)$ (as a topological Lie algebra).[185] Thus $\mathcal{C}l_2^0$ is a central extension of $\mathfrak{g}l(F)$ by k. It equals the Tate extension $\mathfrak{g}l(F)^\flat$ from 2.6.10:

3.8.18 Proposition. There is a canonical isomorphism of topological central extensions

$$\mathcal{C}l_2^0 \xrightarrow{\sim} \mathfrak{g}l(F)^\flat.$$

Proof. According to 2.6.1(ii) and 2.6.10 such an isomorphism amounts to a pair of continuous sections $s_\pm : \mathfrak{g}l_\pm(F) \to \mathcal{C}l_2^0$ such that $s_\pm$ commute with the adjoint action of $\mathfrak{g}l(F)$, $s_+ + s_- : \mathfrak{g}l_+(F) \oplus \mathfrak{g}l_-(F) \to \mathcal{C}l_2^0$ is an open map, and $s_+ - s_- : \mathfrak{g}l_0(F) \to k \subset \mathcal{C}l_2^0$ equals tr. Recall (see Exercise in 2.6.9) that $\mathfrak{g}l_\pm(F)$ are completions of $F \otimes F^*$ with respect to appropriate topologies. We define s_+, s_- restricted to $F \otimes F^*$ as, respectively, compositions $F \otimes F^* = F^*[-1] \otimes F[1] \xrightarrow{\cdot} \mathcal{C}l_2^0$, $F \otimes F^* = F[1] \otimes F^*[-1] \xrightarrow{\cdot} \mathcal{C}l_2^0$ where $\cdot$ is the product map. It follows immediately from Exercise in 2.6.9 that $s_\pm$ are continuous, so they extend to continuous maps $s_\pm : \mathfrak{g}l_\pm(F) \to \mathcal{C}l_2^0$. We leave it to the reader to check that $s_\pm$ satisfy the properties listed above. □

3.8.19. Let L be a topological Lie algebra which is a Tate vector space. The adjoint action yields a continuous morphism of Lie algebras $ad : L \to \mathfrak{g}l(L)$. The *Tate extension* of L is the pull-back $L^\flat$ of the Tate extension $\mathfrak{g}l(L)^\flat$ by ad.

Remark. For every Lie subalgebra $P \subset L$ which is a c-lattice the adjoint action maps P to $\mathfrak{g}l(L)_P$, so, according to Remark in 2.6.11, we have a canonical section $s_P : P \to L^\flat$. If $P' \subset P \subset L$ is another such subalgebra then the restriction of s_P to P' equals $s_{P'} + \mathrm{tr}_{P/P'}$ (the trace of the adjoint action of P' on P/P').

According to 3.8.18 we have a canonical morphism of topological Lie algebras $\beta : L^\flat \to \mathfrak{g}l(L)^\flat \subset \mathcal{C}l^0$ where $\mathcal{C}l$ is the Clifford algebra as in 3.8.17 (for $F = L$). One has $\beta(1^\flat) = 1_{\mathcal{C}l}$.

Let A be a topological associative algebra. A *BRST datum* is a continuous morphism of Lie algebras $\alpha : L^\flat \to A$ such that $\alpha(1^\flat) = -1_A$. Since α and β take opposite values on $1^\flat$ we have a morphism of topological Lie algebras

$$\ell^0 := \alpha + \beta : L \to A\hat{\otimes}\mathcal{C}l^0. \tag{3.8.19.1}$$

[185] Use Exercise in 2.6.9.

Let $L_\dagger$ be the topological DG Lie algebra with non-zero components $L_\dagger^0 = L, L_\dagger^{-1} = L[1]$, the differential $d = id_L$, and the Lie bracket on $L_\dagger^0$ equal to the bracket on L. The above ℓ^0 extends to a morphism of topological graded Lie algebras

$$\ell : L_\dagger \to A\hat{\otimes}\mathcal{C}l, \tag{3.8.19.2}$$

where ℓ^{-1} is the composition $L[1] \to \mathcal{C}l^{-1} \to A\hat{\otimes}\mathcal{C}l^{-1}$.

3.8.20. We are going to define *the BRST current* $\mathfrak{d}$ which is an odd element of $A\hat{\otimes}\mathcal{C}l^1$. Let $\mathfrak{d}_1^{\sim} \in A\hat{\otimes}\mathcal{C}l^0\hat{\otimes}L^*$ be the element that corresponds[186] to the linear map ℓ^0, and $\mathfrak{d}_1 \in A\hat{\otimes}\mathcal{C}l[1]^0$ the image of $\mathfrak{d}_1^{\sim}$ by the product map $\mathcal{C}l\hat{\otimes}L^*[-1] \to \mathcal{C}l$. Let $\mathfrak{d}_2^{\sim} \in L\hat{\otimes}L^*\hat{\otimes}L^*$ be the element that corresponds to the Lie bracket $L \times L \to L$, and $\mathfrak{d}_2 \in \mathcal{C}l[1]^0 \subset A\hat{\otimes}\mathcal{C}l[1]^0$ its image by the product map $L\hat{\otimes}L^*\hat{\otimes}L^* = (L[1]\hat{\otimes}L^*[-1]\hat{\otimes}L^*[-1])[1] \xrightarrow{\cdot} \mathcal{C}l[1]$. Set

$$\mathfrak{d} := \mathfrak{d}_1 - \frac{1}{2}\mathfrak{d}_2 \in A\hat{\otimes}\mathcal{C}l[1]^0. \tag{3.8.20.1}$$

The adjoint action of $\mathfrak{d}$ is *the BRST differential*. This is a continuous odd derivation of $A\hat{\otimes}\mathcal{C}l$ of degree 1.

3.8.21 Proposition. (i) One has $[\mathfrak{d}, \mathfrak{d}] = 0$, hence $d^2 = 0$.

The topological DG associative algbebra $(A \otimes \mathcal{C}l, d)$ is denoted by $\mathcal{C}_{\mathrm{BRST}}(L, A)$.

(ii) ℓ commutes with differentials, so we have a morphism of DG Lie* algebras

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}(L, A). \tag{3.8.21.1}$$

This is *the BRST property* of $\mathfrak{d}$ or d. It determines $\mathfrak{d}$ and d uniquely:[187]

(iii) $\mathfrak{d}$ is a unique odd element in $A\hat{\otimes}\mathcal{C}l^1$ such that $ad_{\mathfrak{d}}\ell^{-1} = \ell^0$. Similarly, d is a unique continuous odd derivation of degree 1 of $A\hat{\otimes}\mathcal{C}l$ such that $d\ell^{-1} = \ell^0$.

Proof (cf.the proof of 3.8.10). (a) An immediate computation shows that $d\ell^{-1} = \ell^0$.

(b) It is clear that the centralizer F of $L[1] \subset \mathcal{C}l \subset A\hat{\otimes}\mathcal{C}l$ equals $A\hat{\otimes}\mathrm{Sym}(L[1]) \subset A\hat{\otimes}\mathcal{C}l$. In particular, F sits in degrees ≤ 0.

[186] See the beginning of 3.8.17.

[187] This is a version of a theorem of Akman [A].

(c) Let us check (iii). Property $[\mathfrak{d}, \ell^{-1}] = \ell^0$ determines $\mathfrak{d}$ uniquely up to an element of F^1. So $\mathfrak{d}$ is unique. Similarly, property $d\ell^{-1} = \ell^0$ determines d up to a derivation ∂ which kills the image of $L[1]$. Such ∂ sends $L^*[-1] \subset \mathcal{C}l^1 \subset A\hat{\otimes}\mathcal{C}l^1$ and $A \subset A\hat{\otimes}\mathcal{C}l^0$ to F. Since ∂ has degree 1 it kills $L^*[-1]$ and A according to (a). Thus $\partial = 0$, i.e., d is determined uniquely.

(d) (i) follows from (b); the argument repeats 3.8.13. (ii) follows from (i) and (a). □

3.8.22. For every graded discrete $A\hat{\otimes}\mathcal{C}l$-module N the action of $\mathfrak{d}$ on N is an odd d-derivation of degree 1 and square 0, so it defines on N a canonical structure of DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. Fix a graded discrete $\mathcal{C}l$-module C; then for a discrete A-module M the tensor product $M \otimes C$ becomes a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module called *the BRST, or semi-infinite homology,* complex of M (relative to C).

Remarks. (i) If L is purely even then an irreducible $\mathcal{C}l$-module is unique up to a twist by a 1-dimensional (super) vector space. So the BRST complex with respect to irreducible C is uniquely defined up to a twist by 1-dimensional graded (super) vector space.

(ii) BRST complexes behave in the obvious manner with respect to morphisms of A's. In particular, the BRST complex of M depends (as a complex) only on the $L^\flat$-module structure on M (coming from α).

3.8.23. Now we can compare the chiral and the Tate linear algebra versions of BRST. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ its complement.

Let L be a $\mathcal{D}$-vector bundle on U_x, L' the dual $\mathcal{D}$-vector bundle. According to 2.6.12 the Tate vector space $L'_{(x)} := \hat{h}_x(j_{x*}L')$ is dual to $L_{(x)}$, i.e., $L'_{(x)} = L^*_{(x)}$.

Let $\mathcal{C}l^{\cdot}$ be the graded chiral Clifford algebra generated by $L'[-1] \oplus L[1]$ (see 3.8.6), and $\mathcal{C}l^{\cdot}_{(x)}$ the topological graded Clifford algebra generated by $L'_{(x)}[-1] \oplus L_{(x)}[1]$ (see 3.8.17). According to 3.7.23 there is a canonical isomorphism of topological graded algebras $\mathcal{C}l_x^{\cdot as} \xrightarrow{\sim} \mathcal{C}l^{\cdot}_{(x)}$.

Here is a simple alternative construction of the identification $r^\flat$ of (2.6.16.1)[188] Consider the Tate extension $\mathfrak{g}l(L)^\flat = \mathcal{C}l_2^0$ (see (3.8.6.1)). The previous isomorphism yields then a

[188] Our L is V of 2.6.16.

morphism of Lie algebras $h_x j_{x*}\mathfrak{g}l(L)^\flat \to \mathcal{C}l^0_{(x)2}$. It is obviously continuous with respect to $\Xi^{sp\flat}_x$-topology (see 2.6.15), so we have a morphism of topological Lie $\mathfrak{g}l(L)^\flat_{(x)} \to \mathcal{C}l^0_{(x)2}$. Using the identification $\mathcal{C}l^0_{(x)2} = \mathfrak{g}l(L_{(x)})^\flat$ of 3.8.18 we can rewrite this morphism as $r^\flat : \mathfrak{g}l(L)^\flat_{(x)} \to \mathfrak{g}l(L_{(x)})^\flat$. By construction this $r^\flat$ is a lifting of the isomorphism $r : \mathfrak{g}l(L)_{(x)} \xrightarrow{\sim} \mathfrak{g}l(L_{(x)})$ of 2.6.14 to our central k-extensions.

Now we assume that our L is a Lie* algebra, so we have the Tate extension $L^\flat$ (see 3.8.7). Then $L_{(x)}$ is a topological Lie algebra (see 2.5.12, 2.5.13(ii)) and $L^\flat_{(x)}$ is a central extension of $L_{(x)}$ by k. By definition of $L^\flat$ the adjoint action of L lifts canonically to a morphism of topological extensions $L^\flat_{(x)} \to \mathfrak{g}l(L)^\flat_{(x)}$. Thus $r^\flat$ provides a canonical identification of $L^\flat_{(x)}$ with the Tate linear algebra extension $(L_{(x)})^\flat$ from 3.8.19.

Assume we have a chiral BRST datum, i.e., a chiral algebra A on U_x and a morphism of Lie* algebras $\alpha : L^\flat \to A$ such that $\alpha(1^\flat) = -1_A$ (see 3.8.8). This yields the topological associative algebra $A^{as}_x = (j_{x*}A)^{as}_x$ (see 3.4.2–3.4.12) and the morphism $\alpha_{(x)} : L^\flat_{(x)} \to A^{as}_x$ which sends $1^\flat$ to -1 which is a Tate linear algebra BRST datum (see 3.8.19).

According to 3.5.17 the functor $A \mapsto A^{as}_x$ commutes with tensor products, so $(A \otimes \mathcal{C}l)^{as}_x = A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$.

3.8.24 Lemma. The canonical morphism (3.4.4.2) of graded Lie algebras

$$h_x j_{x*}A \otimes \mathcal{C}l \to (A \otimes \mathcal{C}l)^{as}_x = A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$$

sends the chiral BRST current (3.8.9.3) to the Tate linear algebra one (3.8.20.1).

Proof. The image of chiral $\mathfrak{d}$ satisfies the BRST property. Now use 3.8.21(i). □

3.8.25. According to 3.4.5 the categories of graded $j_{x*}A \otimes \mathcal{C}l$-modules supported at x and discrete $A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$-modules are canonically identified. Each of these objects carries a canonical BRST differential (defined as the action of the BRST currents), and the above lemma identifies these differentials. This provides a canonical identification of the BRST homology from 3.8.14 (Variant) with those from 3.8.22.

§9 Chiral differential operators

In this section we explain what are enveloping chiral algebras of Lie* algebroids, in particular chiral algebras of differential operators (case of the tangent algebroid). The key difference with the case of Lie* algebras treated in §7 is that the construction of enveloping algebra requires an extra structure (that of chiral extension) on the Lie* algebroid.

3.9.1. Let R be a commutative chiral algebra, so (see (3.3.3.1)) R^ℓ is a commutative $\mathcal{D}_X$-algebra.

We say that a chiral R-module M is an *R-differential module* or simply *$R_{\mathcal{D}if}$-module*, if M is a union of submodules which are successive extensions of commutative R-modules.[189] Equivalently, this means that there is a filtration $\{0\} = M_{-1} \subset M_0 \subset M_1 \subset .., \cup M_i = M$, such that $\mathrm{gr}_i M$ are commutative.[190] Denote by $\mathcal{M}(X,R)^{ch}_{\mathcal{D}if} \subset \mathcal{M}(X,R)^{ch}$ the full pseudo-tensor subcategory of $R_{\mathcal{D}if}$-modules. The subcategory $\mathcal{M}(X,R)_{\mathcal{D}if} \subset \mathcal{M}(X,R)$ is closed under direct sums and subquotients. Any chiral R-module N has the maximal differential submodule $N^{\mathcal{D}if}$. Every chiral operation preserves maximal differential submodules.

Consider the sheaf of commutative Lie algebras $h(R)$ on X. It acts canonically on every chiral R-module M, so M is a $\mathrm{Sym}h(R)$-module. If $M \in \mathcal{M}(X,R)$ is an $R_{\mathcal{D}if}$-module then the $h(R)$-action on M is locally nilpotent which means that every $m \in M$ is killed by some $\mathrm{Sym}^i h(R)$, $i >> 0$. The converse is true if we know that for every $i > 0$ the subsheaf of M that consists of section killed by $\mathrm{Sym}^i h(R)$ is $\mathcal{O}_X$-quasicoherent. This always happens if M is either $\mathcal{O}_X$-flat or supported in finitely many points.

Remark. Let $f : R_1^\ell \to R_2^\ell$ be a morphism of $\mathcal{D}_X$-algebras. If M is an $R_{2\mathcal{D}if}$-module, then it is an $R_{1\mathcal{D}if}$-module via f, so we have a faithful pseudo-tensor functor $f_! : \mathcal{M}(X,R_2)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X,R_1)^{ch}_{\mathcal{D}if}$.

3.9.2. Consider the topological algebra with connection (R^{as}, ∇) defined in the end of 3.3. Since R is commutative R^{as} is also commutative. We have $R^\ell = R^{as}/I_R$ for an open ideal $I_R \subset R^{as}$. Let $R^{\mathcal{D}if}$ be the I_R-adic completion of R^{as}. Precisely, we consider on R^{as}

[189] See 3.3.6, 3.3.9. This definition makes sense for any (not necessary commutative) chiral algebra R.
[190] You may define M_i by induction taking for M_i the preimage of $(M/M_{i-1})^{com}$.

the topology defined by those open ideals $I \subset I_R$ that $I_R^n \subset I$ for $n >> 0$; our $R^{\mathcal{D}if}$ is the corresponding completion. This is a commutative topological $\mathcal{O}_X$-algebra equipped with a connection ∇.

We define discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules in the same way as we defined discrete $R^{as}[\mathcal{D}_X]$-modules in 3.4.6.[191] $R^{\mathcal{D}if}$*-polylinear* chiral operations between such modules are defined as in 3.4.9. So we have the pseudo-tensor category $\mathcal{M}^\ell(X, R^{\mathcal{D}if})^{ch}$.

The equivalence from 3.4.7 identifies $R_{\mathcal{D}if}$-modules with discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules. By 3.4.10 chiral operations between chiral $R_{\mathcal{D}if}$-modules are the same as $R^{\mathcal{D}if}$-polylinear chiral operations. Therefore we have the pseudo-tensor equivalence

$$\mathcal{M}(X, R)^{ch}_{\mathcal{D}if} \xrightarrow{\sim} \mathcal{M}^\ell(X, R^{\mathcal{D}if})^{ch}. \tag{3.9.2.1}$$

3.9.3 Proposition. (i) If $f : R_1^\ell \to R_2^\ell$ is an étale morphism of $\mathcal{D}_X$-algebras then the corresponding morphism of topological algebras $R_1^{\mathcal{D}if} \to R_2^{\mathcal{D}if}$ is also étale.

(ii) $R_{\mathcal{D}if}$-modules and chiral operations between them have étale local nature.

Proof. (i) Let $I_1 \subset R_1^{\mathcal{D}if}$ be an open ideal, $I_2 \subset R_2^{\mathcal{D}if}$ the closed ideal generated by $f(I_1)$. We want to show that I_2 is open and the morphism $R_1^{\mathcal{D}if}/I_1 \to R_2^{\mathcal{D}if}/I_2$ is étale. This follows from the fact that the morphism $R_1^{\mathcal{D}if} \to R_2^{\mathcal{D}if}$ is formally étale, which is an immediate corollary of 3.4.12.

(ii) Follows from (i), 3.9.2, and 3.4.10. □

Remark. The statement (ii) means, in particular, that for étale f the pseudo-tensor functor $f. : \mathcal{M}(X, R_2)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X, R_1)^{ch}_{\mathcal{D}if}$ admits a left adjoint pseudo-tensor functor $f^\cdot : \mathcal{M}(X, R_1)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X, R_2)^{ch}_{\mathcal{D}if}$. One has $f^\cdot M = R_2^{\mathcal{D}if} \underset{R_1^{\mathcal{D}if}}{\otimes} M$.

According to 3.9.3(ii) one has the notion of $\mathcal{O}_{\mathcal{D}if}$-module on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$; we also call these objects $\mathcal{O}$-*modules on* $\mathcal{Y}_{\mathcal{D}if}$. We denote the corresponding abelian pseudo-tensor category by $\mathcal{M}(\mathcal{Y})^{ch}_{\mathcal{D}if}$.

[191] Certainly discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules are the same as discrete $R^{as}[\mathcal{D}_X]$-modules such that the action of R^{as} comes from the discrete action of $R^{\mathcal{D}if}$.

3.9.4. For a commutative R a *chiral $R_{\mathcal{D}if}$-algebra* is a chiral algebra A together with a morphism of chiral algebras $R \to A$ such that A is an $R_{\mathcal{D}if}$-module. According to 3.9.3(ii) these objects have étale local nature, so we have the notion of chiral $\mathcal{O}_{\mathcal{D}if}$-algebra on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$; we also call them *chiral algebras on* $\mathcal{Y}_{\mathcal{D}if}$. The corresponding category is denoted by $\mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$.

Example. Let A be a chiral algebra equipped with a commutative filtration $R = A_0 \subset A_1 \subset ..$ (see 3.3.11). Then A is a chiral $R_{\mathcal{D}if}$-algebra. Notice that in this situation $\mathrm{gr}_1 A$ is a Lie* R-algebroid (see 1.4.10) in the obvious way.

Let A be a chiral $R_{\mathcal{D}if}$-algebra. We say that a chiral A-module M is *R-differential* if such is M considered as an R-module. This notion is étale local, so for any $\mathcal{Y}$ as above and $\mathcal{A} \in \mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$ we know what are *$\mathcal{O}$-differential $\mathcal{A}$-modules on $\mathcal{Y}$* or simply *A-modules on $\mathcal{Y}_{\mathcal{D}if}$*. These objects form an abelian pseudo-tensor category $\mathcal{M}(\mathcal{Y}, \mathcal{A})^{ch}_{\mathcal{D}if}$; we also have the corresponding sheaf of pseudo-tensor categories $\mathcal{M}(\mathcal{Y}_{\text{ét}}, \mathcal{A})^{ch}_{\mathcal{D}if}$ on $\mathcal{Y}_{\text{ét}}$.

3.9.5. Assume for the moment that R is smooth (see 2.3.16). A *chiral algebra of twisted differential operators on* Spec$R$ (we abbreviate it to chiral tdo on Spec$R$ or chiral R-tdo) is a chiral $R_{\mathcal{D}if}$-algebra $\mathcal{D}$ which admits a commutative filtration $\mathcal{D}_0 \subset \mathcal{D}_1 \subset ..$ (see 3.3.11) such that $R = \mathcal{D}_0$ and there is an isomorphism of coisson $\mathbb{Z}$-graded algebras $\mathrm{Sym}^{\cdot}\Theta_R \xrightarrow{\sim} \mathrm{gr}.\mathcal{D}$ identical on R.[192]

Notice that the filtration $\mathcal{D}.$ is uniquely defined: indeed, $\mathcal{D}_0 = R$, $\mathcal{D}_1$ is the normalizer of R in $\mathcal{D}$, and our filtration is 1-generated (see 3.7.13). We call it *the canonical filtration.* The above isomorphism of the coisson graded algebras is also uniquely defined.

It is clear that chiral tdo's are local objects for the étale topology. Thus we know what are chiral tdo's on any smooth algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. We denote the corresponding category by $\mathcal{TDO}(\mathcal{Y}) \subset \mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$. This is a groupoid. We have the corresponding sheaf of groupoids $\mathcal{TDO}(\mathcal{Y}_{\text{ét}})$ on $\mathcal{Y}_{\text{ét}}$.

The principal difference between chiral tdo's and tdo's in the usual setting (see, e.g., [BB]) is that there is no *canonical* ("non-twisted") chiral tdo. We also do not know if chiral

[192] Exercise: show that the existence of such an isomorphism amounts to the folowing conditions: the adjoint $*$ action of $\mathrm{gr}_1\mathcal{D}$ on $\mathcal{D}_0 = R$ identifies it with Θ_R and $\mathrm{gr}_1\mathcal{D}$ generates $\mathrm{gr}.\mathcal{D}$.

tdo's always exist locally in étale topology. This is true (as follows from (3.9.18.2)) if Θ_R is a locally trivial $\mathcal{D}$-vector bundle (which happens when R is a jet $\mathcal{D}_X$-algebra). If chiral tdo's exist locally, then they form a gerbe. We will see that this gerbe need not be trivial.

Just as in the usual setting one handles chiral tdo's realizing them as twisted enveloping algebras of certain Lie algebroids. We explain the chiral Lie algebroid basics in 3.9.6–3.9.14 returning to tdo's in 3.9.15.

3.9.6. Let R be any commutative chiral (super)algebra. An important class of chiral $R_{\mathcal{D}if}$-algebras is formed by chiral enveloping algebras of Lie* R-algebroids. It appears however that, contrary to the case of Lie* algebras, to define the enveloping algebra one needs an extra structure on our algebroid. We describe this structure first; for the definition of chiral enveloping algebra see 3.9.11.

Definition. Let $\mathcal{L}$ be a Lie* R-algebroid (see 1.4.10). A *chiral R-extension* of $\mathcal{L}$ is a $\mathcal{D}$-module extension $0 \to R \xrightarrow{i} \mathcal{L}^\flat \xrightarrow{\sigma} \mathcal{L} \to 0$ together with chiral R-module and Lie* algebra structures on $\mathcal{L}^\flat$ that satisfy the following compatibilities (a)–(d):

(a) i and σ are morphisms of chiral R-modules.

(b) σ is a morphism of Lie* algebras and i is compatible with the Lie* actions of $\mathcal{L}^\flat$.[193]

(c) The R-action $\mu_{\mathcal{L}^\flat} \in P_2^{ch}(\{R, \mathcal{L}^\flat\}, \mathcal{L}^\flat)$ is compatible with Lie* action of $\mathcal{L}^\flat$.[194]

(d) Consider the $*$-pairing $\mu^*_{\mathcal{L}^\flat} \in P_2^*(\{R, \mathcal{L}^\flat\}, \mathcal{L}^\flat)$ that comes from the chiral R-action $\mu_{\mathcal{L}^\flat}$. By (a) we know that it takes values in $i(R) \subset \mathcal{L}^\flat$ and vanishes when the second argument lies in $i(R)$. So $\mu^*_{\mathcal{L}^\flat} \in P_2^*(\{R, \mathcal{L}\}, R)$, and we demand that this operation equals to the structure $*$-action of $\mathcal{L}$ on R.

One calls a pair $(\mathcal{L}^\flat, \mathcal{L})$ as above[195] a *chiral Lie R-algebroid* or a *chiral Lie algebroid on* Spec$R$. These objects form a category $\mathcal{LA}^{ch}(R) = \mathcal{LA}^{ch}(\mathrm{Spec}R)$. By 2.5.15, 3.9.3 they have étale local nature, so we know what are chiral Lie algebroids on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. The corresponding category is denoted by $\mathcal{LA}^{ch}(\mathcal{Y})$, the sheaf of categories by $\mathcal{LA}^{ch}(\mathcal{Y}_{\text{ét}})$. For fixed $\mathcal{L}$ the groupoid of its chiral extensions is denoted by $\mathcal{P}^{ch}(\mathcal{L}) =$

[193] Here $\mathcal{L}^\flat$ acts on R via σ and the $\mathcal{L}$-action, and on $\mathcal{L}^\flat$ via the adjoint action.

[194] See 3.1.13(i).

[195] We will also denote it simply $\mathcal{L}^\flat$.

$\mathcal{P}^{ch}(\mathcal{Y},\mathcal{L})$, the corresponding sheaf of groupoids on $\mathcal{Y}_{\text{ét}}$ by $\mathcal{P}^{ch}(\mathcal{Y}_{\text{ét}},\mathcal{L})$.

Example. In the situation of Example in 3.9.4 A_1 is a chiral Lie R-algebroid. We will see in 3.9.13 that every $L^\flat$ that satisfies certain flatness condition comes in this way.

One usually constructs chiral extensions by means of the following lemma.

Let L be a Lie* algebra, τ an L-action on R, and $L^\flat$ an ω-extension of L (we follow notation of 3.7.20). For a chiral Lie R-algebroid $\mathcal{L}^\flat$ a *morphism* $(L^\flat,\tau)\to\mathcal{L}^\flat$ is a Lie* algebra morphism $\psi^\flat\colon L^\flat\to\mathcal{L}^\flat$ compatible with the actions on R and such that $\psi^\flat(1^\flat)=i(1_R)$. The set $\mathrm{Hom}((L^\flat,\tau),\mathcal{L}^\flat)$ of such morphisms depends on $\mathcal{L}^\flat$ in functorial way.

3.9.7 Lemma. If L is a flat $\mathcal{O}_X$-module[196] then the functor $\mathrm{Hom}((L^\flat,\tau),\cdot)$ is representable.

We denote the universal chiral Lie algebroid by $\mathcal{L}^\flat_R$.

Proof. One constructs $L^\flat_R$ as follows. Set $F:=R\boxtimes L^\flat+j_*j^*R\boxtimes 1^\flat\omega_X\subset j_*j^*R\boxtimes L^\flat$, so $j_*j^*R\boxtimes L^\flat/F=\Delta_*\mathcal{L}_R$ where $\mathcal{L}_R:=R^\ell\otimes L$. Let $\phi\colon F\twoheadrightarrow\Delta_*R$ be the morphism that equals τ on $R\boxtimes L^\flat$ and the canonical projection on $j_*j^*R\boxtimes\omega_X$. We define $\mathcal{L}^\flat_R$ as a plain $\mathcal{D}_X$-module by $\Delta_*L^\flat_R:=j_*j^*R\boxtimes L^\flat/\mathrm{Ker}\phi$, so we have the extension $0\to R\xrightarrow{i}L^\flat_R\xrightarrow{\sigma}L_R\to 0$.

Assume for a second that $R^\ell=\mathcal{O}_X$; then $\tau=0$ and $L^\flat_{\omega_X}=L^\flat$. Therefore[197] for any R the morphism $1_R:\omega_X\to R$ yields the canonical morphism $\psi^\flat:L^\flat\to L^\flat_R$. It is clear that $\psi:=\sigma\psi^\flat:L\to L_R$ is the obvious embedding $l\mapsto 1_R\otimes l$.

The chiral R-action on $L^\flat_R$ comes as follows. It is easy to see that the morphism $j_*j^*R\boxtimes L\to\Delta_*L^\flat_R$ is continuous with respect to the topology on $j_*j^*R\boxtimes L$ described in 3.4.8. Thus we may rewrite our construction saying that $L^\flat_R$ is the push-out of the extension $0\to I_R\otimes L\to R^{as}\otimes L\to R^\ell\otimes L\to 0$ by the morphism $\tau':I_R\otimes L\to R$ coming from the action τ. One easily checks that τ' is R^{as}-linear map, thus $L^\flat_R$ is a discrete $R^{as}[\mathcal{D}_X]$-module, i.e., an R-module (see 3.4.7, 3.4.8). □

As follows from the lemma (and its proof) for a chiral Lie algebroid $\mathcal{L}^\flat$ a morphism $\psi^\flat:(L^\flat,\tau)\to\mathcal{L}^\flat$ such that the corresponding morphism $\psi:L\to\mathcal{L}$ yields an isomorphism

[196] If R is $\mathcal{O}_X$-flat then this condition is irrelevant.

[197] Our construction if functorial with respect to compatible morphisms of R and $L^\flat$'s.

$R^\ell \otimes L \xrightarrow{\sim} \mathcal{L}$ amounts to an isomorphism $L^\flat_R \xrightarrow{\sim} \mathcal{L}^\flat$. Such $\psi^\flat$ is called $(L^\flat, \tau)$-*rigidification* of $\mathcal{L}^\flat$. If $L^\flat = L \oplus \omega_X$ is the trivialized extension then we call $\psi^\flat$ simply (L, τ)-*rigidification*.

3.9.8. It is important to compare chiral extensions with "classical" ones. By definition, a *classical R-extension* of a Lie* R-algebroid $\mathcal{L}$ is an extension $\mathcal{L}^c$ of $\mathcal{L}$ by R in the category of Lie* algebroids. So we have a $\mathcal{D}$-module extension $0 \to R \xrightarrow{i} \mathcal{L}^c \xrightarrow{\sigma} \mathcal{L} \to 0$ together with a Lie* R-algebroid structure on $\mathcal{L}^c$ such that σ is a morphism of Lie* algebroids and i is a morphism of R^ℓ-modules.[198] These objects have étale local nature, so, as above, we have the groupoid of classical extensions $\mathcal{P}^{cl}(\mathcal{L}) = \mathcal{P}^{cl}(\mathcal{Y}, \mathcal{L})$.

The Baer sum of classical extensions makes $\mathcal{P}^{cl}(\mathcal{L})$ a Picard groupoid. We can also form k-linear combinations of classical extensions, so $\mathcal{P}^{cl}(\mathcal{L})$ is a k vector space category.

If $\mathcal{L}^\flat$ is a chiral extension and $\mathcal{L}^c$ is a classical extension of $\mathcal{L}$ then we have the Baer sum of extensions $\mathcal{L}^c + \mathcal{L}^\flat$. This is a chiral R-module and a Lie* algebra in the obvious way. It obviously satisfies compatibilities (a)–(d) of 3.9.6, so $\mathcal{L}^c + \mathcal{L}^\flat$ is a chiral extension of $\mathcal{L}$. Therefore we have defined an Action of the Picard groupoid $\mathcal{P}^{cl}(\mathcal{L})$ on $\mathcal{P}^{ch}(\mathcal{L})$.

3.9.9 Lemma. If $\mathcal{P}^{ch}(\mathcal{L})$ is non-empty then it is a $\mathcal{P}^{cl}(\mathcal{L})$-Torsor in a canonical way.

Proof. For two chiral extensions their Baer difference is a classical extension. □

3.9.10. If $\mathcal{L}^\ell$ is a projective $R[\mathcal{D}_X]$-module then classical extensions of $\mathcal{L}$ may be described in terms of the Chevalley complex $C(\mathcal{L}) = C(\mathcal{L}, R)$ from 1.4.13 as follows. A $C(\mathcal{L})^{\geq 1}$-*torsor* is a $C(\mathcal{L})^1$-torsor S together with a d_C-affine map $d_S : S \to C(\mathcal{L})^2$ such that $d_C d_S = 0$. These objects form a Picard category $C(\mathcal{L})^{\geq 1}$-tors in the obvious way. Now there is a canonical equivalence of Picard categories

$$S : \mathcal{P}^{cl}(\mathcal{L}) \xrightarrow{\sim} C(\mathcal{L})^{\geq 1}\text{-tors} \tag{3.9.10.1}$$

which assigns to $\mathcal{L}^c$ the torsor $S(\mathcal{L}^c)$ of $R[\mathcal{D}_X]$-linear sections $s : \mathcal{L} \to \mathcal{L}^c$, $d_S(s) := [s, s] - s([\ \ ])$.

This description globalizes in the obvious manner. Assume that our $\mathcal{L}$ is a $\mathcal{D}_X$-bundle on $\mathcal{Y}$ (see 2.3.11). According to 2.5.16 we have the complex $\mathcal{C}(\mathcal{L})$ such that the complex of

[198] i automatically commutes with $\mathcal{L}^c$ actions.

sheaves $h(\mathcal{C}(\mathcal{L}))$ equals to the sheafified $C(\mathcal{L})$. We see that the group of automorphisms of a classical extension is[199] $H^1(Y_{\text{ét}}, DR(\mathcal{C}(\mathcal{L})^{\geq 1}))$, and the isomorphism classes of objects of $\mathcal{P}^{cl}(\mathcal{L})$ form the group $H^2(Y_{\text{ét}}, DR(\mathcal{C}(\mathcal{L})^{\geq 1}))$.

Assume in addition that locally on $\mathcal{Y}_{\text{ét}}$ our $\mathcal{L}$ admits a chiral extension. Then, as follows from 3.9.9, our $\mathcal{P}^{ch}(\mathcal{Y}_{\text{ét}}, \mathcal{L})$ is a $C(\mathcal{L})^{\geq 1}$-gerbe. So the obstruction to the existence of a global chiral extension lies in $H^3(Y_{\text{ét}}, DR(\mathcal{C}(\mathcal{L})^{\geq 1}))$.

3.9.11. Let $\mathcal{L}^\flat = (\mathcal{L}, \mathcal{L}^\flat)$ be a chiral Lie R-algebroid. For a chiral algebra A a *chiral morphism* $\mathcal{L}^\flat \to A$ is a morphism of $\mathcal{D}_X$-modules $\varphi : \mathcal{L}^\flat \to A$ compatible with all the structures. Precisely, this means that $\varphi i : R \to A$ is a morphism of chiral algebras and φ commutes with the chiral R-actions and $*$ brackets. The chiral morphisms $\mathcal{L}^\flat \to A$ depend on A in functorial manner, and 3.7.1 immediately implies that there is a universal morphism which we denote by $\mathcal{L}^\flat \to U(R, \mathcal{L})^\flat$. Indeed, $U(R, \mathcal{L})^\flat$ is a quotient of the chiral enveloping algebra[200] $U(\mathcal{L}^\flat)$ modulo the relations needed to make the canonical arrow $\mathcal{L}^\flat \to U(\mathcal{L}^\flat)$ a chiral morphism. We call $U(R, \mathcal{L})^\flat$ the *chiral envelope* of our chiral Lie algebroid.

It is clear that $U(R, \mathcal{L})^\flat$ is a chiral $R_{\mathcal{D}if}$-algebra and the enveloping algebra functor is compatible with étale localization of R. Therefore the above constructions make sense over any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. Notation: $U(\mathcal{L})^\flat_{\mathcal{Y}}$.

Exercise. For a chiral Lie algebroid $\mathcal{L}^\flat$ on $\mathcal{Y}$ and an $\mathcal{O}_{\mathcal{D}if}$-module M on $\mathcal{Y}$ define the notion of chiral action of $\mathcal{L}^\flat$ on M (see 3.7.16). Show that $\mathcal{O}_{\mathcal{D}if}$-modules on $\mathcal{Y}$ equipped with a chiral $\mathcal{L}^\flat$-action are the same as $U(\mathcal{L})^\flat_{\mathcal{Y}}$-modules on $\mathcal{Y}_{\mathcal{D}if}$ (cf. 3.7.18).

We define *the standard*, or *Poincaré-Birkhoff-Witt, filtration* on $U(R, \mathcal{L})^\flat$ as the image of the standard filtration on $U(\mathcal{L}^\flat)$. In other words, this is the filtration generated by the image of R and that of $\mathcal{L}^\flat$ (see 3.7.13). This filtration is commutative, so one has a canonical surjective morphism of $\mathbb{Z}$-graded coisson algebras

$$\text{Sym}^{\cdot}_{R^\ell} \mathcal{L}^\ell \twoheadrightarrow \text{gr.} U(R, \mathcal{L})^{\flat \ell}. \tag{3.9.11.1}$$

[199] Here DR is the de Rham complex along X.

[200] Here we consider $\mathcal{L}^\flat$ as a plain Lie* algebra.

One has the following version of the PBW theorem:

3.9.12 Theorem. If R is a flat $\mathcal{O}_X$-module and $\mathcal{L}$ is a flat R^ℓ-module then (3.9.11.1) is an isomorphism.

Proof. We deduce the statement from the ordinary PBW theorem 3.7.14. The reduction is of quite general nature. It also yields the PBW theorem for the usual Lie algebroids.[201] To make the argument more transparent we first present it in the setting of usual Lie algebroids (see 3.9.13), and then explain the slight modifications needed to cover the chiral situation (see 3.9.14).

The PBW theorem in the setting of usual Lie algebroids looks as follows. Let R be a commutative k-(super)algebra, L a Lie R-algebroid. Denote by $U(R,\mathcal{L})$ its enveloping algebra. This is an associative k-algebra equipped with a morphism of associative k-algebras $i : R \to U(R,\mathcal{L})$ and a morphism of Lie k-algebras $\varphi : \mathcal{L} \to U(R,\mathcal{L})$ such that φ is a morphism of R-modules (here R-acts on $U(R,\mathcal{L})$ via i followed by the left multiplication), and this datum is universal. It carries the standard, or PBW, filtration $U(R,\mathcal{L})_i$.[202] There is an obvious canonical morphism $\pi : \mathrm{Sym}^{\cdot}_R L \twoheadrightarrow \mathrm{gr}.U(R,\mathcal{L})$ of $\mathbb{Z}$-graded commutative (super)algebras. The PBW theorem says that

3.9.13 Theorem. π is an isomorphism if $\mathcal{L}$ is a flat R-module.

Proof. It consists of 3 steps A–C. First we consider the case when $\mathcal{L}$ admits a rigidification, i.e., $\mathcal{L} = R \otimes L$ for a Lie algebra L acting on R. Then our statement follows from the usual PBW for L. Next, in B we construct for any $\mathcal{L}$ a DG associative filtered superalgebra U together with an identification $H^0U = U(R,\mathcal{L})$ such that $\mathrm{gr}U$ is controllable (here step A is used). Finally, we show that if $\mathcal{L}$ is R-flat then U is quasiisomorphic to $U(R,\mathcal{L})$ as a filtered algebra which yields PBW.

A. Assume that $\mathcal{L}$ admits a rigidification, i.e., there exists a Lie k-algebra L and a morphism of Lie k-algebras $L \to \mathcal{L}$ which yields an isomorphism $R \otimes L \xrightarrow{\sim} \mathcal{L}$. Then the Lie

[201] Originally proved in [Rin] for Lie R-algebroids which are free R-modules.

[202] $U(R,\mathcal{L})_0$ is the image of R, $U(R,\mathcal{L})_1$ is the sum of the images of R and $\mathcal{L}$, and $U(R,\mathcal{L})_i = U(R,\mathcal{L})_1^i$ for $i > 0$.

algebra L acts on R. Denote by $U(R,L)$ the universal associative k-algebra equipped with an algebra morphism $i : R \to U(R,L)$ and a Lie algebra morphism $\varphi_L : L \to U(R,L)$ which are compatible with respect to this action.[203] We have the algebra morphism $R \to U(R,\mathcal{L})$ and the Lie algebra morphism $L \to U(R,\mathcal{L})$ which are compatible. It is clear that they satisfy the universality property, i.e., $U(R,L) = U(R,\mathcal{L})$. This leads to the following explicit description of $U(R,\mathcal{L})$.

The L action on R amounts to an action of the Hopf algebra $U(L)$ on the algebra R. Denote by $R \circledast U(L)$ the corresponding twisted tensor product of R and $U(L)$: this is an associative algebra which equals $R \otimes U(L)$ as a k-module with the product $(a \otimes u)\cdot(b \otimes v) = \Sigma a u_i(b) \otimes u_i' v$ where $\Sigma u_i \otimes u_i'$ is the coproduct of u. One checks in a moment that the obvious morphisms $R \to R \circledast U(L)$, $L \to R \circledast U(L)$ satisfy the above universality property, so we have a canonical isomorphism $U(R,L) \xrightarrow{\sim} R \circledast U(L)$. It identifies the standard filtration on $U(R,\mathcal{L})$ with the filtration $R \circledast U(L)_i := R \otimes U(L)_i$. Therefore the PBW theorem for $\mathcal{L}$ follows from the usual PBW for L.

B. Let $\mathcal{L}$ be *any* Lie R-algebroid. Let us define the DG associative filtered algebra U as promised above.

Consider $\mathcal{L}$ as a plain Lie algebra acting on R; let $\mathcal{L}_R = R \underset{k}{\otimes} \mathcal{L}$ be the corresponding rigidified Lie R-algebroid. We have the obvious surjective morphism of Lie R-algebroids $\mathcal{L}_R \twoheadrightarrow \mathcal{L}$; denote by K its kernel. Consider K as a plain R-module and set $\tilde{R} := \mathrm{Sym}_R(K[1])$; this is a $\mathbb{Z}$-graded commutative superalgebra. The Lie algebra $\mathcal{L}$ acts on $\tilde{R}$,[204] so we have the corresponding rigidified Lie $\tilde{R}$-algebroid $\mathcal{L}_{\tilde{R}} = \tilde{R} \underset{k}{\otimes} \mathcal{L}$ and its enveloping algebra $U := U(\tilde{R}, \mathcal{L}_{\tilde{R}})$. So we have the associative algebra morphism $\tilde{i} : \tilde{R} \to U$ and the Lie algebra morphism $\tilde{\varphi} : \mathcal{L}_{\tilde{R}} \to U$. As in step A let us consider $\tilde{i}$ together with the restriction $\tilde{\varphi}_{\mathcal{L}} : \mathcal{L} \to U$ of $\tilde{\varphi}$ to $\mathcal{L}$; these are morphisms of, respectively, associative and Lie algebras compatible via the $\mathcal{L}$-action on $\tilde{R}$, and U is the universal algebra equipped with such morphisms. The $\mathbb{Z}$-grading on $\tilde{R}$ yields $\mathbb{Z}$-gradings on $\mathcal{L}_{\tilde{R}}$ and U (we assume that $\mathcal{L}$ sits in degree 0).

[203] i.e., for $r \in R$, $l \in L$ one has $i(l(r)) = [\varphi_L(l) i(r)]$.

[204] The $\mathcal{L}$-action on $\tilde{R}^{-1} = K$ is the adjoint action of $\mathcal{L} \subset \mathcal{L}_R$ on $K \subset \mathcal{L}_R$.

We defined U as an enveloping algebra, so it carries the PBW filtration which is obviously compatible with the $\mathbb{Z}$-grading. Our U fits into the setting of A, so the corresponding associated graded algebra equals $\tilde{R} \underset{k}{\otimes} \mathrm{Sym}^{\cdot}_k \mathcal{L}$. We want to consider a different filtration: namely, the filtration which on a graded component U^a equals the PBW filtration shifted by a.[205] This is a ring filtration compatible with the grading; we call it *the canonical* filtration. Shifting appropriately the above PBW isomorphism we see that

$$\mathrm{gr}.U = \tilde{R}^{\cdot} \underset{k}{\otimes} \mathrm{Sym}^{\cdot}_k \mathcal{L} = \mathrm{Sym}^{\cdot}_R(K[1]) \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R \mathcal{L}_R = \mathrm{Sym}^{\cdot}_R(K[1] \oplus \mathcal{L}_R). \tag{3.9.13.1}$$

Our U carries a canonical derivation:

Sublemma. There is an odd derivation d of U of degree 1 which kills the images of R and $\mathcal{L}$ and such that on $\tilde{R}^{-1} = K[1]$ it equals the restriction ϕ of the canonical morphism $\mathcal{L}_{\tilde{R}} \xrightarrow{\varphi} U$ to $K \subset \mathcal{L}_R \subset \mathcal{L}_{\tilde{R}}$. Such d is unique.

Proof of Sublemma. The uniqueness of d is clear since our conditions define d on generators of U. It remains to show that d is correctly defined.

Set $\Lambda := \mathrm{Sym}(k[1]) = k \oplus k[1]$; this is a $\mathbb{Z}$-graded commutative (super)algebra. Consider the $\mathbb{Z}$-graded associative Λ-(super)algebra $U_\Lambda = U \otimes \Lambda = U \oplus U[1]$. We have the morphism of graded associative algebras $\tilde{R} \to U_\Lambda$ and that of Lie algebras $\mathcal{L} \to U_\Lambda$ which are compatible via the $\mathcal{L}$-action on $\tilde{R}$. They satisfy the obvious universality property.

As usual, an odd derivation of degree 1 of U is the same as an automorphism of the $\mathbb{Z}$-graded Λ-algebra U_Λ which induces identity on its quotient U. We will construct our d using the universality property of U_Λ.

The conditions on d mean that the corresponding automorphism α_d of U_Λ is identity on the images of R and $\mathcal{L}$, and on $\tilde{R}^{-1} = K[1]$ it equals $\nu := \tilde{i} + \phi : K[1] \to U \oplus U[1]$. Now ν takes values in the centralizer $Z_\Lambda \subset U_\Lambda$ of R; it is a morphism of R-modules, and it is compatible with $\mathcal{L}$-actions. Thus we get α_d from the universality property of U_Λ the moment we extend ν to a morphism of R-algebras $\tilde{R} = \mathrm{Sym}_R(K[1]) \to Z_\Lambda$. Such extension

[205] So the morphism $\tilde{i} : R \to U$ is compatible with filtrations if we equip R with the filtration coming from the grading.

is unique, and its existence amounts to the fact that on the image of ν the product of U_Λ is commutative. One checks this by a direct computation:

Namely, let ε, ε' be a generators of copies of $k[1]$ that have to do with $K[1]$ and Λ. Then for $a \in K$ one has $\nu(a \cdot \varepsilon) = \tilde{i}(a \cdot \varepsilon) + \phi(a)\varepsilon'$. Thus[206] $[\nu(a \cdot \varepsilon), \nu(b \cdot \varepsilon)] = [\phi(a)\varepsilon', \tilde{i}(b \cdot \varepsilon)] + [\tilde{i}(a \cdot \varepsilon), \phi(b)\varepsilon'] = (-1)^{\bar{b}+1}\, \tilde{i}([a,b] \cdot \varepsilon)\varepsilon' + (-1)^{\bar{b}}\, \tilde{i}([a,b] \cdot \varepsilon)\varepsilon' = 0$, q.e.d. □

It is clear that $d^2 = 0$ (check on the generators), so U is a DG algebra which sits in degrees ≤ 0. Our d is R-linear, and it preserves the canonical filtration. Its acts on $\mathrm{gr}_1 U = K[1] \oplus \mathcal{L}_R$ as the canonical embedding $K \to \mathcal{L}_R$, so we can rewrite (3.9.13.1) as a canonical isomorphism of graded DG algebras

$$\mathrm{gr}.U = \mathrm{Sym}^{\cdot}_R \mathcal{C}one(K \to \mathcal{L}_R). \tag{3.9.13.2}$$

The obvious morphisms $\tilde{R} \twoheadrightarrow R$, $\mathcal{L}_{\tilde{R}} \twoheadrightarrow \mathcal{L}$ yield a morphism of the enveloping algebras $U = U(\tilde{R}, \mathcal{L}_{\tilde{R}}) \to U(R, \mathcal{L})$. It identifies $U(R, \mathcal{L})$ (as a filtered algebra) with the quotient of U modulo the ideal generated by $\tilde{R}^{<0}$ and the image of $K \hookrightarrow \mathcal{L}_R \subset \mathcal{L}_{\tilde{R}}$. Therefore $U(R, \mathcal{L}) = H^0 U$.

C. Now assume that $\mathcal{L}$ is R-flat. Then $0 \to K \to \mathcal{L}_R \to \mathcal{L} \to 0$ is a short exact sequence of flat R-modules. Therefore the projection $\mathrm{Sym}^{\cdot}_R \mathcal{C}one(K \to \mathcal{L}) \twoheadrightarrow \mathrm{Sym}^{\cdot}_R \mathcal{L}$ is a quasiisomorphism. By (3.9.13.2) one has $H^a \mathrm{gr}.U = 0$ for $a \neq 0$, hence $H^0 \mathrm{gr}.U \xrightarrow{\sim} \mathrm{gr}.H^0 U$. By (3.9.13.2) one has $H^0 \mathrm{gr}.U = \mathrm{Sym}^{\cdot}_R \mathcal{L}$, so we proved the PBW theorem. □

3.9.14. *Proof of* 3.9.12. Let us return to the chiral setting. The above arguments should be modified as follows. Again, we proceed in three steps.

A. Assume that $\mathcal{L}^\flat$ admits a rigidification in the sense of 3.9.7, i.e., there exists a Lie* algebra L and a morphism of Lie* algebras $\psi^\flat : L \to \mathcal{L}^\flat$ such that $\psi := \sigma\psi^\flat : L \to \mathcal{L}$ yields an isomorphism $R^\ell \otimes L \xrightarrow{\sim} \mathcal{L}$. Then L acts on R via the action τ defined as the composition of ψ and the structure action of $\mathcal{L}$ on R.

We have the morphism of chiral algebras $i : R \to U(R, \mathcal{L})^\flat$ and the morphism of Lie* algebras $L \to U(R, \mathcal{L})^\flat$;[207] the morphism i is compatible with L-actions. One checks easily

[206] Here $\bar{a} \in \mathbb{Z}/2$ is the pairity of a.

[207] Defined as composition of $\psi^\flat$ and the structure morphism $\mathcal{L}^\flat \to U(R, \mathcal{L})^\flat$.

(cf. the proof of uniqueness in 3.9.7) that these morphisms satisfy universality property, i.e., $U(R,\mathcal{L})^\flat$ equals the algebra $R \circledast U(L)$ from 3.7.24.

Now 3.9.12 follows from the PBW result of 3.7.24.

Steps B,C literally repeat those of 3.9.13. □

3.9.15 Corollary. For smooth $\mathcal{Y}$ there is a canonical equivalence of groupoids (see 3.9.5)

$$\mathcal{TDO}^{ch}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{P}^{ch}(\mathcal{Y}, \Theta_{\mathcal{Y}}).$$

Proof. We assign to a tdo $\mathcal{D}$ the term $\mathcal{D}_1$ of the canonical filtration with its obvious structure of the chiral algebroid. To a chiral algebroid $\Theta^\flat_{\mathcal{Y}}$ we assign its enveloping algebra $\mathcal{D}^\flat_{\mathcal{Y}} := U(\Theta_{\mathcal{Y}})^\flat_{\mathcal{Y}}$; this is a tdo by 3.9.12. □

The above results generalize to the DG setting in the obvious way. For example, for a DG commutative $\mathcal{D}_X$-superalgebra R[208] we know what are DG chiral $R_{\mathcal{D}if}$-algebras and DG chiral Lie R-algebroids. If R is smooth then we have the groupoid of chiral DG R-tdo's identified canonically with the groupoid of chiral DG R-extensions of Θ_R.

The following proposition is inspired by [MSV].[209] It may be considered as a version of 3.9.7 for L replaced by a single supersymmetry.

Let R^ℓ be a smooth $\mathcal{D}_X$-(super)algebra. Then $\Omega^\ell = \Omega_{R^\ell}$ is a smooth DG commutative $\mathcal{D}_X$-superalgebra.

3.9.16 Proposition. A DG Ω-tdo $\mathcal{D}_\Omega$ exists; it is unique up to a unique isomorphism.

Proof. (a) We need some facts about the structure of the DG Lie* Ω-algebroid Θ_Ω.

Set $\tilde{\Theta} = \mathcal{C}one(id_{\Theta_R})$, so $\tilde{\Theta}^{-1} = \tilde{\Theta}^0 = \Theta_R$. This complex carries a unique DG Lie* algebra structure such that the bracket on $\tilde{\Theta}^0$ is the usual Lie bracket on Θ_R.

Our $\tilde{\Theta}$ acts on Ω. The action morphism $\tau : \tilde{\Theta} \to \Theta_\Omega$ is constructed as follows. Notice that Ω sits in degrees ≥ 0 and Θ_Ω sits in degrees ≥ -1. Derivations in Θ_Ω^{-1} are R^ℓ-linear, and their action on Ω^1 yields an isomorphism $\Theta_\Omega^{-1} \xrightarrow{\sim} \mathcal{H}om_R(\Omega^1, R) = \Theta_R$. Its inverse is

[208] This means that R is a commutative $\mathcal{D}_X$-superalgebra equipped with an extra $\mathbb{Z}$-grading and an odd derivation of degree $+1$.

[209] The authors of [MSV] consider the case of a jet $\mathcal{D}$-scheme, and they work in the universal setting. We hope that in this situation our construction is equivalent to theirs.

a canonical embedding $\Theta_R[1] \hookrightarrow \Theta_\Omega$. Our τ is the morphism of complexes that extends this canonical embedding. One checks in a moment that $\tau^0 : \Theta_R \to \Theta^0_\Omega$ is the usual action of Θ_R on R extended to Ω by functoriality.[210] Hence τ is a morphism of DG Lie* algebras.

Let $\Theta_{\Omega/R} \subset \Theta_\Omega$ be the centralizer of $R^\ell \subset \Omega^\ell$; this is a graded Ω-submodule of Θ_Ω. The action of Θ_Ω on $R \subset \Omega$ yields the isomorphism $\Theta_\Omega/\Theta_{\Omega/R} \xrightarrow{\sim} \Omega^\ell \underset{R^\ell}{\otimes} \Theta_R$. The embedding $\tau^{-1} : \Theta_R[1] \hookrightarrow \Theta_{\Omega/R}$ extends to the isomorphism $\Omega^\ell \underset{R^\ell}{\otimes} \Theta_R[1] \xrightarrow{\sim} \Theta_{\Omega/R}$. Therefore we have the short exact sequence

$$0 \to \Omega^\ell \underset{R^\ell}{\otimes} \Theta_R[1] \to \Theta_\Omega \to \Omega^\ell \underset{R^\ell}{\otimes} \Theta_R \to 0 \tag{3.9.16.1}$$

of graded Ω-modules. The composition $\Theta_{\Omega/R} \xrightarrow{d} \Theta_\Omega \twoheadrightarrow \Theta_\Omega/\Theta_{\Omega/R}$ equals $id_{\Omega\otimes\Theta_R}$.

(b) *Uniqueness.* As follows from the DG version of 3.9.9, 3.9.15 the groupoid of DG Ω-tdo's is a Torsor over the category of classical DG Ω-extensions of Θ_Ω. The uniqueness amounts to triviality of the latter category.

So we want to prove that every such classical extension Θ^c_Ω admits a unique splitting. Since $\Omega^{-1} = 0$ we see that the projection σ yields the isomorphism $\Theta^{c-1}_\Omega \xrightarrow{\sim} \Theta^{-1}_\Omega$. Since Θ_Ω is generated as a DG Ω-module by Θ^{-1}_Ω we see that our splitting is unique.

Let us construct it. We see that τ lifts in a unique way to a morphism of complexes $\tau^c : \tilde{\Theta} \to \Theta^c_\Omega$. This is a morphism of Lie* algebras.[211] Let $\Theta'_\Omega \subset \Theta^c_\Omega$ be the Ω-submodule generated by the image of τ^c (i.e., the DG Ω-submodule generated by Θ^{c-1}_Ω). Now σ yields an isomorphism $\Theta'_\Omega \xrightarrow{\sim} \Theta_\Omega$. Indeed, τ^{c-1} is R^ℓ-linear, hence it yields a morphism $\Omega^\ell \underset{R^\ell}{\otimes} \Theta_R[1] \to \Theta'_\Omega$, and τ^{c0} is R^ℓ-linear modulo the image $\Theta'_{\Omega/R}$ of this morphism, so we have $\Omega^\ell \underset{R^\ell}{\otimes} \Theta_R \twoheadrightarrow \Theta'_\Omega/\Theta'_{\Omega/R}$; now use (3.9.16.1). The desired splitting is the inverse morphism; it is obviously compatible with all the structures.

(c) *Existence.* Consider the functor that assigns to a chiral DG algebra B the set of pairs (φ, ψ) where $\varphi: \Omega \to B$ is a morphism of chiral DG algebras and $\psi: \tilde{\Theta} \to B$ is a morphism

[210] It identifies Θ_R with $\mathrm{Ker} d : \Theta^0_\Omega \to \Theta^1_\Omega$.

[211] Take $\alpha, \beta \in \tilde{\Theta}^0$. Then α, β are exact, so $\alpha = d(\gamma)$ for $\gamma \in \tilde{\Theta}^{-1}$. One has $[\tau^c(\alpha), \tau^c(\beta)] = [d\tau^c(\gamma), \tau^c(\beta)] = d[\tau^c(\gamma), \tau^c(\beta)] = d\tau^c([\gamma, \beta]) = \tau^c d([\gamma, \beta]) = \tau^c([\alpha, \beta])$ (the third equality follows since the adjoint action of Θ^{c0}_Ω on $\Theta^{c-1}_\Omega = \Theta^{-1}_\Omega$ factors through Θ^0_Ω).

of graded Lie* algebras such that φ is compatible with $\tilde{\Theta}$-actions[212] and $\psi^{-1} : \Theta_R[1] \to B$ is R-linear.[213] This functor is representable by an appropriate quotient[214] $\mathcal{D}_\Omega$ of the chiral DG algebra $\Omega \circledast U(\tilde{\Theta})$. It remains to prove that $\mathcal{D}_\Omega$ is an Ω-tdo.

(d) Consider the DG Lie* Ω-algebroid $\tilde{\Theta}_\Omega := \Omega \underset{k}{\otimes} \tilde{\Theta}$ that corresponds to the action τ (see 1.4.12(i)). Let $K_\Omega \subset \tilde{\Theta}_\Omega$ be the kernel of the canonical morphism $\tau_\Omega : \tilde{\Theta}_\Omega \twoheadrightarrow \Theta_\Omega$. This is a DG Lie* Ω-subalgebra of $\tilde{\Theta}_\Omega$ that looks as follows.

Set $\tilde{\Theta}_{\Omega/R} := \Omega \underset{k}{\otimes} \tilde{\Theta}^{-1} = \Omega \otimes \Theta_R[1] \subset \tilde{\Theta}_\Omega$, $K_{\Omega/R} := K_\Omega \cap \tilde{\Theta}_{\Omega/R}$, and let K be the kernel of the product map $R^\ell \otimes \Theta_R \to \Theta_R$. One has $\tilde{\Theta}_\Omega^{-1} = R^\ell \otimes \Theta_R$ which yields identification $K_\Omega^{-1} = K$. It follows from (3.9.16.1) that $K_{\Omega/R} = \Omega^\ell \underset{R^\ell}{\otimes} K[1]$, we have a short exact sequence of graded Ω-modules $0 \to \Omega^\ell \underset{R^\ell}{\otimes} K[1] \to K_\Omega \to \Omega^\ell \underset{R^\ell}{\otimes} K \to 0$, and the composition $K_{\Omega/R} \xrightarrow{d} K_\Omega \twoheadrightarrow K_\Omega/K_{\Omega/R}$ is $id_{\Omega \otimes K}$.

(e) Let $\tilde{\Theta}_\Omega^\flat$ be the $(\tilde{\Theta}, \tau)$-rigidified chiral extension of $\tilde{\Theta}_\Omega$ (see 3.9.7). This is a DG chiral Lie Ω-algebroid. The projection σ yields the isomorphism $\tilde{\Theta}_\Omega^{\flat -1} \xrightarrow{\sim} \Theta_\Omega^{-1}$. Let $K_\Omega^\flat \subset \tilde{\Theta}_\Omega^\flat$ be the DG chiral Ω-submodule generated by $K_\Omega^{\flat -1} :=$ the preimage of K_Ω^{-1} in $\tilde{\Theta}_\Omega^{\flat -1}$. This is a Lie* ideal in $\tilde{\Theta}_\Omega^\flat$.[215]

Our $\mathcal{D}_\Omega$ is the quotient of $U(\Omega, \tilde{\Theta}_\Omega)^\flat$ modulo the ideal generated by the image of $K_\Omega^\flat$. We will prove in (f) that $i(\Omega) \cap K_\Omega^\flat = \mathrm{Ker}(K_\Omega^\flat \xrightarrow{\sigma} K_\Omega) = 0$. This implies that $\Theta_\Omega^\flat := \tilde{\Theta}_\Omega^\flat / K_\Omega^\flat$ is a chiral extension of Θ_Ω by Ω. So $\mathcal{D}_\Omega = U(\Omega, \Theta_\Omega)^\flat$ which is an Ω-tdo by 3.9.15, q.e.d.

(f) We want to show that the surjective map $K_\Omega^\flat \xrightarrow{\sigma} K_\Omega$ is an isomorphism. Our $K_\Omega^\flat$ lies in the centralizer of Ω,[216] so $K_\Omega^\flat$ is the Ω^ℓ-submodule generated by $K^{\flat -1} = K$ and $d(K^{\flat -1})$. Since $K^{\flat -1}$ is an R^ℓ-submodule, we have the map $\Omega^\ell \underset{R^\ell}{\otimes} K[1] \to K_\Omega^\flat$. This map is injective;[217] denote its image by $K_{\Omega/R}^\flat$. We have $\sigma : K_{\Omega/R}^\flat \xrightarrow{\sim} K_{\Omega/R}$. The composition $K = K_\Omega^{\flat -1} \xrightarrow{d} K_\Omega^{\flat 0} \twoheadrightarrow K_\Omega^{\flat 0}/K_{\Omega/R}^{\flat 0}$ is R^ℓ-linear, so it yields the morphism

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[212] Here $\tilde{\Theta}$ acts on B via ψ and the adjoint action.

[213] Here R acts on B via $R \subset \Omega \xrightarrow{\varphi} B$.

[214] Imposed to assure R-linearity of ψ^{-1}.

[215] The Lie* subalgebra $i(\Omega) + \tilde{\Theta}_\Omega^{\flat -1} + d(\tilde{\Theta}_\Omega^{\flat -1})$ obviously normalizes $K_\Omega^\flat$ and generates $\tilde{\Theta}_\Omega^\flat$ as an Ω-module. Since the adjoint action of $K_\Omega^\flat$ on $\tilde{\Theta}_\Omega^\flat$ commutes with the chiral Ω-action we see that $\tilde{\Theta}_\Omega^\flat$ normalizes $K_\Omega^\flat$.

[216] Which equals the kernel of the projection $\tilde{\Theta}_\Omega^\flat \to \tilde{\Theta}_\Omega \to \Theta_\Omega$.

[217] Since such its composition with σ.

$\Omega^\ell \underset{R^\ell}{\otimes} K \to K^\flat_\Omega / K^\flat_{\Omega/R}$. This is an isomorphism,[218] hence $\sigma: K^\flat_\Omega / K^\flat_{\Omega/R} \xrightarrow{\sim} K_\Omega / K_{\Omega/R}$, and we are done. □

3.9.17. We will use $\mathcal{D}_\Omega$, or rather a part of it, to give a "classical" description of R-tdo's.

The part we need is $\mathcal{L}^\natural := \mathcal{D}^0_{\Omega 1} = \Theta^{\flat 0}_\Omega$.[219] This is a chiral Lie R-algebroid that can be described explicitly as follows.

Let $\Theta^\flat_{\Omega/R} \subset \Theta^\flat_\Omega$, $\mathcal{D}_{\Omega/R} \subset \mathcal{D}_\Omega$ be the centralizers of $R \subset \Omega$. Both $\Theta^{\flat -1}_\Omega = \Theta_R$ and $\Omega^1 \subset \Theta^{\flat 1}_\Omega$ are commutative Lie* subalgebras of $\Theta^\flat_{\Omega/R}$; the commutator between them takes values in $R \subset \Omega$ and it is equal to the canonical R-duality. These subalgebras generate $\mathcal{D}_{\Omega/R}$, so $\mathcal{D}_{\Omega/R}$ is the graded Clifford R-algebra generated by Θ_R and Ω^1_R. In particular, the image of the chiral pairing $j_* j^* \Theta_R \boxtimes \Omega^1_R \to \mathcal{D}^0_{\Omega 1} = \mathcal{L}^\natural$ is the Tate R-extension $\mathfrak{g}l(\Theta_R)^\flat$ of the Lie* R-algebra $\mathfrak{g}l(\Theta_R)$ (see ??). One checks immediately that this is the centralizer of R in $\mathcal{L}^\natural$. So we have the exact sequence $0 \to \mathfrak{g}l(\Theta_R)^\flat \to \mathcal{L}^\natural \xrightarrow{\sigma} \Theta_R \to 0$. There is a canonical Lie* algebra splitting $s^\natural : \Theta_R \to \mathcal{L}^\natural$ equal to the morphism ψ^0 from the part (c) of the proof of 3.9.16; so $s^\natural$ is the composition $\Theta_R \xrightarrow{\sim} \Theta^{\flat -1}_{\Omega 1} \xrightarrow{d} \Theta^{\flat 0}_{\Omega 1}$. The $s^\natural$-adjoint action of Θ_R on $\mathfrak{g}l(\Theta_R)^\flat$ coincides with the canonical action. The section $s^\natural$ is *not* R-linear: since $\Theta_R \xrightarrow{\sim} \Theta^{\flat -1}_{\Omega 1}$ is R-linear, the Leibnitz formula shows that the chiral operation $\mu_{\mathcal{L}^\natural}(id_R, s^\natural) - s^\natural \mu_{\Theta_R} \in P^{ch}_2(\{R, \Theta_R\}, \mathfrak{g}l(\Theta_R)^\flat)$ is equal to the composition of the Clifford pairing $\mu_{\mathcal{C}l} \in P^{ch}_2(\{\Omega^1, \Theta_R\}, \mathfrak{g}l(\Theta_R))$ with $d : R \to \Omega^1$.

Summing up: as a Lie* algebra $\mathcal{L}^\natural$ is the semi-direct product of Θ_R and $\mathfrak{g}l(\Theta_R)^\flat$ with respect to the canonical action of Θ_R on $\mathfrak{g}l(\Theta_R)^\flat$. The chiral action of R on $\mathcal{L}^\natural$ equals the obvious R^ℓ-action on $\Theta_R \oplus \mathfrak{g}l(\Theta_R)^\flat$ plus $\mu_{\mathcal{C}l}(d, id_{\Theta_R}) \in P^{ch}_2(\{R, \Theta_R\}, \mathfrak{g}l(\Theta_R))$.

We see that $\mathcal{L}^\natural$ is a chiral R-extension of the Lie* R-algebroid equal to the semi-direct product of Θ_R and $\mathfrak{g}l(\Theta_R)$. This Lie* R-algebroid identifies canonically with the Lie* R-algebroid $\mathcal{E}(\Theta_R)$ of infinitesimal symmetries of the $\mathcal{D}_X$-vector bundle Θ_R (see 2.3.16). Indeed, $\mathcal{E}(\Theta_R)$ is an extension of Θ_R by $\mathfrak{g}l(\Theta_R)$, and the Lie action of Θ_R on the $\mathcal{D}_X$-vector bundle Θ_R provides a Lie algebra section $s : \Theta_R \to \mathcal{E}(\Theta_R)$ which is compatible with the

[218] It is injective since such is the composition $\Omega^\ell \underset{R^\ell}{\otimes} K \to K^\flat_\Omega / K^\flat_{\Omega/R} \to K_\Omega / K_{\Omega/R}$.

[219] The upper index is the grading, the lower one is the canonical filtration. Here $\Theta^\flat_\Omega$ is the DG chiral Lie Ω-algebroid that corresponds to $\mathcal{D}_\Omega$.

R^ℓ-action exactly as $s^\natural \mathrm{mod} i(R)$ does.

3.9.18. Let now $\Theta_R^\flat$ be a chiral R-extension of Θ_R. Denote by $\mathcal{E}(\Theta_R)^\flat$ the extension of $\mathcal{E}(\Theta_R)$ by R defined as the Baer difference of $\mathcal{L}^\natural$ and the pull-back of the extension Θ_R by $\mathcal{E}(\Theta_R) \xrightarrow{\tau} \Theta_R$. So a section of $\mathcal{E}(\Theta_R)^\flat$ may be written as a pair (l, θ) where $l \in \mathcal{L}^\flat$, $\theta \in \Theta_R^\flat$ are sections such that the images of l, θ in Θ_R coincide, and $(l, \theta) = 0$ if and only if $\theta = i(r) = l$ for some $r \in R$.

The obvious chiral R-action on $\mathcal{E}(\Theta_R)^\flat$ is *commutative*, so $\mathcal{E}(\Theta_R)^\flat$ is an R^ℓ-module. It is also a Lie* algebra with bracket $[(l, \theta), (l', \theta')] = ([l, l'], [\theta, \theta'])$. So $\mathcal{E}(\Theta_R)^\flat$ is a Lie* R-algebroid which is an R-extension of $\mathcal{E}(\Theta_R)$. The kernel of the projection $\mathcal{E}(\Theta_R)^\flat \xrightarrow{\tau} \Theta_R$ is the Tate extension $\mathfrak{g}l(\Theta_R)^\flat$. Therefore $\mathcal{E}(\Theta_R)^\flat$ is a Tate structure on Θ_R (see 2.6.7).

We have defined a functor

$$\mathcal{P}^{ch}(\Theta_R) \to \mathcal{T}ate(\Theta_R), \quad \Theta_R^\flat \mapsto \mathcal{E}(\Theta_R)^\flat. \tag{3.9.18.1}$$

Both our categories are $\mathcal{P}^{cl}(\Theta_R)$-Torsors (see 3.9.9, 2.6.8). By construction our functor *anti*commutes with $\mathcal{P}^{cl}(\Theta_R)$-translations. In particular, it is an equivalence of groupoids. Using 3.9.15 we may rewrite it as a canonical equivalence

$$\mathcal{TDO}^{ch}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{T}ate(\Theta_{\mathcal{Y}}) \tag{3.9.18.2}$$

valid for every smooth algebraic $\mathcal{D}_X$ space $\mathcal{Y}$.

§10 The universal setting

§11 Chiral homology

The chiral homology is a "quantum" version of the space of global horizontal sections of a $\mathcal{D}_X$-scheme.[220] One defines it as follows. We have seen in section 3.4 that chiral algebras amount to factorization algebras which are $\mathcal{D}$-modules on Ran's space $\mathcal{R}(X)$ (see 3.5.1). Now the chiral homology of X with coefficients in a chiral algebra is the de Rham homology of $\mathcal{R}(X)$ with coefficients in the corresponding $\mathcal{D}$-module.

[220] i.e., global solutions of a system of non-linear differential equations.

The section divides in three parts. First we make the above informal definition rigorous and establish some basic properties chiral homology (such as the compatibility with tensor and direct products); a key tool is the structure of Batalin-Vilkovisky algebra on the chiral homology complex. Then we "compute" chiral homology of chiral envelopes of Lie* algebras and algebroids, and explain the compatibility with the BRST reduction. Finally we show that when our picture varies in a way that the variation is governed by certain symmetries of chiral algebras then these symmetries yield a "connection" on the chiral homology bundles;[221] a typical example is the Knizhnik-Zamolodchikov equations.

3.11.1. We begin with motivational remarks.

The only problem with the above definition is that $\mathcal{R}(X)$ is *not* an ind-scheme in the strict sense[222] (recall that $\mathcal{R}(X)_i$ are not algebraic varieties for $i \geq 3$, see 3.5.2), so we are outside of the documented grounds of algebraic geometry. We know however that $\mathcal{R}(X)$ is an ind-scheme in a broad sense: it is an inductive limit of X^I's with respect to all diagonal embeddings (see 3.5.1(ii)). Luckily this inductive system satisfies an essential "cohomological descent" property (see lemma below) which permits to treat sheaves on $\mathcal{R}(X)$ in the same manner as if $\mathcal{R}(X)$ were an ind-scheme in the strict sense.

To make the sheaf-theoretic constructions clear in this subsection we explain them in plain topological setting of 3.5.1. What follows is basically an adaptation of the general formalism of SGA4 $\mathrm{V}^{\mathrm{bis}}$, except that we treat sheaves in a dual manner more suitable for the $\mathcal{D}$-module setting we have in mind. We consider just a few basic definitions; the more interesting subjects like duality and perverse sheaves are not touched.

So our X is any separated topological space. We have the topological space $\mathcal{R}(X)$; it is the union of the increasing sequence of closed subspaces $\mathcal{R}(X)_i$. On the other hand, we have an $\mathcal{S}^\circ$-diagram[223] $X^{\mathcal{S}}$, $I \mapsto X^I$, of topological spaces. The obvious projections $r_I : X^I \to \mathcal{R}(X)$ identify $\mathcal{R}(X)$ with the inductive limit of this diagram.

(i) We want to study sheaves on $\mathcal{R}(X)$ in terms of sheaves on $X^{\mathcal{S}}$. However, sheaves on

[221] A physicist would, probably, call it "Ward identities".

[222] i.e., $\mathcal{R}(X)$ is not a union of a directed system of closed subschemes.

[223] Recall that $\mathcal{S}$ is the category of finite non-empty sets and surjective maps.

a diagram of topological spaces have to do with sheaves on the *homotopy* direct limit of the diagram. We are saved by the next key lemma.

Denote by $\tilde{\mathcal{R}}(X)$ the homotopy inductive limit of our $\mathcal{S}^\circ$-diagram $X^{\mathcal{S}}$. By definition, this is the realization of the simplicial topological space whose l-simplices are pairs $(y, \tilde{n})$ where $\tilde{n}$ is an l-simplex of the classifying simplicial set $\mathcal{B}$ of $\mathcal{S}^\circ$ as defined in 2.2.19(iii), and $y \in X^{I_0}$. There is a canonical projection $\rho : \tilde{\mathcal{R}}(X) \to \mathcal{R}(X)$, $(y, \delta) \mapsto r_{I_0}(y)$.

In the above definition we have to replace $\mathcal{S}$ by a *small* category equivalent to it. Up to a canonical homotopy the choice of the small category is irrelevant. *From now on we replace* $\mathcal{S}$ *by a small category which we again denote by* $\mathcal{S}$ *by abuse of notation.*

Lemma. (i) The fibers of ρ sre contractible.

(ii) If X is a CW complex then ρ is a homotopy equivalence.

Proof. (i) For $v = [T] \in \mathcal{R}(X)$ the fiber $\rho^{-1}(v)$ is the realization of the simplicial set $B(\mathcal{S}/T)$ where $\mathcal{S}/T$ is the category whose objects are pairs (I, φ), $I \in \mathcal{S}$, $\varphi : I \twoheadrightarrow T$, the morphisms are the morphisms between I's compatible with φ's. This category is contractible since it has final object (T, id_T)).

(ii) Use [QK]??. □

Now let us pass to formal sheaf-theoretic definitions.

(ii) Let $\mathcal{I}$ be a small category and Y, $I \mapsto Y_I$, an $\mathcal{I}$-diagram of topological spaces.[224] We assume that every Y_I has finite cohomological dimension[225] (to be able to consider infinite complexes of sheaves without complications) and for each $\varphi : I \to J$ the map $\varphi_Y : Y_I \to Y_J$ is a closed embedding. A *!-sheaf on* Y is a rule P that assigns to each $I \in \mathcal{I}$ a sheaf of k-modules P_{Y_I} on Y_I and to each $\varphi : I \to J$ a morphism $\theta^{(\varphi)} : \varphi_{Y!} P_{Y_I} \to P_{Y_J}$. We demand that θ's are compatible with composition of φ's, and $\theta^{(id)}$ is the identity map. These objects form an abelian k-category category $\mathcal{S}h^!(Y)$. We denote by $D(Y)$ the corresponding derived category.

We say that a complex P of !-sheaves on Y is *admissible* if for each morphism $\varphi : I \to J$

[224] i.e., a functor on $\mathcal{I}$ with values in the category of topological spaces.

[225] i.e. there is $N = N_I < \infty$ such that for every open $U \subset Y_I$ and a sheaf F on U one has $H^{>N}(U, F) = 0$.

the morphism $\theta^{(\varphi)}$ yields a quasiisomorphism $P_{Y_I} \xrightarrow{\sim} R\varphi^!_Y P_{Y_J}$. Admissible complexes form a full triangulated subcategory $D_{adm}(Y) \subset D(Y)$.

A !-sheaf P on Y yields a functor ΓP on $\mathcal{I}$ with values in k-modules, $\Gamma P(I) := \Gamma(Y_I, P_{Y_I})$. Denote by $\Gamma(Y, P)$ the homotopy inductive limit $C(\mathcal{I}, \Gamma P)$ (see 3.12.1). The functor $\Gamma(Y, \cdot)$ admits a right derived functor: for a complex P of !-sheaves $R\Gamma(Y, P) = \Gamma(Y, \tilde{P})$ where $\tilde{P}$ is a flabby resolution of P.

(iii) We apply the above considerations to $\mathcal{I} = \mathcal{S}^\circ$ and the diagram $X^{\mathcal{S}}$, $I \mapsto X^I$. We assume that X is locally compact, separated and has finite cohomological dimension.

The sheaves on $\mathcal{R}(X)$ we will consider are sheaves of k-modules which are unions of subsheaves supported on $\mathcal{R}(X)_i$'s; they form a category $\mathcal{S}h(\mathcal{R}(X))$. The maps $r_I : X^I \to \mathcal{R}(X)$ are finite, so the push-forward functor $r_{I!} : \mathcal{S}h(X^I) \to \mathcal{S}h(\mathcal{R}(X))$ is exact. So it admits a right adjoint functor $r_I^! : \mathcal{S}h(\mathcal{R}(X)) \to \mathcal{S}h(X^I)$. We have a left exact functor $r^! : \mathcal{S}h(\mathcal{R}(X)) \to \mathcal{S}h^!(X^{\mathcal{S}})$, where $r^! F_{X^I} := r_I^! F$, and θ's are the obvious canonical morphisms.

Denote by $D(\mathcal{R}(X))$ the derived category of $\mathcal{S}h(\mathcal{R}(X))$. The functor $r^!$ admits the right derived functor $Rr^! : D(\mathcal{R}(X)) \to D(X^{\mathcal{S}})$. For a complex C of sheaves on $\mathcal{R}(X)$ one has $Rr^!(C) = r^!(\tilde{C})$ where $\tilde{C}$ is a flabby resolution of C. It is clear that $Rr^!(C)$ is an admissible complex.

The functor $r^!$ admits a left adjoint functor $r_! : \mathcal{S}h^!(X^{\mathcal{S}}) \to \mathcal{S}h(\mathcal{R}(X))$. Namely, a !-sheaf P on $X^{\mathcal{S}}$ defines a functor $P_{\mathcal{R}} : \mathcal{S}^\circ \to \mathcal{S}h(\mathcal{R}(X))$, $P_{\mathcal{R}}(I) := r_{I!} P_{X^I}$, $P_{\mathcal{R}}(\pi) := r_{J!}(\theta^{(\pi)})$, and $r_! P$ is the $\mathcal{S}^\circ$-inductive limit of $P_{\mathcal{R}}$.

The functor $r_!$ admits a left derived functor $Lr_! : D(X^{\mathcal{S}}) \to D(\mathcal{R}(X))$ which is left adjoint to $Rr^!$. It assigns to a complex P of !-sheaves on $X^{\mathcal{S}}$ the complex $C(\mathcal{S}^\circ, P_{\mathcal{R}})$. We leave it to the reader to define the adjunction morphisms and check the adjunction property.

Notice that $\Gamma(X^{\mathcal{S}}, P) = \Gamma(\mathcal{R}(X), C(\mathcal{S}^\circ, P_{\mathcal{R}}))$, so one has $R\Gamma(X^{\mathcal{S}}, P) = R\Gamma(\mathcal{R}(X), Lr_! P)$.

(iv) *The key fact:* The adjoint functors

$$D_{adm}(X^{\mathcal{S}}) \underset{Rr^!}{\overset{Lr_!}{\rightleftarrows}} D\mathcal{R}(X). \tag{3.11.1.1}$$

are mutually inverse equivalences of categories.

We leave the proof (which is an algebraic version of the lemma in 3.11.1(i)) as an exercise to the reader. One may also look at 3.11.4 below.

(v) For $n \geq 1$ let $\mathcal{S}_n \subset \mathcal{S}$ be the subcategory of sets of order $\leq n$. There is an obvious "truncated" version of all the above constructions with $\mathcal{S}$ replaced by $\mathcal{S}_n$ and $\mathcal{R}(X)$ replaced by $\mathcal{R}(X)_n$.

Denote by i_n the embedding $\mathcal{R}(X)_n \hookrightarrow \mathcal{R}(X)$, so we have the functors $i_n^!$, $i_{n!}$ between the categories of sheaves on $\mathcal{R}(X)_n$, $\mathcal{R}(X)$ and the corresponding derived functors. The $X^{\mathcal{S}}$ counterpart for these functors look as follows. The embedding $\mathcal{S}_n \hookrightarrow \mathcal{S}$ yields an obvious exact "truncation" functor $i_n^! : \mathcal{S}h^!(X^{\mathcal{S}}) \to \mathcal{S}h^!(X^{\mathcal{S}_n})$. It admits the left adjoint functor $i_{n!} : \mathcal{S}h^!(X^{\mathcal{S}_n}) \to \mathcal{S}h^!(X^{\mathcal{S}})$; for $Q \in \mathcal{S}h^!(X^{\mathcal{S}_n})$ and $I \in \mathcal{S}$ the sheaf $i_{n!}Q_{X^I}$ is the inductive limit of the sheaves $\Delta_*^{(I/T)}Q_{X^T}$, $T \in Q(I, \leq n)$ (:= the ordered set of quotients of I of order $\leq n$). The left derived functor $Li_{n!}$ has a similar description with the inductive limit replaced by the homotopy inductive limit. Both derived functors $Ri_n^!$ and $Li_{n!}$ send admissible complexes to admissible ones.

(vi) For every complex P of sheaves on $X^{\mathcal{S}}$ the complex $\Gamma(X^{\mathcal{S}}, P)$ has a canonical filtration $\Gamma(X^{\mathcal{S}}, P)_1 \subset \Gamma(X^{\mathcal{S}}, P)_2 \subset \ldots$ where $\Gamma(X^{\mathcal{S}}, P)_n := \Gamma(X^{\mathcal{S}_n}, i_n^! P) = C(\mathcal{S}_n^\circ, \Gamma P)$. Passing to the right derived functors we get a canonical filtration[226] on $R\Gamma(X^{\mathcal{S}}, P)$. If P is admissible then this filtration coincides with the filtration on $R\Gamma(\mathcal{R}(X), Lr_! P)$ coming from the filtration $\mathcal{R}(X)_n$ on $\mathcal{R}(X)$, so $\mathrm{gr}_n R\Gamma(X^{\mathcal{S}}, P) = R\Gamma(\mathcal{R}(X)_n^o, i_n^! Lr_! P)$.[227]

Summing up: we have identified the derived category of sheaves on $\mathcal{R}(X)$ with the derived category of admissible complexes of !-sheaves on $X^{\mathcal{S}}$. The cohomology functor $R\Gamma(\mathcal{R}(X), \cdot)$ is identified with the functor $R\Gamma(X^{\mathcal{S}}, \cdot)$.

[226] A filtration on a complex G, considered as an object of the derived category, is a filtered complex $G.$, considered as an object of the filtred derived category, together with an isomorphism (in the derived category) between G and the plain complex that corresponds to $G.$. The filtered derived category is formed by complexes F equipped with an increasing filtration $F.$ such that $F_a = 0$ for $a << 0$ and $\cup F_a = F$; the morphisms are morphisms of complexes preserving filtrations localized by filtered quasiisomorphisms (:= the morphisms that induce quasiisomorphisms between the successive quotients).

[227] Recall that $\mathcal{R}(X)_n^o$ is the locally closed stratum $\mathcal{R}(X)_n \smallsetminus \mathcal{R}(X)_{n-1}$.

3.11.2. Let us turn to the algebro-geometric setting. Now X is our curve. "Sheaf" means sheaf on the étale topology. The definition of !-sheaves on X^{S} from 3.11.1(ii) makes perfect sense in the present setting, so we have the abelian k-category $\mathcal{S}h^!(X^{\mathrm{S}})$, its derived category $D(X^{\mathrm{S}})$, the DG functor $\Gamma(X^{\mathrm{S}},\cdot)$ on the category of complexes $C(\mathcal{S}h^!(X^{\mathrm{S}})$, and its derived functor $R\Gamma(X^{\mathrm{S}},\cdot)$ together with the canonical filtration (3.11.1(vi)). The notion of admissible complex of !-sheaves from 3.11.1(ii) also makes sense in the present situation. In view of 3.11.1(iv) we denote the derived category of admissible complexes by $D(\mathcal{R}(X))$; admissible complexes will be also called complexes of sheaves on $\mathcal{R}(X)$. Thus $D(\mathcal{R}(X))$ is a full subcategory of $D(X^{\mathrm{S}})$. The restriction of the functor $R\Gamma(X^{\mathrm{S}},\cdot)$ to $D(\mathcal{R}(X))$ is denoted by $R\Gamma(\mathcal{R}(X),\cdot)$; the terms of its canonical filtration by $R\Gamma(\mathcal{R}(X),\cdot)_n = R\Gamma(\mathcal{R}(X)_n,\cdot)$.

3.11.3. Let us describe the successive quotients of the canonical filtration on $R\Gamma(\mathcal{R}(X),\cdot)$ as we did in 3.11.1(vi) for the topological setting.

For $n>0$ consider the space $\mathcal{R}(X)^o_n$ of configurations of n distinct points in X (see 3.5.1(i)). This is a smooth algebraic variety: we have a canonical projection $r_n : U^{(n)} \to \mathcal{R}(X)^o_n$ which identifies $\mathcal{R}(X)^o_n$ with the quotient of $U^{(n)}$ modulo the action of the symmetric group.[228]

Let P be a !-sheaf on X^{S}. The sheaf P_{X^n} is equivariant with respect to the action of the symmetric group, so its restriction to $U^{(n)}$ yields, by descent, the sheaf on $\mathcal{R}(X)^o_n$ which we denote by $P_{\mathcal{R}(X)^o_n}$. The functor $P \mapsto P_{\mathcal{R}(X)^o_n}$ is exact, so it extends to the derived categories in the obvious way.

Below we write $\Gamma(X^n,P) := \Gamma(X^n,P_{X^n})$, $\Gamma(\mathcal{R}(X)^o_n,P) := \Gamma(\mathcal{R}(X)^o_n,P_{\mathcal{R}(X)^o_n})$, etc.

Consider the canonical morphism of complexes $\Gamma(X^n,P) \to \Gamma(\mathcal{R}(X)^o_n,P)$ defined by the trace map $r_{n!}(P_{U^{(n)}}) \to P_{\mathcal{R}(X)^o_n}$. Composing it with the obvious projector[229] $\Gamma(X^{\mathrm{S}},P)_n = C(\mathcal{S}^\circ_n,\Gamma P) \to \Gamma(X^n,P)$ we get a morphism of complexes that vanishes on $\Gamma(X^{\mathrm{S}},P)_{n-1}$, i.e., a morphism $\mathrm{gr}_n\Gamma(X^{\mathrm{S}},P) \to \Gamma(\mathcal{R}(X)^o_n,P)$. Passing to derived functors (say, using flabby resolutions) we get a canonical morphism

$$\mathrm{gr}_n R\Gamma(X^{\mathrm{S}},P) \to R\Gamma(\mathcal{R}(X)^o_n,P). \tag{3.11.3.1}$$

[228] Recall that $U^{(n)}$ is the complement to the diagonal divisor in X^n.

[229] This projector is *not* compatible with the differential.

3.11.4 Lemma. If P is admissible then this arrow is a quasiisomorphism.

Proof. We assume that P is flabby. Consider a filtration $P.$ of P where P_a is a subcomplex whose sections over X^I are sections of P_{X^I} supported on diagonals of dimension $\leq a$. Notice that P_a, as well as P_a/P_b, $b \leq a$, are flabby admissible complexes. So our claim for P follows from the corresponding statements for $\mathrm{gr}_a P$'s. Therefore we may assume that $P = P_a$, $P_{a-1} = 0$.

The complexes $P_{\mathcal{R}(X)^o_n}$ vanish unless $n = a$. We have to consider 3 situations:

(i) $n < a$: Since $\Gamma(X^{\mathcal{S}}, P)_{a-1} = 0$ our claim is true for trivial reason.

(ii) $n = a$: Our $\Gamma(X^{\mathcal{S}}, P)_n$ is the chain complex of the symmetric group Σ_n acting on $\Gamma(X^n, P) = \Gamma(U^{(n)}, P)$. Since our complex is flabby the trace morphism $\mathrm{gr}_n \Gamma(X^{\mathcal{S}}, P) = \Gamma(X^{\mathcal{S}}, P)_n \to \Gamma(\mathcal{R}(X)^o_n, P)$ is a quasiisomorphism.

(iii) $n > a$: To prove our claim it suffices to show that $\Gamma(X^{\mathcal{S}}, P)_a \hookrightarrow \Gamma(X^{\mathcal{S}}, P)_n$ is a quasiisomorphism. We do it in 2 steps:

(a) Consider the complex P' of !-sheaves with $P'_{X^I} := \underset{T \in Q(I,a)}{\oplus} \Delta_.^{(I/T)} P_{X^T}$, the $\theta^{(\varphi)}$ are the obvious embeddings. There is a canonical map $P' \to P$ identical over X^a. Admissibility means that this is a quasiisomorphism, so we may assume that $P = P'$.

(b) Any Σ_a-module R defines a functor $\mathrm{Ind}R$ on $\mathcal{S}^\circ_n$, $I \mapsto \mathrm{Ind}R(I) := \underset{T \in Q(I,a)}{\oplus} R_T$ where R_T is the T-twisted copy of R. Consider the canonical map

$$C(\Sigma_a, R) = C(\mathcal{S}^\circ_a, \mathrm{Ind}R) \hookrightarrow C(\mathcal{S}^\circ_n, \mathrm{Ind}R). \tag{3.11.4.1}$$

Sublemma. This is a quasiisomorphism.

For $R = \Gamma(X^a, P)$ one has $\mathrm{Ind}R = \Gamma P$, so the sublemma implies our claim.

Proof of the sublemma. Denote by $\mathcal{Q}_n$ the category whose objects are pairs (I, T) where $I \in \mathcal{S}_n$, $T \in Q(I, a)$; the morphisms are surjections of I's that induce bijections between T's. Let $\tilde{\mathcal{Q}}_n$ be the category whose objects are pairs (I, φ), $I \in \mathcal{S}_n$, $\varphi : I \twoheadrightarrow \{1, .., a\}$, the morphisms are surjections of I's compatible with φ's. There are obvious functors $\tilde{\mathcal{Q}}_n \to \mathcal{Q}_n \to \mathcal{S}_n$; the group Σ_a acts on $\tilde{\mathcal{Q}}_n$. The projection $B\tilde{\mathcal{Q}}_n \to B\mathcal{Q}_n$ is a Σ_a-torsor; let $\tilde{R}$ be the local system on $B\mathcal{Q}_n$ defined by our Σ_a-module R and this covering. In other words, $\tilde{R}$ is the local system defined by functor $(I, T) \mapsto R_T$ on $\mathcal{Q}_n$.

Notice that $B\tilde{\mathcal{Q}}_n$ is contractible. So the embedding $B\Sigma_a = B\tilde{\mathcal{Q}}_a \subset B\tilde{\mathcal{Q}}_n$ yields a quasi-isomorphism of the chain complexes $C(\Sigma_a, R) \hookrightarrow C(B\tilde{\mathcal{Q}}_n, \tilde{R})$. The projection $\mathcal{Q}_n \to \mathcal{S}_n$ identifies the system of coefficients for $\mathrm{Ind}R$ with the push-forward of $\tilde{R}$. Therefore it yields an isomorphism of the chain complexes $C(B\tilde{\mathcal{Q}}_n, \tilde{R}) \xrightarrow{\sim} C(\mathcal{S}_n^\circ, \mathrm{Ind}R)$. The composition of these arrows is (3.11.4.1); we are done. □

3.11.5. So for $P \in D(\mathcal{R}(X))$ one has a canonical quasiisomorphism

$$\mathrm{gr}_n R\Gamma(\mathcal{R}(X), P) \xrightarrow{\sim} R\Gamma(\mathcal{R}(X)^o_n, P). \tag{3.11.5.1}$$

Consider the spectral sequence $E_r^{p,q}$ for our filtration. It converges to $H^\cdot(\mathcal{R}(X), P)$, and, by 3.11.4, one has

$$E_1^{p,q} = H^{p+q}(\mathcal{R}(X)^o_{-p}, P). \tag{3.11.5.2}$$

3.11.6. A *right $\mathcal{D}$-module M on $X^{\mathcal{S}}$* is a rule that assigns to $I \in \mathcal{S}$ a right $\mathcal{D}$-module M_{X^I} on X^I and to $\pi : J \twoheadrightarrow I$ a morphism of $\mathcal{D}$-modules $\theta^{(\pi)} : \Delta_*^{(\pi)} M_{X^I} \to M_{X^J}$ which satisfy same properties as θ's in the definition of !-sheaf. These objects form an abelian k-category $\mathcal{M}(X^{\mathcal{S}})$. A complex M of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ (a.c.a. *right $\mathcal{D}$-complex on $X^{\mathcal{S}}$*) is *admissible* if for each π the morphism $M_{X^I} \to R\Delta^{(\pi)!} M_{X^J}$ defined by $\theta^{(\pi)}$ is a quasiisomorphism. We also call an admissible right $\mathcal{D}$-complex on $X^{\mathcal{S}}$ a *right $\mathcal{D}$-complex on $\mathcal{R}(X)$*. The corresponding derived category is denoted by $D\mathcal{M}(\mathcal{R}(X))$. Below we skip word "right" when it is clear that we deal with right $\mathcal{D}$-modules.

An increasing filtration on a $\mathcal{D}$-complex M is *admissible* if $\mathrm{gr}_i M$ are admissible complexes. The corresponding filtered derived category is denoted by $DF\mathcal{M}(\mathcal{R}(X))$.

Notice that the above definitions make sense for any diagram of varieties and closed embeddings.

Remark. If L is a left $\mathcal{D}$-module on $\mathcal{R}(X)$ in the sense of 3.5.2 and M is a right $\mathcal{D}$-module on $X^{\mathcal{S}}$ then we have a right $\mathcal{D}$-module $L \otimes M$ on $X^{\mathcal{S}}$ such that $(L \otimes M)_{X^I} = L_{X^I} \otimes M_{X^I}$ and its structure morphisms $\theta^{(\pi)}$ come from the corresponding morphisms $\nu^{(\pi)}$ and $\theta^{(\pi)}$ for L and M via (2.1.3.1). If L is flat[230] and M is an admissible complex then the

[230] As an $\mathcal{O}_{\mathcal{R}(X)}$-module.

complex $L \otimes M$ is admissible. So the tensor category $\mathcal{M}^\ell(\mathcal{R}(X))$ acts on $\mathcal{M}(X^{\mathcal{S}})$, and the tensor DG category of complexes $C(\mathcal{M}^\ell(\mathcal{R}(X)))$ acts on $C(\mathcal{M}(X^{\mathcal{S}}))$. The action of the tensor subcategory of complexes of flat modules[231] preserves admissible complexes and quasiisomorphisms, so it yields an action on $D\mathcal{M}(\mathcal{R}(X))$.

A $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ yields a complex $DR(M)$ of !-sheaves on $X^{\mathcal{S}}$, $DR(M)_{X^I} := DR(M_{X^I})$. The functor DR preserves quasiisomorphisms and sends admissible complexes to admissible ones,[232] so we have the exact functors $DR : D\mathcal{M}(X^{\mathcal{S}}) \to D(X^{\mathcal{S}})$, $D\mathcal{M}(\mathcal{R}(X)) \to D(\mathcal{R}(X))$. For a $\mathcal{D}$-complex M we set $\Gamma_{DR}(X^{\mathcal{S}}, M) := \Gamma(X^{\mathcal{S}}, DR(M))$ and $R\Gamma_{DR}(X^{\mathcal{S}}, M) := R\Gamma(X^{\mathcal{S}}, DR(M))$.[233] These complexes carry a canonical filtration coming from the canonical filtration on Γ and $R\Gamma$. For admissible M we write $H^i_{DR}(\mathcal{R}(X), M) := H^i R\Gamma_{DR}(\mathcal{R}(X), M)$, $H^i_{DR}(\mathcal{R}(X)_n, M) := H^i(R\Gamma_{DR}(\mathcal{R}(X), M)_n)$.

Repeating the construction from 3.11.3 we see that a $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ defines a complex $M_{\mathcal{R}(X)^o_n}$ of $\mathcal{D}$-modules on every $\mathcal{R}(X)^o_n$. One has $DR(M_{\mathcal{R}(X)^o_n}) = DR(M)_{\mathcal{R}(X)^o_n}$, so, by 3.11.4, for $M \in D\mathcal{M}(\mathcal{R}(X))$ we get a canonical quasiisomorphism

$$\mathrm{gr}_n R\Gamma_{DR}(\mathcal{R}(X), M) \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X)^o_n, M) \tag{3.11.6.1}$$

and the spectral sequence $E_r^{p,q}$ converging to $H^{\cdot}_{DR}(\mathcal{R}(X), M)$ with the first term

$$E_1^{p,q} = H^{p+q}_{DR}(\mathcal{R}(X)^o_{-p}, M). \tag{3.11.6.2}$$

3.11.7. There is an exact fully faithful functor

$$j^{(n)}_{\mathcal{R}*} : \mathcal{M}(\mathcal{R}(X)^o_n) \hookrightarrow \mathcal{M}(X^{\mathcal{S}}) \tag{3.11.7.1}$$

defined by $(j^{(n)}_{\mathcal{R}*}N)_{X^I} := \underset{T \in Q(I,n)}{\oplus} \Delta_*^{(I/T)} j_*^{(I)} N_{U^{(T)}}$, $N \in \mathcal{M}(\mathcal{R}(X)^o_n)$. Here $N_{U^{(T)}}$ is the pull-back of N by the étale projection $U^{(T)} \to \mathcal{R}(X)^o_n$. Note that $N = (j^{(n)}_{\mathcal{R}*}N)_{\mathcal{R}(X)^o_n}$.

A *Cousin $\mathcal{D}$-complex on* $\mathcal{R}(X)$ is a complex M of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ such that $M^{-n} \in j^{(n)}_{\mathcal{R}*}(\mathcal{M}(\mathcal{R}(X)^o_n))$ (in particular, $M^a = 0$ for $a \geq 0$). Such M is automatically admissible. Cousin $\mathcal{D}$-complexes form an abelian category $\mathcal{C}ous(\mathcal{R}(X))$.

[231] We can play with infinite length complexes without worries since the Tor dimension is finite.

[232] DR commutes with $\Delta^{(\pi)!}$, see 2.1.6.

[233] For an explicit way to compute $R\Gamma_{DR}$ see 3.12.6.

Remark. The functor $\mathcal{C}ous(\mathcal{R}(X)) \to D\mathcal{M}(\mathcal{R}(X))$ is a fully faithful embedding. In fact, $\mathcal{C}ous(\mathcal{R}(X))$ is the core of a certain canonical t-structure on $D\mathcal{M}(\mathcal{R}(X))$.

Let M be a Cousin $\mathcal{D}$-complex. According to (3.11.6.1) we have a canonical quasiisomorphism

$$\mathrm{gr}_n R\Gamma_{DR}(\mathcal{R}(X), M) \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X)^o_n, M^{-n})[n]. \tag{3.11.7.2}$$

In particular, since $\dim \mathcal{R}(X)^o_n = n$, the cohomology groups $H^i_{DR}(\mathcal{R}(X), M)$[234] vanish for $i > 0$.

3.11.8 Lemma. The morphisms $H^a_{DR}(\mathcal{R}(X)_n, M) \to H^a_{DR}(\mathcal{R}(X), M)$ are surjective for $a > -n$ and are isomorphisms for $a > -n+1$. If X is affine then this is true, respectively, for $a > -n-1$ and $a > -n$.

Proof. We use (3.11.7.2). If X is affine then such are the varieties $\mathcal{R}(X)^o_m$. So the groups $H^a \mathrm{gr}_m R\Gamma_{DR}(\mathcal{R}(X), M)$ vanish for $a > -m$, which implies our statement. If X is compact, then we consider a coordinate projection $U^{(m)} \to X$. It is affine and X is a curve, so $H^a_{DR}(U^{(m)}, M^{-m}) = 0$ for $a > 1$. Since $H^a_{DR}(\mathcal{R}(X)^o_m, M^{-m}) \subset H^a_{DR}(U^{(m)}, M^{-m})$ we see that $H^a \mathrm{gr}_m R\Gamma_{DR}(\mathcal{R}(X), M) = 0$ for $a > -m+1$, and we are done. □

3.11.9. According to 3.5.11, a DG chiral (super) algebra A on X yields a (super) $\mathcal{D}$-complex $C(A)$ on $X^{\mathcal{S}}$ (the morphisms $\theta^{(\pi)}$ are defined by (3.5.11.3)). This complex is obviously admissible, i.e., $C(A)$ is a $\mathcal{D}$-complex on $\mathcal{R}(X)$. If A is a plain chiral algebra then $C(A)$ is a Cousin complex.

For the rest of §11 we assume that X is compact and connected.

Definition. We set $R\Gamma^{ch}(X, A) := R\Gamma_{DR}(\mathcal{R}(X), C(A))$ and $H^{ch}_a(X, A) = H^{ch}_a(A) := H^{-a} R\Gamma(X, A)$; this is the *chiral homology of X with coefficients in A* or, simply, the chiral homology of A.

We also set $\Gamma^{ch}(X, A) := \Gamma_{DR}(\mathcal{R}(X), A)$. According to 3.12.6 for a Dolbeaut algebra $\mathcal{Q}_X$ one has $R\Gamma^{ch}(X, A) = \Gamma^{ch}(X, \mathcal{Q}_X \otimes A)$.

[234] As well as $H^i_{DR}(\mathcal{R}(X)^o_n, M)$ and $H^i_{DR}(\mathcal{R}(X)_n, M)$.

One has $C(A)_{\mathcal{R}(X)^o_n} = \mathrm{Sym}^n(A[1])_{\mathcal{R}(X)^o_n}$, so the spectral sequence (3.11.6.2) converging to $H^{ch}_n(X,A)$ looks as

$$E^1_{p,q} = H^{-p-q}_{DR}(\mathcal{R}(X)^o_p, \mathrm{Sym}^p(A[1])). \tag{3.11.9.1}$$

The following notation is useful. For a right $\mathcal{D}$-module M on a connected smooth n-dimensional variety Y we set $H^{DR}_a(Y,M) := H^{n-a}_{DR}(Y,M)$; this is the *de Rham homology* of Y with coefficients in M. Notice that $H^{DR}_a(Y,M)$ vanish unless $a \in [-2n, 0]$; if Y is affine the vanishing holds unless $a \in [-2n, n]$. The vector space $H^{DR}_0(Y,M)$ vanishes if Y is non-compact; if Y is compact then there is a canonical morphism $M \to H^{DR}_0(X,M) \otimes \omega_Y$ which identifies the latter $\mathcal{D}$-module with the maximal "constant" quotient of M.[235]

So for a plain chiral algebra A we may rewrite our spectral sequence as

$$E^1_{p,q} = H^{DR}_{p+q}(\mathcal{R}(X)^o_p, \Lambda^p_{ext}A). \tag{3.11.9.2}$$

Here the $\mathcal{D}$-module $\Lambda^p_{ext}A$ on $\mathcal{R}(X)^o_p$ is $\mathrm{Sym}^p(A[1])_{\mathcal{R}(X)^o_p}[-p]$. Notice that $H^{ch}_a(X,A) = 0$ for $a < 0$.

Remarks. (i) The definition of $C(A)$ makes no use of the unit in A, so A can actually be any Lie^{ch} algebra.

(ii) We do not know if the chiral homology equals the left derived functor of H^{ch}_0. This is true if we deal with commutative chiral algebras (see 3.11.?).

3.11.10. For a plain chiral algebra A we will also denote the 0th chiral homology group $H^{ch}_0(A)$ by $\langle A\rangle = \langle A\rangle(X)$.[236] By construction and 3.5.12, 3.5.13 this is a plain $\mathcal{S}^\circ$ direct limit of vector spaces $H^{DR}_0(X^I, A_{X^I})$. So for every $I \in \mathcal{S}$ one has a canonical morphism of $\mathcal{D}$-modules

$$\langle\,\rangle_I : A^\ell_{X^I} \to \langle A\rangle \otimes \mathcal{O}_{X^I}. \tag{3.11.10.1}$$

These morphisms are compatible with pull-backs to the diagonals, so they form a morphism $\langle\,\rangle : A^\ell_{\mathcal{R}(X)} \to \langle A\rangle \otimes \mathcal{O}_{\mathcal{R}(X)}$ of the left $\mathcal{D}$-modules on $\mathcal{R}(X)$ (see 3.5.2 for terminology).

[235] See the proof of 2.4.2.

[236] According to 2.4.4 this notation is compatible with that of 2.4.1 when A is commutative.

Set $1_A^{ch} = 1^{ch} = \langle 1 \rangle \in \langle A \rangle$ (here 1 is the unit (horizontal) section of $A^{\ell}_{X^I}$).

Restricting $\langle\,\rangle$ to the complement $\mathcal{R}(X)^o_n$ of the diagonal divisor on $\mathrm{Sym}^n X$ we get the *correlator maps*

$$\langle\,\rangle_n : (\mathrm{Sym}^n A^{\ell})_{\mathcal{R}(X)^o_n} \to \langle A \rangle \otimes \mathcal{O}_{\mathcal{R}(X)^o_n}. \tag{3.11.10.2}$$

Remarks. (i) It follows easily from 3.11.8 that $H_0^{ch}(A) = \mathrm{Coker}(H_0^{DR}(U, j^*A^{\boxtimes 2}) \to H_0^{DR}(X, A))$ where the arrow comes from the chiral product $\mu : j_*j^*A \boxtimes A \to \Delta_* A$.

(ii) Assume that A is commutative. Then, by (i) and 2.4.4, $\langle A \rangle$ coincides with the same noted vector space from 2.4.1.

Recall that $\langle A \rangle$ is a commutative unital algebra (see 2.4.1). Its unit equals 1^{ch}, and the product $\cdot$ can be described (see, e.g., the proof of 2.4.4) in present terms as the quotient map of the composition $H_0^{DR}(X, A)^{\otimes 2} \xrightarrow{\boxtimes} H_0^{DR}(X \times X, A^{\boxtimes 2}) \to H_0^{DR}(X \times X, A_{X^2}) \to \langle A \rangle$ (the last arrow is the canonical morphism, the middle one comes since $A^{\boxtimes 2} \subset A_{X^2}$).

Example. For the unit algebra ω_X the obvious identification $\langle \omega_X \rangle = k$ (see 2.4.1) comes from the trace isomorphisms $H_0^{DR}(X^I, \omega_{X^I}) \xrightarrow{\sim} k$.

3.11.11. A filtration $A_0 \subset A_1 \subset ..$ on a chiral DG algebra A (see 3.3.11) yields an admissible filtration $C(A)_0 \subset C(A)_1 \subset ..$ on $C(A)$. Namely, for $I \in \mathcal{S}$ set $(A[1])^{\boxtimes I}_n := \sum \bigotimes_{i \in I} (A_{l_i}[1]) \subset (A[1])^{\boxtimes I}$, the summation is over the set of all collections $(l_i) \in \mathbb{Z}^I_{\geq 0}$ such that $\sum_{i \in I} l_i \leq n$. Now one has $C(A)_{nX^I} := \bigoplus_{T \in Q(I)} \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T}_n$ (see (3.5.11.1)).

Our filtration on $C(A)$ yields the filtrations on $\Gamma^{ch}(X, A)$ and $R\Gamma^{ch}(X, A)$. Since $\mathrm{gr}C(A) = C(\mathrm{gr}A)$ one has $\mathrm{gr}R\Gamma^{ch}(X, A) = R\Gamma^{ch}(X, \mathrm{gr}A)$. So one has a spectral sequence converging to $H^{ch}_{\cdot}(X, A)$ with

$$E^1_{p,q} = H^{ch}_{p+q}(X, \mathrm{gr}A)^p. \tag{3.11.11.1}$$

Here the upper index p is the grading on $H^{ch}(X, \mathrm{gr}A)$ that comes from the grading $\mathrm{gr}.A$.

Remark. The above construction makes sense if A is any DG Lie^{ch} algebra (see Remark (i) in 3.11.9) and the filtration need not be unital. For example, on any A we have a stupid canonical non-unital filtration $A_0 = 0$, $A_1 = A$. Then $C(A)_n \subset$

$C(A)$ is the submodule of sections supported on diagonals of dimension $\leq n$. It is clear that $\Gamma_{DR}(\mathcal{R}(X), C(A))_n \subset \Gamma_{DR}(\mathcal{R}(X), C(A)_n)$, and 3.11.4 (or its proof) shows that $R\Gamma_{DR}(\mathcal{R}(X), C(A))_n \to R\Gamma_{DR}(\mathcal{R}(X), C(A)_n)$ is a quasiisomorphism. Thus our filtration on $R\Gamma^{ch}(X, A)$ coincides with the canonical filtration and (3.11.11.1) equals (3.11.9.1).

Notice that this filtration is commutative (see 3.3.11).

We refer the reader to 3.12.7 for a review of needed facts on Batalin-Vilkovisky algebras.

3.11.12 Proposition. (i) For a DG chiral algebra[237] A the complex $\Gamma^{ch}(X, A)$ carries a canonical structure of a homotopy BV algebra.

(ii) For any *commutative* filtration on A the corresponding filtration on $\Gamma^{ch}(X, A)$ is compatible with the homotopy BV algebra structure.

Proof. (i) For $I \in \mathcal{S}$ set $\Gamma_{DR}C(A)_I := \Gamma_{DR}(X^I, C(A))$, so $\Gamma_{DR}C(A)$ is a functor on $\mathcal{S}^\circ$ with values in the DG category $C(k)$ of (super) k-complexes.

The category $C(k)^{\mathcal{S}}$ of all such functors is a DG (super) k-category. It carries a canonical tensor structure: for $F_i \in C(k)^{\mathcal{S}}$, $i \in I$, one has $(\otimes F_i)_J := \underset{J \twoheadrightarrow I}{\oplus} \otimes F_{iJ_i}$, the functor structure is the obvious one.

Notice that the pseudo-tensor structure on $C(k)^{\mathcal{S}}$ equals the one defined in 3.12.3(ii) (for $\mathcal{P} = \mathcal{S}^\circ$, $\mathcal{A} = C(k)$; the tensor operation Σ on $\mathcal{S}^\circ$ is the disjoint sum).

(a) Let us show that $\Gamma_{DR}C(A) \in C(k)^{\mathcal{S}}$ carries a canonical structure of BV algebra.

Namely, the product morphism $\Gamma_{DR}C(A) \otimes \Gamma_{DR}C(A) \stackrel{\cdot}{\to} \Gamma_{DR}C(A)$ at $J \in \mathcal{S}$ is the map $\underset{J=J_1 \sqcup J_2}{\oplus} \Gamma_{DR}(X^{J_1}, C(A)) \otimes \Gamma_{DR}(X^{J_2}, C(A)) \to \Gamma_{DR}(X^J, C(A))$ coming from the exterior tensor product maps $C(A)_{J_1} \boxtimes C(A)_{J_2} \to C(A)_J$. It is obviously commutative and associative. Let us show that $\cdot$ defines a BV algebra structure.

Let $\Gamma_{DR}C(A)[-1] \otimes \Gamma_{DR}C(A)[-1] \xrightarrow{\{\,\}} \Gamma_{DR}C(A)[-1]$ be the differential of $\cdot$. According to 3.5.11 it can be described as follows. For $J = J_1 \sqcup J_2 \in \mathcal{S}$ the component $\{\,\}_{J_1 J_2} : \Gamma_{DR}(X^{J_1}, C(A))[-1] \otimes \Gamma_{DR}(X^{J_2}, C(A))[-1] = \Gamma_{DR}(X^J, C(A)_{X^{J_1}}[-1] \boxtimes C(A)_{X^{J_2}}[-1]) \to \Gamma_{DR}(X^J, C(A))[-1]$ comes from a morphism $[\quad]_{J_1 J_2}$ of $\mathcal{D}$-complexes $C(A)_{X^{J_1}}[-1] \boxtimes C(A)_{X^{J_2}}[-1] \to C(A)_{X^J}[-1]$. For $T_i \in Q(J_i), T \in Q(J)$ the corresponding component

[237] The proposition remains valid in the non-unital setting of Remark in 3.11.11.

of this morphism $\Delta_*^{(J_1/T_1)} j_*^{(T_1)} j^{(T_1)*}(A[1])^{\boxtimes T_1}[-1] \boxtimes \Delta_*^{(J_2/T_2)} j_*^{(T_2)} j^{(T_2)*}(A[1])^{\boxtimes T_2}[-1] \to \Delta_*^{(J/T)} j_*^{(T)} j^{(T)*}(A[1])^{\boxtimes T}[-1]$ can be non-zero only if T is the quotient of $T_1 \sqcup T_2$ modulo a relation $t_1 = t_2$ where $t_1 \in T_1$, $t_2 \in T_2$. In this case our morphism is the $*$ bracket $A_{t_1} \boxtimes A_{t_2} \to \Delta_* A$ tensored by the identity map for $(A[1])^{\boxtimes T'}$ where $T' := T \setminus \{t\} = T_1 \setminus \{t_1\} \sqcup T_2 \setminus \{t_2\}$.

It is clear that $\{\,\}$ is an odd Poisson bracket with respect to $\cdot$, so $\Gamma_{DR} C(A)$ is a BV algebra in $C(k)^{\mathcal{S}}$ as promised.

(b) Let $\mathcal{T} = \mathcal{N}orm\, k[T]$ be the DG operad defined in 3.12.3(ii) for the tensor category $\mathcal{P} = \mathcal{S}^\circ$ (so T_I is the simplex whose vertices is the set of all "disjoint sum" functors $\Sigma_I : \mathcal{S}^I \to \mathcal{S}$). The projection $T \to$ (point) yields a canonical projection $\mathcal{T} \to \mathcal{C}om$ which is a quasiisomorphism.

One has a homotopy BV operad BV$\otimes\mathcal{T}$. According to (3.12.3.5) our BV algebra structure on $\Gamma_{DR} C(A)$ yields a BV$\otimes\mathcal{T}$ algebra structure on the complex $C(\mathcal{S}^\circ, \Gamma_{DR} C(A)) = \Gamma^{ch}(X, A)$. We are done.

(ii) Clear from the construction. □

3.11.13. (i)[238] Let A be a DG chiral algebra equipped with a commutative filtration $A_\cdot$. We get a filtered complex $R\Gamma^{ch}(X, A)$. Replacing $A_\cdot$ by $\Omega_X \otimes A_\cdot$ as in 3.12.6 we get on $R\Gamma^{ch}(X, A)$ a canonical structure of homotopy BV algebra. It is clear that $R\Gamma^{ch}$ sends filtered quasiisomorphisms to filtered quasiisomorphisms.

Denote by $\mathcal{FCA}(X)$ the category of DG chiral algebras equipped with a commutative filtration; let ${}^c\mathcal{FCA}(X)$ be the full subcategory of algebras with commutative filtrations. Below we denote by $\mathcal{H}o$ the localization of our categories by (filtered) quasiisomorphisms.[239] We see that $R\Gamma^{ch}$ lifts canonically to a functor

$$R\Gamma^{ch} : \mathcal{H}o\, {}^c\mathcal{FCA}(X) \to \mathcal{H}o\mathcal{BV}. \tag{3.11.13.1}$$

(ii) If A is a commutative DG algebra then the bracket $\{\,\}$ on its chiral complex vanishes, so $R\Gamma^{ch}(X, A)$ is a homotopy commutative algebra. So $H^{ch}_\cdot(A)$ is a commutative (super)

[238] This subsection makes sense in the non-unital setting.

[239] We did not check that our categories are closed model categories, so this definition is a bit formal.

algebra. We denote its product by $\cdot$. According to Remark (ii) from 3.11.10 on $H_0^{ch}(A)$ this product coincides with the product on $\langle A\rangle$ from 2.4.1. Thus $H_0^{ch}(A)$ is a unital algebra with unit 1^{ch}. We will see in 3.11.16 that 1^{ch} is, in fact, the unit for the whole $H_\cdot^{ch}(A)$.

3.11.14. Let A, B be DG chiral algebras. Choose a "disjoint union" functor $\sqcup \in T_2$ (see section (b) of the proof of 3.11.12 and 3.12.3(ii));[240] it yields a product $\cdot = \cdot_\sqcup$ on $\Gamma^{ch}(X, A\otimes B)$. This product need not commute with the differential. However, since the canonical morphisms $A, B \to A\otimes B$ mutually commute, their composition with $\cdot$

$$\circ_{A,B} : \Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)\to\Gamma^{ch}(X,A\otimes B) \tag{3.11.14.1}$$

is a morphism of complexes.

Passing as above to derived functors (i.e., replacing A by $\Omega_X\otimes A$, etc.) we can replace Γ^{ch} by $R\Gamma^{ch}$. The morphism $\circ_{A,B}$ for $R\Gamma^{ch}$ does not depend on the choice of $\sqcup$.

Remarks. (i) Assume that A, B carry filtrations and let us equip $A\otimes B$ with the tensor product of this filtrations. Then $\circ_{A,B}$ is a morphism of filtered complexes.

(ii) To make (3.11.14.1) independent of the choice of $\sqcup$ one writes it as a canonical morphism $\Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)\otimes\mathcal{T}_2\to\Gamma^{ch}(X,A\otimes B)$ (for $\mathcal{T}_2$ see the proof of 3.11.12(i)(b)).

(iii) $\circ_{A,B}$ is a morphism of homotopy BV algebras up to canonical homotopies. Precisely, for a finite non-empty I let the simplicial set T'_I be the joint of $T_I\times T_I$ and T_I (the simplex T_I was defined in the proof of 3.11.12(i)(b)). This is a contractible simplicial set equipped with embeddings $\nu_0 : T_I\times T_I\hookrightarrow T'_I$, $\nu_1 : T_I\hookrightarrow T'_I$. Set $\mathcal{T}'_I := \mathcal{N}ormk[T'_I]$. Now there is a canonical morphism $\alpha : \mathrm{BV}_I\otimes\mathcal{T}'_I\otimes\Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)\to\Gamma^{ch}(X,A\otimes B)$ such that $\alpha\nu_0 = \circ_{A,B}a_0$ and $\alpha\nu_1 = a_1\circ_{A,B}$. Here a_0 is the action of $\mathrm{BV}\otimes\mathcal{T}\otimes\mathcal{T}$ on $\Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)$ (see 3.12.6), and a_1 is the action of $\mathrm{BV}\otimes\mathcal{T}$ on $\Gamma^{ch}(X,A\otimes B)$. In the filtered setting the above canonical homotopies are compatible with filtrations. The compatibility with compositions of operations is left to the reader.

3.11.15 Theorem. (i) Assume that A, B are $\mathcal{O}_X$-flat.[241] Then the product morphism $\circ_{A,B} : R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,B)\to R\Gamma^{ch}(X,A\otimes B)$ is a quasiisomorphism. If A, B

[240] Recall that $\sqcup$ is defined up to a canonical isomorphism, and we need to choose a concrete functor.

[241] It suffices to assume that A and B are Tor-independent, i.e., the supports of their $\mathcal{O}_X$-torsion are disjoint.

are equipped with filtrations[242] such that $\mathrm{gr}A$, $\mathrm{gr}B$ are $\mathcal{O}_X$-flat then this is a filtered quasiisomorphism.

(ii) One has $H^{ch}_{\neq 0}(X,\omega)=0$, so (see Example in 3.11.10) one has a canonical identification $R\Gamma^{ch}(X,\omega)=H^{ch}_0(\omega)=k$.

(iii) The product endomorphism $a\mapsto a\circ_{A,\omega} 1^{ch}_{\omega}$ of $R\Gamma^{ch}(X,A)$ is identity.

Remark. The vanishing statement (ii) follows from the contractability of $\mathcal{R}(X)$ in classical topology (see 3.5.1(iv)) combined with 3.11.1(iv) and usual comparison between the de Rham and topological homology. The argument below is an algebraic version of the topological proof of 3.5.1(iv).

Proof. Since $\mathrm{gr}R\Gamma^{ch}(X,A)=R\Gamma^{ch}(X,\mathrm{gr}A)$ (see 3.11.11) and $\mathrm{gr}(A\otimes B)=\mathrm{gr}A\otimes\mathrm{gr}B$ the filtered setting follows from the non-filtered one. So we forget about filtrations.

Let us construct the inverse to $\circ_{A,B}$ explicitely.

Consider the $\mathcal{S}^\circ\times\mathcal{S}^\circ$-diagram of varieties $X^{\mathcal{S}}\times X^{\mathcal{S}}$. Consider $\mathcal{D}$-complexes $C(A,B)$ and $C(A)\boxtimes C(B)$ on $X^{\mathcal{S}}\times X^{\mathcal{S}}$ where $C(A,B)_{X^I\times X^J}$ is the Cousin complex of $A_{X^I}\boxtimes B_{X^J}[|I\sqcup J|]$ with respect to the diagonal stratification of $X^{I\sqcup J}$ and $(C(A)\boxtimes C(B))_{X^I\times X^J}:=C(A)_{X^I}\boxtimes C(B)_{X^J}$. The flatness property of A, B assures that the obvious embedding $C(A)\boxtimes C(B)\hookrightarrow C(A,B)$ is a quasiisomorphism. The cohomology of $C(A)\boxtimes C(B)$ are computed by Kunneth formula, so one has a canonical identification $R\Gamma_{DR}(X^{\mathcal{S}}\times X^{\mathcal{S}},C(A,B))=R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,B)$.

Consider the diagonal and disjoint union functors $\mathcal{S}\xrightarrow{\delta}\mathcal{S}\times\mathcal{S}\xrightarrow{\sqcup}\mathcal{S}$, and their liftings to the morphisms of the diagrams $X^I\xrightarrow{\delta}X^I\times X^I$, $X^I\times X^J\xrightarrow{\sim}X^{I\sqcup J}$ and of the $\mathcal{D}$-complexes $\delta_*C(A\otimes B)_{X^I}\xrightarrow{\delta_{A,B}}C(A,B)_{X^I\times X^I}$, $C(A,B)_{X^I\times X^J}\xrightarrow{\circ_{A,B}}C(A\otimes B)_{X^{I\sqcup J}}$. Here $\delta_{A,B}$ comes from the canonical identification $\delta^!C(A,B)_{X^I\times X^I}=C(A\otimes B)$ and $\circ_{A,B}$ from the morphism $1_B^{\boxtimes I}\boxtimes 1_A^{\boxtimes J}: A_{X^I}\boxtimes B_{X^J}\to(A\otimes B)_{X^{I\sqcup J}}$. Passing to $R\Gamma_{DR}$'s we get the morphisms of complexes

$$R\Gamma^{ch}(X,A\otimes B)\underset{\circ_{A,B}}{\overset{\delta_{A,B}}{\rightleftarrows}}R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,B). \tag{3.11.15.1}$$

It is clear that this $\circ_{A,B}$ coincides with the morphism (3.11.14.1).

[242] The filtrations are assumed to be unital.

We claim that $\delta_{A,B}$ and $\circ_{A,B}$ are mutual inverses. By the definition of Γ the composition $\circ_{A,B}\delta_{A,B}$ is the identity endomorphism of $R\Gamma^{ch}(X, A\otimes B)$. To finish we need first to prove statements (ii) and (iii) of the theorem:

(ii) Consider the case $A = B = \omega_X$. We are playing with a commutative chiral algebra, so $\circ_{\omega,\omega}$ is the commutative and associative product $\cdot$ on $H^{ch}_\cdot(\omega)$ from 3.11.13(ii).

To prove the vanishing take a to be the minimal number > 0 such that $H^{ch}_a(\omega) \neq 0$. Since $H^{ch}_0(\omega) = k$ we can rewrite, by Kunneth, the H_a maps of (3.11.15.1) as

$$H^{ch}_a(\omega) \xrightarrow{\delta_a} H^{ch}_a(\omega) \oplus H^{ch}_a(\omega) \xrightarrow{\cdot_a} H^{ch}_a(\omega).$$

Here δ_a is the diagonal embedding (since its composition with the projections to the multiples is identity). Both components of $\cdot_a$ are equal to the endomorphism $\gamma \mapsto 1^{ch}\cdot\gamma$ of $H^{ch}_a(\omega)$. We will prove that it is the identity map. Then $\cdot_a\delta_a$ is multiplication by 2, which contradicts to the fact that $\cdot_a\delta_a$ is the identity map.

Indeed, we know that 1^{ch} is the unit in $H^{ch}_0(\omega_X)$. Therefore one has $1^{ch}\cdot(1^{ch}\cdot\gamma) = (1^{ch}\cdot 1^{ch})\cdot\gamma = 1^{ch}\cdot\gamma$. So, since also $\gamma = \cdot_a\delta_a(\gamma) = 2(1^{ch}\cdot\gamma)$, we see that $1^{ch}\cdot\gamma = \gamma$, q.e.d.

(iii) Assume that A is arbitrary, $B = \omega$. Recall that the identification $\mathrm{tr} : R\Gamma^{ch}(X,\omega) \xrightarrow{\sim} k$ of (ii) is given by the trace maps $R\Gamma_{DR}(X^I, C(\omega)_{X^I}) = R\Gamma_{DR}(X^I, \omega_{X^I})[I] \to k$. Therefore the composition $R\Gamma^{ch}(X,A) \xrightarrow{\delta_{A,\omega}} R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,\omega) \xrightarrow{id\otimes\mathrm{tr}} R\Gamma^{ch}(X,A)$ is the identity morphism. Thus $\delta_{A,\omega}$ is an isomorphism. Since $\circ_{A,\omega}$ is left inverse to $\delta_{A,\omega}$, we see that $\delta_{A,\omega}$, $\circ_{A,\omega}$ are mutually inverse and $a \mapsto a\circ_{A,\omega} 1^{ch}$ is the identity endomorphism of $R\Gamma^{ch}(X,A)$.

Now we are ready to prove that $\circ_{A,B}$ and $\delta_{A,B}$ are mutually inverse for arbitrary DG chiral algebras A, B. The composition $\delta_{A,B}\circ_{A,B}$ comes from the composition of morphisms $\delta_* C(A)_{X^I}\boxtimes\delta_* C(B)_{X^J} \to C(A,\omega)_{X^I\times X^I}\boxtimes C(\omega,B)_{X^J\times X^J} \leftarrow [C(A)_{X^I}\boxtimes C(\omega)_{X^I}]\boxtimes [C(\omega)_{X^J}\boxtimes C(B)_{X^J}] = [C(A)_{X^I}\boxtimes C(\omega)_{X^J}]\boxtimes[C(\omega)_{X^I}\boxtimes C(B)_{X^J}] \to C(A)_{X^{I\sqcup J}}\boxtimes C(B)_{X^{I\sqcup J}}$. Here the first arrow is $\delta_{A,\omega}\boxtimes\delta_{\omega,B}$, the second (left) arrow is the tensor product of the quasiisomorphisms from (i), the last arrow is $\circ_{A,\omega}\boxtimes\circ_{\omega,B}$. These morphisms of $\mathcal{D}$-complexes lift the obvious morphisms of diagrams $X^S\times X^S \to X^S\times X^S\times X^S\times X^S \to X^S\times X^S\times X^S\times X^S \to X^S\times X^S$. Now apply $R\Gamma_{DR}$; we are done by (iii). $\square$

3.11.16 Remarks. (i) The definition of $\circ_{A,B}$ from 3.11.14 generalizes immediately to any number of DG chiral algebras. It is obviously homotopy associative and commutative. So for any finite collection of DG chiral algebras $\mathcal{A}_\alpha$ one has a canonical quasiisomorphism

$$\circ = \circ_{\{A_\alpha\}} : \otimes R\Gamma^{ch}(X, A_\alpha) \xrightarrow{\sim} R\Gamma^{ch}(X, \otimes A_\alpha). \tag{3.11.16.1}$$

If A_α are equipped with filtrations and $\otimes A_\alpha$ with the tensor product of these filtrations then (3.11.16.1) is a filtered quasiisomorphism. So we have tensor functors

$$R\Gamma^{ch} : \mathcal{H}o\mathcal{CA}(X) \to D(k), \quad R\Gamma^{ch} : \mathcal{H}o\mathcal{FCA}(X) \to DF(k). \tag{3.11.16.1}$$

(ii) According to Remark (ii) in 3.11.14 and 3.12.? we have a tensor functor

$$R\Gamma^{ch} : \mathcal{H}o^c\mathcal{FCA}(X) \to \mathcal{H}o\mathcal{BV}. \tag{3.11.16.2}$$

(iii)

(iv) One may identify $H_0^{ch}(X, \omega)$ with the 0th de Rham cohomology supported at any point of $\mathcal{R}(X)$. This gives the following description of the map $\circ_{\omega,A}$. Take any finite non-empty $T \subset X$, so we have $[T] \in \mathcal{R}(X)$ and the corresponding point $[T] \in X^T$. Let 1_T be the generator of $H^0_{DR\,[T]}(X^T, C(\omega)) = k$ (cohomology with support at $[T]$). The morphisms $1_A^{\boxtimes T} : C(\omega)_{X^T} \boxtimes C(A)_{X^I} \to C(A)_{X^{T\sqcup I}}$ yield the morphisms of complexes $R\Gamma(X^{\mathrm{S}}, C(A)) \to R\Gamma_{DR\,[T]}(X^T, C(\omega)) \boxtimes R\Gamma(X^{\mathrm{S}}, C(A)) \xrightarrow{\sim} R\Gamma_{DR}(X^T \times X^{\mathrm{S}}, C(\omega)_{X^T} \boxtimes C(A)) \to R\Gamma_{DR}(X^{\mathrm{S}}, C(A))$ where the first arrow is $1_T \boxtimes \cdot$. Our $\circ_{\omega,A}$ is the composition of these maps.

Consider the case of 0th chiral homology. Let $S, T \subset X$ be disjoint finite subsets. Then, by 3.11.15(iii), for every $v \in \otimes A_s^\ell$ one has[243]

$$\langle (\otimes 1_t) \otimes (\otimes v_s) \rangle_{T\sqcup S} = \langle \otimes v_s \rangle_S. \tag{3.11.16.3}$$

[243] See 3.11.10 for notation.

3.11.17 Lemma. For every affine open $j : U \hookrightarrow X$ and a DG chiral algebra A one has $H^{\cdot}_{DR}(\mathcal{R}(U), C(A)) = 0$.

Proof. The operator $\circ = \circ_{\omega, j_* j^* A}$ acts on $H^a_{DR}(\mathcal{R}(X), C(j_* j^* A)) = H^{ch}_{-a}(X, j_* j^* A)$ as identity by 3.11.15(iii). On the other hand, we can repeat the construction of $\circ$ from 3.11.14 replacing X by U. We get a map $\circ_U : H^0_{DR}(\mathcal{R}(U), C(\omega)) \otimes H^a_{DR}(\mathcal{R}(U), C(A)) \to H^a_{DR}(\mathcal{R}(U), C(A))$. It is clear that the restriction isomorphism $H^a_{DR}(\mathcal{R}(X), C(j_* j^* A)) \xrightarrow{\sim} H^a_{DR}(\mathcal{R}(U), C(A))$ identifies $\circ$ and $\circ_U$, i.e., for $v \in H^a_{DR}(\mathcal{R}(X), C(j_* j^* A))$ one has $\circ(v)|_U = \circ_U(1^{ch}|_U \otimes v|_U)$. Since U is affine one has $1^{ch} = 0$[244] hence $\circ = 0$. We are done. □

Remark. The above lemma has the following relative version. Let Y be a smooth variety and $j : U \hookrightarrow X \times Y$ an open embedding such that U is affine over Y. An obvious relative version of the above constructions provides a $\mathcal{D}$-complex $C(j_* j^* A)$ on $\mathcal{R}(X) \times Y$. We claim that $p_{Y*} C(j_* j^* A) \in D\mathcal{M}(Y)$ vanishes. The proof coincides with that of the lemma.

3.11.18 Theorem. For DG chiral algebras A, B the projections $A \times B \to A, B$ yield a canonical isomorphism $R\Gamma^{ch}(X, A \times B) \xrightarrow{\sim} R\Gamma^{ch}(X, A) \times R\Gamma^{ch}(X, B)$.

Proof. We have an obvious embedding $C(A) \oplus C(B) \hookrightarrow C(A \times B)$, so the $\mathcal{D}$-complex $C(A) \oplus C(B)$ is canonically a direct summand of $C(A \times B)$. Let F be its complement, i.e., the kernel of the projection $C(A \times B) \to C(A) \oplus C(B)$. It suffices to show that $H^{\cdot}_{DR}(\mathcal{R}(X), F)$ vanish.

One has $R\Gamma_{DR}(\mathcal{R}(X), F) = R\Gamma_{DR}(\mathcal{R}(X) \times \mathcal{R}(X), P)$ where P is the following $\mathcal{D}$-complex P on $\mathcal{R}(X) \times \mathcal{R}(X)$ (i.e., on $X^{\mathcal{S}} \times X^{\mathcal{S}}$). Let $j^{[I,J]} : U^{[I,J]} \hookrightarrow X^I \times X^J$ be the complement to the diagonals $x_i = x_j$, $i \in I$, $j \in J$. Then $P_{X^I \times X^J} := j^{[I,J]}_* j^{[I,J]*} C(A, B)_{X^I \times X^J}$ where the complex $C(A, B)$ was defined in the proof of 3.11.15. We compute the cohomology by means of our usual spectral sequence with respect to the second variable. Its terms are $H^a_{DR}(\mathcal{R}(X) \times \mathcal{R}(X)^o_n, P)$. Let $\tilde{U}_n \subset \mathcal{R}(X) \times \mathcal{R}(X)^o_n$ be the open subset of pairs $([S], [T])$ such that $S \cap T = \emptyset$; then $H^a_{DR}(\mathcal{R}(X) \times \mathcal{R}(X)^o_n, P) = H^a_{DR}(\tilde{U}_n, C(A) \boxtimes \mathrm{Sym}^n B)$. Let $p_1, p_2 : \tilde{U}_n \to \mathcal{R}(X), \mathcal{R}(X)^o_n$ be the projections. We may rewrite our group as $H^a_{DR}(\mathcal{R}(X)^o_n, (Rp_{2*} p_1^* C(A)) \overset{L}{\otimes} \mathrm{Sym}^n B)$. According to the remark after 3.11.17 one has

[244] Indeed, $H^0_{DR}(\mathcal{R}(U), C(\omega)) = 0$ say, by 3.11.8.

$Rp_{2*}p_1^*C(A) = 0$, and we are done. □

3.11.19. Let us formulate the "localization principle" for chiral homology.

Let $S \subset X$ be a finite non-empty subset. Consider the subspace $i_S : \mathcal{R}(X,S) \hookrightarrow \mathcal{R}(X)$ of those points that the corresponding finite subsets of X contain S.

Let $i_S : X^{\mathrm{S}}_S \hookrightarrow X^{\mathrm{S}}$ be the r preimage of $\mathcal{R}(X,S)$; this is a diagram of closed subvarieties of X^{S}. The !-sheaves, right $\mathcal{D}$-modules, etc., on X^{S}_S are the same as corresponding objects on X^{S} supported on X^{S}_S. We have the standard functors $Ri^!_S$, i_{S*} between the corresponding derived categories.

Let A be a DG chiral algebra. Consider the canonical arrow

$$R\Gamma_{DR}(\mathcal{R}(X,S), Ri^!_S C(A)) \to R\Gamma_{DR}(\mathcal{R}(X), C(A)). \tag{3.11.19.1}$$

3.11.20 Proposition. This is a quasiisomorphism.

Proof. Idea of the argument: consider the complementary open subset $\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S) \subset \mathcal{R}(X)$. By the standard exact sequence our claim amounts to the vanishing of the cohomology $H^{\cdot}_{DR}(\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S), C(A))$.

Notice that $\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S)$ is covered by open subsets $\mathcal{R}(U_s)$, $s \in S$, where $U_s := X \smallsetminus \{s\}$. Any intersection of these subsets is $U_T := X \smallsetminus T$ for some non-empty $T \subset S$. All cohomology groups $H^{\cdot}_{DR}(\mathcal{R}(U_T), C(A))$ vanish by 3.11.15, so, by the Cech spectral sequence, the cohomology $H^{\cdot}_{DR}(\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S), C(A))$ also vanish.

To make the above argument precise one has to play with $H^{\cdot}_{DR}(\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S), C(A))$ using the sheaves on $X^{\mathrm{S}} \smallsetminus X^{\mathrm{S}}_S$. The details are left to the reader. □

Remark. As follows from 3.11.16(iv) the inverse map to (3.11.19.1) equals the composition of arrows (see 3.11.16(iv)) $R\Gamma(X^{\mathrm{S}}, C(A)) \to R\Gamma_{DR[S]}(X^S, C(\omega)) \boxtimes R\Gamma(X^{\mathrm{S}}, C(A)) \xrightarrow{\sim} R\Gamma_{DR[S]\times X^{\mathrm{S}}}(X^S \times X^{\mathrm{S}}, C(\omega)_{X^S} \boxtimes C(A)) \to R\Gamma_{DR\,X^{\mathrm{S}}_S}(X^{\mathrm{S}}, C(A))$.

3.11.21. Assume we are in situation 3.11.19. Set $U = U_S := X \smallsetminus S$. We have a retraction $\rho_S : \mathcal{E}xp(X) \to \mathcal{R}(X,S)$, $[T] \mapsto [S \cup T]$. It induces a bijective map[245] $\mathcal{E}xp(U) \to \mathcal{R}(X,S)$.

[245] We replaced $\mathcal{R}$ by $\mathcal{E}xp$ for this reason.

The restriction of ρ_S to each stratum $\mathcal{E}xp(U)^o_n$ is a closed embedding of smooth algebraic varieties $\rho_{Sn} : \mathcal{E}xp(U)^o_n \hookrightarrow \mathcal{R}(X)^o_{n+|S|}$; its image is $\mathcal{R}(X,S) \cap \mathcal{R}(X)^o_{n+|S|}$.

Assume that our A is flat at points of S. According to 3.11.20 the chiral cohomology of A are de Rham cohomology of the $\mathcal{D}$-complex $C(A,S) := i_{S*}Ri^!_S C(A)$. Look at the spectral sequence (3.11.6.2). The complex $C(A,S)_{\mathcal{R}(X)^o_{n+|S|}}$ is supported on the image of ρ_{Sn}. One has $i^!_{Sn}C(A,S) = (\otimes A^\ell_s) \otimes \mathrm{Sym}^n(A[1])_{\mathcal{E}xp(U)^o_n}$. Therefore our spectral sequence converging to $H^{ch}_n(X,A)$ is (we shifted the p index in (3.11.6.2) by $|S|$)

$$E^1_{p,q} = (\otimes A^\ell_s) \otimes H^{-p-q}_{DR}(\mathcal{E}xp(U)^o_p, \mathrm{Sym}^p(A[1])). \tag{3.11.21.1}$$

If A is a plain chiral algebra we may rewrite it as (cf. (3.11.9.2))

$$E^1_{p,q} = (\otimes A^\ell_s) \otimes H^{DR}_{p+q}(\mathcal{E}xp(U)^o_p, \Lambda^p_{ext}A). \tag{3.11.21.2}$$

Since $\mathcal{E}xp(U)^o_i$ are affine varieties of dimension i one has $E^1_{p,q} = 0$ unless $p \geq q \geq 0$.

Now we can describe the 0th chiral homology group $\langle A \rangle$ for a plain chiral algebra A. Consider the Lie algebra $h(U,A) = H^0_{DR}(U,A)$. It acts canonically on each vector space A^ℓ_s (recall that $i_{s*}A^\ell_s$ is a $j_{U*}j^*_U A$-module, see 3.4.3(i)), hence $\otimes A^\ell_s$ is an $h(U,A)$-module.

3.11.22 Proposition. (i) $\langle A \rangle$ is canonically isomorphic to the coinvariants of the $h(U,A)$-action on $\otimes A^\ell_s$.

(ii) Let $P \subset A_U$ be any $\mathcal{D}_U$-submodule which generates A_U as a chiral algebra. Then the coinvariants for the actions of $h(U,P)$ and $h(U,A)$ on $\otimes A^\ell_s$ coincide.[246] Thus

$$\langle A \rangle = (\otimes A^\ell_s)_{h(U,P)}. \tag{3.11.22.1}$$

Proof. (i) As follows from (3.11.21.2) one has $\langle A \rangle_X = E^2_{0,0} = \mathrm{Coker}(d : E^1_{1,0} \to E^1_{1,0})$. The latter differential is the arrow $h(U,A) \otimes (\otimes A^\ell_s) \to \otimes A^\ell_s$. One checks immediately that this is the action map, and we are done.

(ii) We want to show that the projection $\langle A \rangle_{h(U,P)} \twoheadrightarrow \langle A \rangle_{h(U,A)}$ is an isomorphism. Let F be the image of the action map $h(U,P) \otimes (\otimes A_s) \to \otimes A_s$. We use induction by $|S|$.

[246] i.e., the images of the action maps $h(U,P) \otimes (\otimes A_s) \to \otimes A_s$ and $h(U,A) \otimes (\otimes A_s) \to \otimes A_s$ coincide.

(a) Assume that $|S| = 1$, $S = \{s\}$. Set $Q := \mu(j_*j^*P \boxtimes P)$. It suffices to show that the image G of the action map $h(U,Q) \otimes A_s^\ell \to A_s^\ell$ is contained in F. The map $h(\mu) : h(U \times U, j_*j^*P \boxtimes P) \to h(U,Q)$ is surjective, so G equals the image of the composition $h(U \times U, j_*j^*P \boxtimes P) \otimes A_s \xrightarrow{\mu \otimes id_{A_s}} h(U,Q) \otimes A_s \to A_s$. By Jacobi this map equals the sum of two maps each of them is composition $h(U \times U, j_*j^*P \boxtimes P) \otimes A_s \to h(U,P) \otimes A_s \to A_s$ where the first arrow is the action along the first or the second variable and the second one is the action map. The images of these maps are contained in F, and we are done.

(b) Assume that $|S| > 1$. Pick $s_0 \in S$; set $S' = S \setminus \{s_0\}$, so we have $U \subset U' := X \setminus S'$. Let $P' \subset A_{U'}$ be a $\mathcal{D}_{U'}$-submodule which equals P on U. We can assume that P' generates $A_{U'}$. Indeed, let $A' \subset A$ be the chiral subalgebra that equals A on $X \setminus \{s_0\}$ and is the chiral subalgebra generated by P' on U'. Since $\underset{s \in S}{\otimes} A_s$ is a quotient of $\underset{s \in S}{\otimes} A'_s$ and we consider coinvariants of the same vector space of operators the desired statement for (A,S) follows from that for (A',S). We arrive to our assumption replacing A by A'.

By (i) we want to prove that the embedding $F \subset K := \mathrm{Ker}(\underset{s \in S}{\otimes} A_s \twoheadrightarrow \langle A \rangle)$ is an isomorphism. Since $h(U,P)$ generates $A_{s_0}^{a\,s}$ one has $\underset{s \in S}{\otimes} A_s = F + \underset{s \in S'}{\otimes} A_s$. Since $\underset{s \in S'}{\otimes} A_s \twoheadrightarrow \langle A \rangle$ it suffices to check that $F \cap (\underset{s \in S'}{\otimes} A_s) \hookrightarrow K \cap (\underset{s \in S'}{\otimes} A_s)$ is an isomorphism.

By induction assumption and (i) applied to S' we know that the image F' of $h(U',P') \otimes (\underset{s \in S'}{\otimes} A_s) \to \underset{s \in S'}{\otimes} A_s$ equals the kernel of the projection $\underset{s \in S'}{\otimes} A_s \twoheadrightarrow \langle A \rangle$, i.e., to $K \cap (\underset{s \in S'}{\otimes} A_s)$. Since $F \supset F'$ we are done. □

Remark. The above canonical projection $\otimes A_s^\ell \to \langle A \rangle$ coincides with the value at $[S] \in \mathcal{R}(X)^o_{|S|}$ of the correlator map $\langle\,\rangle_{|S|}$ (see (3.11.10.2)).

3.11.23. Let now A be a DG chiral algebra and $\{M_s\}$, $s \in S$, a finite family of DG chiral A-modules. We define *the chiral homology* $H_a^{ch}(A, \{M_s\})$ *of* A *with coefficients in* $\{M_s\}$ as follows.

Consider the chiral algebra $\tilde{A} := A \oplus (\oplus M_s[-1])$ (see 3.3.5). This chiral algebra carries an obvious $\mathbb{G}_m^S$-action, therefore $C(\tilde{A})$ is a $\mathbb{Z}^S$-graded complex. Denote by $C(A, \{M_s\})$ its component of degree 1^S. Set

$$R\Gamma^{ch}(X, A, \{M_s\}) := R\Gamma_{DR}(\mathcal{R}(X), C(A, \{M_s\})) \tag{3.11.23.1}$$

and $H^{ch}_a(X, A, \{M_s\}) = H^{ch}_a(A, \{M_s\}) := H^{-a}R\Gamma^{ch}(X, A, \{M_s\})$. If $S = \emptyset$ these are the chiral homology of A as defined in 3.11.9.

Let us describe $R\Gamma^{ch}(X, A, \{M_s\})$ a bit more explicitly. Denote by $\mathfrak{S}_S$ the category whose objects are non-empty finite sets equipped with an embedding $S \hookrightarrow I$, morphisms are surjections identical on S. We have a $\mathfrak{S}^\circ_S$-diagram of closed embeddings $X^{\mathfrak{S}_S}$, $I \mapsto X^I$ which carries a $\mathcal{D}$-complex $C'(A, \{M_s\})$ defined as follows. The graded $\mathcal{D}$-module $C'(A, \{M_s\})_{X^I}$ has an additional grading by $Q(I, S) :=$ the set of[247] morphisms $\pi : I \to T$ in $\mathfrak{S}_S$. For such π the corresponding component is $\Delta^{(\pi)}_* j^{(I)}_* j^{(I)*}((A[1])^{\boxtimes T \smallsetminus S} \boxtimes (\boxtimes M_s))[|T \smallsetminus S|]$. The differential is the sum of two components: the first one comes from μ_A and μ_{M_s}, the second one from the differentials on A and M_s. We have an obvious canonical identification $R\Gamma_{DR}(X^{\mathfrak{S}_S}, C'(A, \{M_s\}) = R\Gamma^{ch}(X, A, \{M_s\})$.

It is clear that $R\Gamma^{ch}(X, A, \{M_s\})$ is a homology functor of the arguments $\{M_s\}$. Moreover, chiral homology are functorial with respect to chiral operations: for $S \twoheadrightarrow T$ and a T-family of A-modules N_t every $\varphi \in P^{ch}_{A\,S/T}(\{M_s\}, \{N_t\}) := \otimes P^{ch}_{A\,S_t}(\{M_s\}, N_t)$ (see 3.3.4) yields an obvious morphism $R\Gamma^{ch}(\varphi) : R\Gamma^{ch}(X, A, \{M_s\}) \to R\Gamma^{ch}(X, A, \{N_t\})$; one has $R\Gamma^{ch}(\varphi\psi) = R\Gamma^{ch}(\varphi)R\Gamma^{ch}(\psi)$.

We leave it to the reader to generalize 3.11.21 to the present setting. Notice that the analog of 3.11.20 for $C(A, \{M_s\})$ is true since $C(A, \{M_s\})$ is a direct summand of $C(\tilde{A})$.

3.11.24. Consider the situation when $S \neq \emptyset$ and each M_s is supported at a (closed) point $s \in X$.[248] One may assume that these points are pairwise different (otherwise $C(A, \{M_s\}) = 0$), so we have $S \subset X$. Let $j_S : U = U_S = X \setminus S \hookrightarrow X$ be the complementary embedding. According to 3.4.3(i) the A-action on M_s extends canonically to a $j_{S*}j^*_S A$-action. The obvious morphism $C(A, \{M_s\}) \to C(j_{S*}j^*_S A, \{M_s\})$ is an isomorphism, so we can write

$$H^{ch}_a(U, A, \{M_s\}) := H^{ch}_a(X, A, \{M_s\}). \tag{3.11.24.1}$$

[247] isomorphisms classes of

[248] Notice that the general situation of 3.11.23 reduces to that one: for arbitrary $\{M_s\}$ the complex $R\Gamma^{ch}(A, \{M_s\})$ is equal to the de Rham cohomology of $U^{(S)}$ with coefficients in a $\mathcal{D}$-complex with fibers $R\Gamma^{ch}(A, \{M_{x_s}\})$.

Recall (see 3.4.5) that an A-module M_s supported at s amounts to a discrete $A_s^{a\,s}$-module $h(M_s) = i_s^!(M_s)$. In formulas we will often replace M_s by $h(M_s)$.

The complex $C(A, \{M_s\})$ is supported on $\mathcal{R}(X, S) \subset \mathcal{R}(X)$. As in 3.11.21, the spectral sequence (3.11.6.2) (the first index is shifted by $|T|$) converging to $H_n^{ch}(A, \{M_s\})$ looks as

$$E^1_{p,q} = (\otimes h(M_s)) \otimes H_{DR}^{-p-q}(\mathcal{E}xp(U)_p, \mathrm{Sym}^p(A[1])). \tag{3.11.24.2}$$

Assume that A is a plain chiral algebra and M_s are plain modules. Then

$$E^1_{p,q} = (\otimes h(M_s)) \otimes H^{DR}_{p+q}(\mathcal{E}xp(U)_p, \Lambda^p_{ext} A). \tag{3.11.24.3}$$

As in 3.11.21 we see that $E^1_{p,q} = 0$ unless $p \geq q \geq 0$. In particular, $H_a^{ch}(A, \{M_s\}) = 0$ for $a < 0$.

Let us compute $\langle h(M_s)\rangle := H_0^{ch}(A, \{M_s\})$. The Lie algebra $h(U, A)$ acts on each $h(M_s)$, so $\otimes h(M_s)$ is an $h(U, A)$-module. One has $E^2_{0,0} = H_0^{ch}(A, \{M_s\})$, so, as in 3.11.22(i), we see that the projection $E^1_{0,0} = \otimes h(M_s) \twoheadrightarrow E^2_{0,0}$ identifies $\langle h(M_s)\rangle$ with the coinvariants of the $h(U, A)$-action on $\otimes h(M_s)$:

$$(\otimes h(M_s))_{h(U,A)} \xrightarrow{\sim} \langle h(M_s)\rangle. \tag{3.11.24.4}$$

More generally, let $P \subset A_U$ be any $\mathcal{D}_U$-submodule that generates A_U as a chiral algebra. Then the coinvariants of the actions of $h(U, P)$ and $h(U, A)$ on $\otimes h(M_s)$ coincide,[249] so we have

$$(\otimes h(M_s))_{h(U,P)} \xrightarrow{\sim} \langle h(M_s)\rangle. \tag{3.11.24.5}$$

Example. Assume that A is flat at S and $M_s = j_{s*}j_s^* A/A$, i.e., $h(M_s) = A_s^\ell$ is the vacuum module. Then $C(A, \{A_s^\ell\}) = i_{S*}Ri_S^! C(A)$ (see 3.11.19 for notation). Therefore, by 3.11.20, one has

$$H_a^{ch}(A, \{A_s^\ell\}) = H_a^{ch}(A). \tag{3.11.24.6}$$

[249] Proof: the class of collections $\{M_s\}$ for which our statement is true is closed under quotients. As follows from 3.11.22(ii) it contains any $\{M_s\}$ such that every $h(M_s)$ is isomorphic to a direct sum of $A_s^{a\,s}$-modules $A_{\xi\,s}$ where $A_\xi \in \Xi_s^{a\,s}$. Since every $A_s^{a\,s}$-module is a quotient of one of these modules we are done.

More generally, if $S' \subset X$ is another finite subset such that $S \cap S' = \emptyset$ and A is flat at S' then $C(A, \{h(M_s), A^\ell_{s'}\}_{S \sqcup S'}) = i_{S'*} Ri^!_{S'} C(A, \{M_s\})$. By the variant of 3.11.20 for $C(A, \{M_s\})$ (see the end of 3.11.23) we have a canonical isomorphism (we use notation of (3.11.24.1))

$$H^{ch}_a(X \smallsetminus S, A, \{M_s\}) = H^{ch}_a(X \smallsetminus (S \sqcup S'), A, \{M_s, A^\ell_{s'}\}). \tag{3.11.24.7}$$

3.11.25. Let L be a DG Lie* algebra on X which we assume to be flat as an $\mathcal{O}_X$-module. Let $S \subset X$ be a finite (possibly empty) subset, $U = U_S := X \smallsetminus S$, M_s a DG chiral L-module (see 3.7.16–3.7.18) supported at $s \in S$.

We are going to compute the chiral homology $R\Gamma^{ch}(U(L), \{M_s\})$. It appears that they coincide with the homology of the DG Lie algebra $R\Gamma_{DR}(U, L)$ with coefficients in $\otimes h(M_s)$.

To make sense to this statement we have to explain first why $R\Gamma_{DR}(U, L)$ is a DG Lie algebra. Precisely, we want to define a canonical object in the homotopy category of DG Lie algebras (see e.g. [H]) which identifies with $R\Gamma_{DR}(U, L)$ as an object of plain derived category of k-modules. Of course[250] $R\Gamma_{DR}(U, L)$ is a Lie algebra in the tensor category $D(k)$, but this does not suffice for our purpose.

The nest point we want to explain is how to write down the Chevalley complex of $R\Gamma_{DR}(U, L)$. The problem is that the construction of 2.2.19 yields directly not a Lie DG algebra, but a homotopy one (there was a twist by a contractible operad $\mathcal{T}$). Of course, the corresponding objects are the same as Lie DG algebras in the homotopy category, but we want to write down the Chevalley complex explicitly.

Let us return to 2.2.19(iv). According to (2.2.19.1), (2.2.19.2) $DR^{\mathcal{S}}(L)$ is a Lie DG algebra in the pseudo-tensor DG category $\mathcal{C}(\mathcal{B}, C(X))$. We say that an object $F \in \mathcal{C}(\mathcal{B}, C(X))$ is a *quasicomplex on* X if for every simplex $\tilde{n} \in \mathcal{B}_l$ and a structure "simplicial map" $\partial : \mathcal{B}_l \to \mathcal{B}_m$ the corresponding morphism of complexes $F_{\tilde{n}} \to F_{\partial \tilde{n}}$ is a quasiisomorphism.

is a Lie algebra in the derived category of complexes, and one has yet to explain how to lift it to an object of the derived category of DG Lie algebras. Instead of doing this we formulate our theorem without mentioning explicitly this DG Lie algebra structure. The

[250] Since $R\Gamma_{DR}$ is a pseudo-tensor functor $D(\mathcal{M}(X))^* \to D(k)$.

homology of the graded Lie algebra $H^{\cdot}_{DR}(U,L)$ appear then on the level of the spectral sequence (see 3.11.26).

Let us define a $\mathcal{D}$-subcomplex[251]

$$C(L,\{M_s\}) \hookrightarrow C(U(L),\{M_s\}). \tag{3.11.23.1}$$

If $S=\emptyset$ then $C(L)=C(\omega)\oplus \bar{C}(L)$. Here the embedding $C(\omega)\hookrightarrow C(U(L))$ is given by $1_{U(L)}$, and the $\mathcal{D}$-subcomplex $\bar{C}(L)\subset C(U(L))$ is defined as follows. Each $\bar{C}(L)^{\cdot}_{X^I}$ is a $Q(I)$-graded submodule of $C(U(L))^{\cdot}$ (see 3.5.11); for $T\in Q(I)$ its T-component is $\Delta_*^{(I/T)}(L[1])^{\boxtimes T}\subset \Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}(U(L)[1])^{\boxtimes T}$. As follows from 3.5.11 the "interesting" components of the differential on $\bar{C}(L)$ are given by the $*$ bracket on L.

If $S\neq\emptyset$ then $C(L,\{M_s\})^{\cdot}_{X^I}$ is a $Q(I)$-graded submodule of $C(U(L),\{M_s\})^{\cdot}_{X^I}$; for $T\in Q(I)$ its T-component is $\Delta_*^{(I/T)}(\underset{S\hookrightarrow T}{\oplus}(\boxtimes M_s)\boxtimes (j_{U*}j_U^*L[1])^{\boxtimes T\smallsetminus S})$. We see that the "interesting" components of the differential are given by the $*$ bracket on L and the chiral action of L on M_s.

Our $\mathcal{D}$-complex $C(L,\{M_s\})$ carries an increasing filtration $C(L,\{M_s\})_{\cdot}$ where the subcomplex $C(L,\{M_s\})_{nX^I}$ is the sum of T-components for $|T|\leq n+|S|$. The complex $C(U(L),\{M_s\})$ also carries an increasing filtration defined by the canonical filtration on $U(L)$ (and the trivial filtration on M_s). The embedding (3.11.23.1) is compatible with the filtrations.

Passing to de Rham cohomology we get a morphism of filtered complexes

$$R\Gamma_{DR}(X^S,C(L,\{M_s\}))\to R\Gamma^{ch}(X,U(L),\{M_s\}). \tag{3.11.23.2}$$

3.11.24 Theorem. This is a filtered quasiisomorphism.

For a proof see 3.11.27.

3.11.25 Corollary. There is a canonical identification

$$\mathrm{gr}_n R\Gamma^{ch}(X,U(L),\{M_s\})\xrightarrow{\sim}(\otimes h(M_s))\otimes \mathrm{Sym}^n(R\Gamma(U,L)[1]). \tag{3.11.25.1}$$

[251] Unlike $C(U(L),\{M_s\})$ the complex $C(L,\{M_s\})$ on X^S is *not* admissible.

The "extension" morphism $\mathrm{gr}_n R\Gamma^{ch}(X, U(L), \{M_s\}) \to \mathrm{gr}_{n-1} R\Gamma^{ch}(X, U(L), \{M_s\})[1]$ is the Chevalley differential $(\otimes h(M_s)) \otimes \Lambda^n R\Gamma(U, L) \to (\otimes h(M_s)) \otimes \Lambda^{n-1} R\Gamma(U, L)$ defined by the Lie bracket on $R\Gamma(U, L)$ and the action of $R\Gamma(U, L)$ on $\otimes h(M_s)$.

Proof of the corollary. As follows from the definition of $C(L, \{M_s\})$ and the sublemma in 3.11.4 the canonical morphism of complexes $(\otimes h(M_s)) \otimes \mathrm{Sym}^n(R\Gamma(U, L)[1]) = R\Gamma_{DR}(X^{n+|S|}, C(L, \{M_s\}))_{\Sigma_{n+|S|}} \to R\Gamma_{DR}(X^S, \mathrm{gr}_n C(L, \{M_s\}))$ is a quasiisomorphism. Now use 3.11.24 to get (3.11.25.1). The second statement is left to the reader. □

3.11.26. The canonical filtration on $U(L)$ yields a spectral sequence converging to the chiral homology $H_n^{ch}(U(L), \{M_s\})$. The above corollary computes it. Namely, consider the graded Lie algebra $H^{\cdot}_{DR}(U, L)$. It acts on $H^{\cdot}(\otimes h(M_s))$. Then the second term of our spectral sequence equals to the Lie algebra homology of $H^{\cdot}_{DR}(U, L)$ with coefficients in $H^{\cdot}(\otimes h(M_s))$.

Important particular cases: if $H^{\neq 0}_{DR}(U, L) = 0$ and M_s are plain chiral L-modules then

$$H_n^{ch}(U(L), \{M_s\}) = H_n(H^0_{DR}(U, L), \otimes h(M_s)). \tag{3.11.25.2}$$

If $S = \emptyset$ and $H^{\neq 1}_{DR}(X, L) = 0$ then

$$H_0^{ch}(X, U(L)) = \mathrm{Sym} H^1_{DR}(X, L), \quad H^{ch}_{\neq 0}(X, U(L)) = 0. \tag{3.11.25.3}$$

3.11.27. *Proof of* 3.11.24. For simplicity of notation we assume that $S = \emptyset$. The similar argument in case $S \neq \emptyset$ is left to the reader.

(i) Denote by L^{ab} the $\mathcal{D}_X$-module L considered as a Lie* algebra with trivial bracket. According to 3.7.14 we have $\mathrm{gr}U(L) = \mathrm{Sym}L = U(L^{ab})$. It is clear that $\mathrm{gr}C(U(L)) = C(\mathrm{Sym}L)$.

Therefore our theorem will follow if we prove that (3.11.23.2) is a quasiisomorphism in case when the $*$ bracket on L is trivial.[252]

(ii) From now on we assume this condition. Then the action of homotheties on L yields $\mathbb{Z}_{\geq 0}$-gradings on our complexes which splits the filtrations. We denote the graded components by $C(L)^{(n)} \subset C(\mathrm{Sym}L)^{(n)}$.

[252] In case $S \neq \emptyset$ you assume in addition that the chiral action of L on M_s is trivial.

We will define on $C(\mathrm{Sym}L)$ a decreasing filtration $F^n = F^nC(\mathrm{Sym}L)$, $n \geq 0$, that satisfies the following properties:

(a) Our filtration is compatible with the grading; the filtration on each graded component $F^0C(\mathrm{Sym}L)^{(n)}$ is finite.

(b) One has $R\Gamma_{DR}(X^{\mathcal{S}}, C(\mathrm{Sym}L)/F^0) = 0$.

(c) One has $C(L)^{(n)} \subset F^n$, and this embedding yields a quasiisomorphism of complexes $\epsilon^n : R\Gamma_{DR}(X^{\mathcal{S}}, C(L)^{(n)}) \xrightarrow{\sim} R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^n_F C(\mathrm{Sym}L))$.

This will clearly prove our theorem.

(iii) The filtration F we are going to define is compatible with the $Q(I)$-grading on each $C(\mathrm{Sym}L)^{\cdot}_{X^I}$. So we need to determine for every $T \in Q(I)$ the filtration F^n_T on the T-component of $C(\mathrm{Sym}L)^{\cdot}_{X^I}$. This T-component equals $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}((\mathrm{Sym}L)[1])^{\boxtimes T}$ (see 3.5.11).

Set $F^1_T := \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}((\mathrm{Sym}^{>0}L)[1])^{\boxtimes T}$, $F^0 := C(\omega) \oplus F^1$. To define the next terms of the filtration notice that $j_*^{(T)} j^{(T)*}((\mathrm{Sym}L)[1])^{\boxtimes T} = (j_*^{(T)} j^{(T)*} \omega_X^{\boxtimes T}) \otimes ((\mathrm{Sym}L^\ell)[1])^{\boxtimes T}$. The $\mathcal{D}_T$-module $j_*^{(T)} j^{(T)*} \omega_X^{\boxtimes T}$ carries the canonical filtration W^T_n defined in 3.1.5, 3.1.6. For $n \geq 1$ we set $F^n_T := \Delta_*^{(I/T)}(W^T_{-n} \otimes ((\mathrm{Sym}^{>0}L^\ell)[1])^{\boxtimes T})$.

It is clear that F^n is a filtration by subcomplexes of $C(\mathrm{Sym}L)$.

(iv) It remains to show that our F^n satisfies conditions (a)–(c) of (ii). Condition (a) is obvious.

Let us check (b). The $\mathcal{D}$-complex $[C] := C(\mathrm{Sym}L)/F^0$ carries an obvious filtration $V_1 \subset V_2 \subset ..$ such that the components of $\mathrm{gr}^V_n[C]^{\cdot}_{X^I}$ are $\Delta_*^{(I/S\sqcup T)} j_*^{(S\sqcup T)} j^{(S\sqcup T)*}((\omega[1])^{\boxtimes S} \boxtimes ((\mathrm{Sym}^{>0}L)[1])^{\boxtimes T})$ where $S, T \in \mathcal{S}$, $|T| = n$. It suffices to show that $R\Gamma_{DR}(\mathcal{R}(X), \mathrm{gr}^V_n[C]) = 0$. We have $R\Gamma_{DR}(\mathcal{R}(X), \mathrm{gr}^V_n[C]) = R\Gamma_{DR}(\hat{U}_n, C(\omega) \boxtimes G_n)$ where $\hat{U}_n \subset \mathcal{R}(X) \times \mathcal{R}(X)^o_n$ is the open subset of pairs $([S], [T])$ such that $S \not\subseteq T$ and G_n denotes the $\mathcal{D}$-complex[253] $\mathrm{Sym}^n((\mathrm{Sym}^{>0}L)[1])$ on $\mathcal{R}(X)^o_n$. Let $p_1, p_2 : \hat{U}_n \to \mathcal{R}(X)$, $\mathcal{R}(X)^o_n$ be the projections. A relative version of the argument used in the proof of 3.11.18 shows that $p_{2*}p_1^* C(\omega) = 0$. By projection formula this implies the desired vanishing, and we are done.

[253] Here the first Sym is exterior symmetric product, while the second one is "interior" symmetric algebra.

Let us turn to condition (c). The first statement is obvious, so we have to check that ϵ^n is a quasiisomorphism. For $n = 0$ our statement is obvious. Consider the case $n = 1$. Notice that $\mathrm{gr}^1_F C(\mathrm{Sym}L)_{X^I}$ is supported on the diagonal $X \subset X^I$, so we can rewrite $R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^1_F C(\mathrm{Sym}L)$ as the de Rham cohomology of an $\mathcal{S}^\circ$-diagram Φ of $\mathcal{D}$-complexes on X. Using the canonical identification $\mathrm{gr}^W_{-1} j^{(T)}_* j^{(T)*} \omega_X^{\boxtimes T} = (P^{ch}_T(\omega_X))^* \otimes \Delta^{(T)}_* \omega_X = \mathcal{L}ie^*_T \otimes \Delta^{(T)}_* \omega_X$ (see 3.1.4–3.1.6) we see that $\Phi(I) = \underset{T \in Q(I)}{\oplus} \mathcal{L}ie^*_T \otimes ((\mathrm{Sym}^{>0} L)[1])^{\otimes T}$, i.e., $\Phi(I)^\ell = \underset{T \in Q(I)}{\oplus} \mathcal{L}ie^*_T \otimes (\mathrm{Sym}^{>0} L^\ell)^{\otimes T} \lambda_T[|T|]$. Here the differential in Φ comes from the differential in $\mathrm{Sym}^{>0} L^\ell$ and the maps $\mathcal{L}ie^*_T \otimes (\mathrm{Sym}^{>0} L^\ell)^{\otimes T} \to \mathcal{L}ie^*_{T'} \otimes (\mathrm{Sym}^{>0} L^\ell)^{\otimes T'}$ for $T' \subset T$, $|T'| = |T| - 1$, equal to the tensor product of the "cobracket map" $\mathcal{L}ie^*_T \to \mathcal{L}ie^*_{T'}$ and the product map $(\mathrm{Sym}^{>0} L^\ell)^{\otimes T} \to (\mathrm{Sym}^{>0} L^\ell)^{\otimes T'}$.

Denote by $\bar{\Phi}$ the $\mathcal{S}^\circ$-inductive limit of Φ. Consider the canonical map $C(\mathcal{S}^\circ, \Phi) \to \bar{\Phi}$ (we use notation of 3.11.1(iii)). This is a quasiisomorphism[254], so $R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^1_F C(\mathrm{Sym}L) = R\Gamma_{DR}(X, \bar{\Phi})$. Notice that $\bar{\Phi}^\ell$ is just the Harrison complex of the commutative algebra[255] $(\mathrm{Sym}^{>0} L^\ell)[1]$. Therefore, by ???, the embedding $L \hookrightarrow \bar{\Phi}$ is a quasiisomorphism, and we are done.

The case $n > 1$ is just exterior tensor product of n copies of the previous situation. □

3.11.27.

§12 Utensils

This section collects some implements needed in §11.

3.12.1 Homotopy direct limits. Below $\mathcal{A}$ is a category which we tacitly assume to be closed under direct sums of sufficiently high cardinality.

(i) Let B be a simplicial set. So for every $n \geq 0$ we have a set B_n and for every monotonous map of intervals $\alpha : [0, .., m] \to [0, .., n]$ we have the corresponding map $\partial_\alpha : B_n \to B_m$ compatible with compositions of α's.

[254] To see this consider an increasing filtration on Φ such that $\Phi_m(I)$ is formed by the summands in $\Phi(I)$ with $|T| \leq m$ and the corresponding filtration on $\bar{\Phi}$. Our morphism is, in fact, a filtred quasiisomorphism; this follows from the sublemma from 3.11.4.

[255] In the tensor category $\mathcal{M}^\ell(X)$.

Denote by $\mathcal{C}(B,\mathcal{A})$ the category of homology type coefficient systems on B with coefficients in $\mathcal{A}$. Thus $F \in \mathcal{C}(B,\mathcal{A})$ is a rule that assigns to every simplex $b \in B_n$ an object $F_b \in \mathcal{A}$ and to every α as above a morphism $\partial_\alpha = \partial_\alpha^F : F_b \to F_{\partial_\alpha(b)}$ compatible with composition of α's.

Denote by $\mathcal{C}_s(\mathcal{A})$ the category of simplicial objects in $\mathcal{A}$. We have a functor

$$C_s : \mathcal{C}(B,\mathcal{A}) \to \mathcal{C}_s(\mathcal{A}) \tag{3.12.1.1}$$

where $C_s(F)_n := \underset{b\in B_n}{\oplus} F_b$ and the structure maps $\partial_\alpha : C_s(F)_n \to C_s(F)_m$ are direct sums of the corresponding morphisms ∂_α^F.

(ii) Let $\mathcal{P}$ be a small category. It yields a simplicial set $B\mathcal{P}$ (see e.g. [QK]); an n-simplex $\tilde{p} \in B\mathcal{P}_n$ is a diagram $(p_0 \to .. \to p_n)$ in $\mathcal{P}$. Consider the category $\mathcal{F}un(\mathcal{P},\mathcal{A})$ of functors $F : \mathcal{P} \to \mathcal{A}$, $p \mapsto F_p$. There is a fully faithful embedding

$$\mathcal{F}un(\mathcal{P},\mathcal{A}) \hookrightarrow \mathcal{C}(B\mathcal{P},\mathcal{A}) \tag{3.12.1.2}$$

which identifies a functor F with the system $F_{(p_0\to..\to p_n)} = F_{p_0}$.

(iii) Assume that $\mathcal{A}$ is a k-category. Let $C(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. By Dold-Puppe one has the fully faithful embedding

$$\mathcal{N}orm : C_s(\mathcal{A}) \hookrightarrow C(\mathcal{A}) \tag{3.12.1.3}$$

which identifies $C_s(\mathcal{A})$ with the full subcategory of complexes sitting in degrees ≤ 0.

(iv) Now assume that $\mathcal{A}$ is a DG (super) k-category. We have the functor tot: $C(\mathcal{A}) \to \mathcal{A}$ which sends a complex $C = (C^n, d)$ to an object $\mathrm{tot}C \in \mathcal{A}$ such that $\mathrm{tot}C = \oplus C^n[-n]$ as a plain object without differential, the structure differential is the sum of d and the structure differentials of $C^n[-n]$. Set

$$\mathrm{Norm} := \mathrm{tot}\mathcal{N}orm : C_s(\mathcal{A}) \to \mathcal{A} \tag{3.12.1.4}$$

For a functor $F : \mathcal{P} \to \mathcal{A}$ set $C(\mathcal{P},F) := \mathrm{Norm}C_s(B\mathcal{P},F)$; this is *the homotopy direct limit* of F.

Notice that the DG structure on $\mathcal{A}$ yields DG structures on all the above categories, and (3.12.1.1)–(3.12.1.4) are DG functors.

Remark. If $\mathcal{A} = C(\mathcal{B})$ and F takes values in $\mathcal{B} \subset \mathcal{A}$ that the complex $C(\mathcal{P}, F)$ lives in degrees ≤ 0 and $H^0C(\mathcal{P}, F) \in \mathcal{B}$ is the plain $\mathcal{P}$-direct limit of F.

3.12.2 Operations I. Below $\mathcal{A}$ is pseudo-tensor category; we assume that direct sums in $\mathcal{A}$ are compatible with operations.[256]

(i) Consider the category of k-modules. This is a tensor category, so both categories of simplicial k-modules and k-complexes are tensor categories. $\mathcal{N}orm$ is *not* a tensor functor. However for any finite family of simplicial k-modules M_i there is a canonical morphism of complexes $c = c_{\{N_i\}} : \otimes \mathcal{N}orm(\mathcal{N}_i) \to \mathcal{N}orm(\otimes N_i)$ uniquely determined by condition that it equals $id_{\otimes N_i}$ if N_i are constant simplicial k-modules.[257] Therefore $\mathcal{N}orm$ is a pseudo-tensor functor.

Now if $\mathcal{C}$ is any pseudo-tensor simplicial k-category we define its *normalization* as a pseudo-tensor DG k-category $\mathcal{N}orm\mathcal{C}$ having same objects as $\mathcal{C}$ and operations $P^{\mathcal{N}orm\mathcal{C}} := \mathcal{N}orm P^{\mathcal{C}}$. The composition of operation is defined using the above c.

Remark. The plain k-categories corresponding to $\mathcal{C}$ and $\mathcal{N}orm\mathcal{C}$ coincide.

(ii) $C_s(\mathcal{A})$ is a simplicial pseudo-tensor catregory. Namely, for $F_i, G \in C_s(\mathcal{A})$ the simplicial set $P_I(\{F_i\}, G)_s$ is defined as follows. Consider $P^a_b := P_I(\{F_{i\,a}\}, G_b)$; they form a cosimplicial-simplicial set P. Our $P_I(\{F_i\}, G)_s$ is the corresponding total simplicial set $\mathcal{T}otP$ (for the definition of $\mathcal{T}ot$ see e.g. [BK]).

Remark. If $\mathcal{A}$ is a tensor category then $C_s(\mathcal{A})$ is a tensor simplicial category, and this tensor structure coincides with the obvious one.

(iii) If $\mathcal{A}$ is a pseudo-tensor k-category then $C(\mathcal{A})$ is a pseudo-tensor DG k-category. The normalization functor extends in the obvious way to a pseudo-tensor DG functor

$$\mathcal{N}orm : \mathcal{N}orm C_s(\mathcal{A}) \to C(\mathcal{A}). \tag{3.12.2.1}$$

[256] i.e., $P_I(\{\underset{\alpha \in A_i}{\oplus} F_{\alpha i}\}, G) = \prod P_I(\{F_{\alpha i}\}, G)$.

[257] Which amounts to the fact that $\mathcal{N}orm(N_i)$ are complexes supported in degree 0.

(iv) Assume that $\mathcal{A}$ is a pseudo-tensor DG (super) k-category. Then the categories in (3.11.2.1) are biDG categories and $\mathcal{N}orm$ is a pseudo-tensor biDG functor. Let us consider them as DG categories (with respect to the total grading and differential).

The functor tot: $C(\mathcal{A}) \to \mathcal{A}$ is a pseudo-tensor DG functor in the obvious way. We get a pseudo-tensor DG functor

$$\text{Norm} := \text{tot}\mathcal{N}orm : C_s(\mathcal{A}) \to \mathcal{A}. \tag{3.12.2.2}$$

3.12.2 Operations II. (i) Let B be a simplicial set equipped with an action of a simplicial operad T. Then $\mathcal{C}(B,\mathcal{A})$ is a pseudo-tensor category: an operation $\varphi \in P_I(\{F_i\}, G)$ is a rule that attaches to every $t \in T_{In}$ and $(b_i) \in B_n^I$ an operation $\varphi_t \in P(\{F_{b_i}\}, G_{t(b_i)})$ such that for every structure map ∂_α (see 3.12.1(i)) one has $\varphi_{\partial_\alpha(t)}(\partial_\alpha^{F_i}) = \partial_\alpha^G \varphi_t$. The composition of operations is the obvious one.

The functor C_s lifts to a simplicial pseudo-tensor functor

$$C_s : \mathcal{C}(B,\mathcal{A}) \times T \to \mathcal{C}_s(\mathcal{A}) \tag{3.12.3.1}$$

In the k-linear setting we can rewrite it as a simplicial pseudo-tensor k-functor

$$C_s : \mathcal{C}(B,\mathcal{A}) \otimes k[T] \to \mathcal{C}_s(\mathcal{A}) \tag{3.12.3.2}$$

Set $\mathcal{T} := \mathcal{N}ormk[T]$; this is a DG operad. If $\mathcal{A}$ is a DG k-category then composing $\mathcal{N}ormC_s$ with tot we get a pseudo-tensor DG functor

$$C : \mathcal{C}(B,\mathcal{A}) \otimes \mathcal{T} \to \mathcal{A}. \tag{3.12.3.3}$$

In particular, for any DG k-operad $\mathcal{B}$ if F is a $\mathcal{B}$ DG algebra is $\mathcal{C}(B,\mathcal{A})$ then $C(F)$ is a DG $\mathcal{B} \otimes \mathcal{T}$ algebra in $\mathcal{A}$.

(ii) Let $\mathcal{P}$ be a small tensor (:= commutative monoidal) category; we denote the operation in $\mathcal{P}$ by Σ. Then for every non-empty finite set I we have a functor $\Sigma_I : \mathcal{P}^I \to \mathcal{P}$ defined up to a canonical isomorphism. All such functors Σ_I form a small category equivalent to the trivial one; let T_I be the corresponding contractible simplicial set (= the simplex with

the set of vertices Σ_I). These T_I's form a simplicial operad T which acts on $B\mathcal{P}$ in the obvious way.

One defines a pseudo-tensor structure on $\mathcal{F}unct(\mathcal{P},\mathcal{A})$ as follows. An operation $\varphi \in P_I(\{F_i\},G)$ is a rule that assigns to every $(p_i) \in \mathcal{P}^I$ an operation $\varphi_{(p_i)} \in P_I(\{F_{ip_i}\},G_{\Sigma p_i})$ such that for every morphism $(f_i):(q_i)\to(p_i)$ in $\mathcal{P}^I$ the operations $\varphi_{(p_i)}(f_i),(\Sigma f_i)\varphi_{(q_i)} \in P_I(\{F_{iq_i}\},G_{\Sigma p_i})$ coincide. It is clear that (3.12.1.2) extends to a fully faithful pseudo-tensor embedding

$$\mathcal{F}unct(\mathcal{P},\mathcal{A}) \hookrightarrow \mathcal{C}(B\mathcal{P},\mathcal{A}). \tag{3.12.3.4}$$

If $\mathcal{A}$ is a pseudo-tensor DG category then this is a pseudo-tensor DG embedding. Composing it with (3.12.3.3) we get a pseudo-tensor DG functor (here $\mathcal{T}=\mathcal{N}ormk[T]$)

$$C(\mathcal{P},\cdot):\mathcal{F}unct(\mathcal{P},\mathcal{A})\otimes\mathcal{T}\to\mathcal{A}. \tag{3.12.3.5}$$

3.12.4.

3.12.5 Dolbeault algebras. In 3.11 we deal with de Rham homology of systems of $\mathcal{D}$-modules on powers of X connected by diagonal embeddings. We will present explicit complexes computing this homology which are compatible with the diagonal Gysin maps and exterior tensor products. These complexes are an algebraic version of the $\bar\partial$-complex. We begin with preliminaries.

Definition. Let X be any scheme. A *Dolbeault $\mathcal{O}_X$-algebra* is a commutative DG $\mathcal{O}_X$-algebra[258] $\mathcal{P}_X$ sitting in degrees ≥ 0 which satisfies the following conditions:

(i) The structure morphism $\mathcal{O}_X \xrightarrow{\alpha} \mathcal{P}_X$ is a quasiisomorphism.

(ii) $\mathcal{P}_X$ is $\mathcal{O}_X$-flat.

(iii) $\mathrm{Spec}\mathcal{P}_X^0$ is an affine scheme.

A Dolbeault $\mathcal{D}_X$-algebra is a commutative DG $\mathcal{D}_X$-algebra which is a Dolbeault $\mathcal{O}_X$-algebra.

Lemma (i) Dolbeault $\mathcal{O}_X$-algebras on a quasiprojective X exist.

[258] As always, quasicoherent as an $\mathcal{O}_X$-module.

(ii) If, in addition, X is smooth then it admits a Dolbeault $\mathcal{D}_X$-algebra.

Proof (i) Let $\pi : Y \to X$ be a Jounalou map, i.e., Y is a torsor over X with respect to an action of some vector bundle such that Y is an affine scheme. Recall that one can construct such Y as follows. Choose an open embedding $X \subset \bar{X}$ such that $\bar{X}$ is a projective variety. Blowing $\bar{X} \smallsetminus X$ up if necessary we can assume that $\bar{X} \smallsetminus X \subset \bar{X}$ is locally defined by one equation. Choose an embedding $\bar{X} \hookrightarrow \mathbb{P}^n$. Let $V \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ be the complement to the incidence divisor. Set $Y := X \underset{\mathbb{P}^n}{\times} V$.

Consider now the relative de Rham complex $DR(Y/X)$. It is clear that $\mathcal{P}_X := \pi_* DR_{Y/X}$ is a Dolbeault $\mathcal{O}_X$-algebra.

(ii) Take $\mathcal{P}_X$ as above. Then the jet algebra $\mathcal{J}\mathcal{P}_X = DR(\mathcal{J}\mathcal{P}_X^0/\mathcal{O}_X)$[259] (see 2.3.2) is a Dolbeault $\mathcal{D}_X$-algebra. Indeed, it obviously satisfies conditions (i), (ii), and (iii) follows since the morphism $\mathrm{Spec}\mathcal{J}\mathcal{P}_X^0 = \mathcal{J}Y \to Y$ is affine. □

One uses Dolbeault algebras to construct canonical resolutions computing the cohomology. In $\mathcal{D}$-module situation the statement looks as follows. Let $\mathcal{Q}_X$ be a Dolbeaut $\mathcal{D}_X$-algebra, M a right $\mathcal{D}_X$-complex.

Lemma. (i) The morphism $\alpha_M = \alpha \otimes id_M : M \to \mathcal{Q} \otimes M$ is a quasiisomorphism.

(ii) One has $R\Gamma_{DR}(X, M) = \Gamma_{DR}(X, \mathcal{Q} \otimes M)$.

Proof (i) If M is a $\mathcal{D}$- (or $\mathcal{O}$-) module then α_M is a quasiisomorphism by (i), (ii).[260] Since $\mathcal{Q}_X$ sits in degrees ≥ 0 this implies that α_M is a quasiisomorphism for every $\mathcal{D}$- (or $\mathcal{O}$-) complex M.

(ii) Let F be a k-sheaf which admits a structure of $\mathcal{O}_X$-quasicoherent $\mathcal{Q}_X^0$-module (we forget about the $\mathcal{D}$-module structure on $\mathcal{Q}_X^0$ here). Then, by condition (iii), one has $H^{>0}(X, F) = 0$. Thus for a complex G of such k-sheaves one has $\Gamma(X, G) = R\Gamma(X, G)$.

Therefore $\Gamma_{DR}(X, \mathcal{Q}_X \otimes M) \xrightarrow{\sim} R\Gamma_{DR}(X, \mathcal{Q}_X \otimes M)$ since $DR(\mathcal{Q}_X \otimes M)$ obviously satisfies this property. By (i) $R\Gamma_{DR}(X, M) \xrightarrow{\sim} R\Gamma_{DR}(X, \mathcal{Q}_X \otimes M)$ and we are done. □

[259] The fact that functors DR and $\mathcal{J}$ commute follows from their universal properties. Precisely, for an $\mathcal{O}_X$-algebra B both $DR(\mathcal{J}B)$ and $\mathcal{J}DR(B)$ represent the same functor $F \mapsto \mathrm{Hom}_{\mathcal{O}_X}(B, F^0)$ on the category of commutative DG $\mathcal{D}_X$-algebras.

[260] Since $\mathcal{O}$ has finite Tor-dimension we can play with infinite complexes without problems.

Remarks. (i) Of course, Dolbeault $\mathcal{D}$-algebras are huge, so the above complexes "computing" $R\Gamma_{DR}(X,M)$ look excessive. A smaller complex doing same job is $\Gamma_{DR}(Y,\pi^*M)$ where $\pi: Y \to X$ is a Jounalou map. However we will actually use Dolbeault $\mathcal{D}$-resolutions for computations of de Rham homology of $\mathcal{D}$-*modules on* $\mathcal{R}(X)$ (see the next subsection), and we do not know how to do this using smaller complexes.

(ii) Let $i: X \hookrightarrow Z$ be a closed embedding of smooth varieties. The for $\mathcal{Q}_X$, M as above one has $\Gamma(Y, DR(i_*(\mathcal{Q}_X \otimes M))) = R\Gamma_{DR}(Z, i_*(\mathcal{Q}_X \otimes M))$. Indeed, terms of the complex $DR(i_*(\mathcal{Q}_X \otimes M))$ admit an increasing bounded below filtration whose successive quotients are $\mathcal{Q}_X^0$-modules; then follow as above.

(iii) According to Remark (i) in 1.4.6 the functor $C\mathcal{M}(X) \to C\mathcal{M}(X, \mathcal{Q}_X)$, $M \mapsto \mathcal{Q}_X \otimes M$, is a compound tensor DG functor (here C denotes the category of complexes or DG modules). The "forgetting of $\mathcal{Q}_X$-action" functor $C\mathcal{M}(X, \mathcal{Q}_X) \to C\mathcal{M}(X)$ is a $*$ pseudo-tensor functor. So $\alpha_M : M \to \mathcal{Q}_X \otimes M$ is a morphism of $*$ pseudo-tensor functors. Thus for $\varphi \in P_I^*(\{M_i\}, N)$ its action on $R\Gamma_{DR}$ coincides with the morphism of complexes $\underset{I}{\boxtimes} \Gamma_{DR}(X, \mathcal{Q}_X \otimes M_i) \to \Gamma_{DR}(X^I, \Delta_*^{(I)}\mathcal{Q}_X \otimes N)$.

3.12.6 Computation of $R\Gamma^{ch}$. We are in the setting of 3.11, so our X is a compact curve. Let $\mathcal{Q}_X$ be a Dolbeault $\mathcal{D}_X$-algebra. According to 3.5.? it yields a commutative factorization DG algebra $\mathcal{Q}$, so we have commutative DG $\mathcal{D}_{X^n}$-algebras $\mathcal{Q}_{X^n}$.

Lemma. Every $\mathcal{Q}_{X^n}$ is a Dolbeault $\mathcal{D}_{X^n}$-algebra.

Proof. Condition (i) holds since it obviously holds on the level of the Cousin complexes. Condition (ii) follows from Remark in 3.5.4. By construction, $\mathcal{Q}_{X^n}$ is a $\mathcal{Q}_X^{\boxtimes n}$-algebra, so one has an affine morphism $\mathrm{Spec}\mathcal{Q}_{X^n} \to (\mathrm{Spec}\mathcal{Q}_X)^n$. The latter is an affine scheme which implies condition (iii). □

Let M be a right $\mathcal{D}$-complex on X^S. We say that M is *convenient* if $\Gamma(X^n, DR(M)) = R\Gamma(X^n, DR(M))$ for every n. A DG chiral algebra A is convenient if such is the $\mathcal{D}$-complex $C(A)$. Thus for convenient M, A one has canonical filtered quasiisomorphisms $R\Gamma_{DR}(X^S, M) = \Gamma_{DR}(X^S, M)$, $R\Gamma^{ch}(X, A) = \Gamma^{ch}(X, A)$.

Corollary. (i) For every right $\mathcal{D}$-complex M on X^S the complex $\mathcal{Q} \otimes M$ (see Remark

in 3.11.6) is convenient.

(ii) For every DG chiral algebra A the chiral algebra $\Omega_X \otimes A$ is convenient.

Proof. (i) is clear, (ii) follows from (i) since $C(\Omega_X \otimes A) = \Omega \otimes C(A)$. □

The canonical morphisms $M \to \Omega \otimes M$, $A \to \Omega_X \otimes A$ are quasiisomorphisms so one has canonical (filtered) quasiisomorphisms

$$R\Gamma_{DR}(X^S, M) = \Gamma_{DR}(X^S, \Omega \otimes M), \quad R\Gamma^{ch}(X, A) = \Gamma^{ch}(X, \Omega_X \otimes A). \tag{3.12.6.1}$$

3.12.7 Batalin-Vilkovisky algebras. For more details and examples see e.g. [Ge]. Below we follow superconventions of 1.1.16 as always dropping the prefix "super".

The Batalin-Vilkovisky operad BV is the odd Poisson k-operad[261] considered as a DG operad. The grading is determined by property that the product $\cdot$ has degree 0 and the odd Poisson bracket $\{\ \ \}$ has degree 1. The differential is determined by property $d(\cdot) = \{\ \ \}$. So for a complex C a BV algebra structure on C is a commutative algebra structure on C considered as a plain graded (super) vector space given by product $\cdot$ together with a Lie bracket $\{\ \ \}$ on $C[-1]$. One demands that the Lie algebra $C[-1]$ acts on C (the adjoint action) by derivations of $\cdot$ and $d(\cdot) = \{\ \ \}$.

The BV operad is acyclic – one has $H^\cdot(\mathrm{BV}_n) = 0$ for $n > 0$. So to get an interesting homotopy theory one has to consider BV as a *filtered* DG operad.

Namely, BV carries a canonical increasing filtration (the stupid one) $BV^{\geq -n}$. A *filtered* BV algebra is a complex C equipped with a BV algebra structure and an increasing filtration which is compatible with the BV algebra structure (i.e., the products $\mathrm{BV}_n \otimes C^{\otimes n} \to C$ are compatible with the filtrations). This amounts to the property that the product $\cdot$ is compatible with filtrations and the induced product on $\mathrm{gr}C$ is compatible with the differential. Therefore $\mathrm{gr}C$ is an odd Poisson DG algebra.[262]

From now on we assume that our filtered BV algebras C satisfy conditions

$$C_{-1} = 0, \quad \cup C_n = C. \tag{3.12.7.1}$$

[261] alias braid, alias Gerstenhaber operad.

[262] The odd Poisson bracket $\{\ \ \} : \mathrm{gr}_i C[-1] \otimes \mathrm{gr}_j C[-1] \to \mathrm{gr}_{i+j-1} C[-1]$ comes from the bracket $\{\ \ \} : C_i[-1] \otimes C_j[-1] \to C_{i+j-1}[-1]$.

The category of such filtered BV algebras is denoted by $\mathcal{BV}$.

The BV operad is augmented in the obvious way. So we have the notion of *unital* BV algebra. Explicitely, a BV algebra C is unital if it has a unit $1 \in C^0$ with respect to $\cdot$ such that $d(1) = 0$ (then 1 lies in the $\{\ \ \}$-center of C). In the filtered setting we assume that $1 \in C_0$. The category of unital BV algebras is denoted by $\mathcal{BV}u$.

The DG operad BV has a canonical coproduct $\delta : \mathrm{BV} \to \mathrm{BV} \otimes \mathrm{BV}$, $\delta(\cdot) = \cdot \otimes \cdot$, so we know what is tensor product of BV algebras.[263] Tensor product of BV compatible filtrations is BV compatible, so we know what is tensor product of filtered BV algebras. Tensor product of unital algebras is obviously unital.

Following [H] one can show that the category $\mathcal{BV}$ is a closed model category in the sense of Quillen: the weak equivalences are filtered quasiisomorphisms, the fibrations are those morphisms f that $\mathrm{gr}f$ is surjective. Therefore one can localize it by filtered quasiisomorphisms in a nice way; the result is the homotopy category $\mathcal{H}o\mathcal{BV}$ of BV algebras. In the unital setting we get the homotopy category $\mathcal{H}o\mathcal{BV}u$.

Remark. The notion of BV algebra makes sense in any DG pseudo-tensor category, that of filtered BV algebra in a filtered DG pseudo-tensor category. Unital algebras make sense in the augmented setting (see 1.2.8).

3.12.8 Enveloping BV algebras. For a Lie DG algebra L its Chevalley homology complex $C(L)$ is a filtered unital BV algebra: as a plain graded algebra it equals $\mathrm{Sym}(L[1])$, the odd Poisson bracket is determined by condition that on $L = \mathrm{Sym}^1(L[1])[-1] \subset C[-1]$ it coincides with the Lie bracket, the filtration is $\mathrm{Sym}^{\leq i}(L[1])$.

The above example can be generalized as follows. Let R be a commutative unital DG algebra. For a Lie DG R-algebroid[264] $\mathcal{L}$ its BV *extension* is an extension of complexes $0 \to R[-1] \xrightarrow{i} \mathcal{L}^\flat \xrightarrow{\pi} \mathcal{L} \to 0$ together with a graded Lie R-algebroid structure on $\mathcal{L}^\flat$ (with differential forgotten) such that

[263] This is the usual tensor product of odd Poisson algebras with the obvious differential.
[264] See 2.8.1.

(a) π is a morphism of graded Lie algebroids, i is a morphism of R-modules, and $k[-1]1^\flat$ belongs to the center of $\mathcal{L}^\flat$; here $1^\flat := i(1) \in \mathcal{L}^\flat[1]$.

(b) The morphism[265] $\mathcal{L}^\flat \otimes R = R \otimes \mathcal{L}^\flat \xrightarrow{d(\cdot)} \mathcal{L}^\flat[1]$ equals $-i$ composed with the structure action of $\mathcal{L}^\flat$ on R.

We call such $(\mathcal{L}, \mathcal{L}^\flat)$ a *BV Lie R-algebroid*, and write it simply as $\mathcal{L}^\flat$. Pairs $(R, \mathcal{L}^\flat)$ form a category in the obvious way.

Now if C is a unital filtered BV algebra then $R_C := C_0 = \mathrm{gr}_0 C$ is a commutative unital DG algebra, $\mathcal{L} := \mathrm{gr}_1 C[-1]$ is a Lie DG R_C-algebroid, and $C_1[-1]$ is a BV extension of $\mathcal{L}$. We have defined a functor from the category of unital BV algebras to that of pairs $(R, \mathcal{L}^\flat)$. It admits a left adjoint functor $(R, \mathcal{L}^\flat) \mapsto C(R, \mathcal{L}^\flat)$. As a plain commutative graded algebra $C(R, \mathcal{L}^\flat)$ equals $\mathrm{Sym}^\flat(\mathcal{L}[1]) :=$ the quotient of $\mathrm{Sym}_R(\mathcal{L}^\flat[1])$ modulo the relation $1^\flat = 1$, the filtration subspaces $C(R, \mathcal{L}^\flat)_a$ are images of $\mathrm{Sym}^{\leq a}(\mathcal{L}^\flat)$, and the odd Poisson bracket and differential are uniquely determined by condition that $\mathcal{L}^\flat \to C(R, \mathcal{L}^\flat)_1[-1]$ is a morphism of Lie DG algebras. If $\mathcal{L}$ is R-flat then $\mathrm{gr}C(R, \mathcal{L}^\flat)$ equals $\mathrm{Sym}(\mathcal{L}[1])$ as an odd Poisson DG algebra.

3.12.9 Homotopy BV algebras. Let $\mathcal{T}$ be a k-operad equipped with a morphism $\mathcal{T} \to \mathcal{C}om$ such that every $\mathcal{T}_I \to \mathcal{C}om_I = k$ is a quasiisomorphism. Then $\mathrm{BV} \otimes \mathcal{T}$ is a filtered DG operad equipped with a morphism of $\mathrm{BV} \otimes \mathcal{T} \to \mathrm{BV}$ which is a filtered quasiisomorphism.

Such objects are called *homotopy BV operads*, and the ones we need are all come from some $\mathcal{T}$ by the above construction.

is a filtered DG operad $\mathrm{BV}^\sim$ equipped with a morphism $\mathrm{BV}^\sim \to \mathrm{BV}$ which is a quasiisomorphism. We assume that the 0th term of the filtration equals $\mathrm{BV}^\sim$ and on every complex of n-operations $\mathrm{BV}^\sim_n$ the filtration is finite. As above the $\mathrm{BV}^\sim$ filtered DG algebras C we consider are assumed to satisfy the condition $C_{-1} = 0$; such algebras form the category $\mathcal{BV}^\sim$. As above, this is is a closed model category in the sense of Quillen (same definition of weak equivalences and fibrations). As in [H] one can show that the obvious functor $\mathcal{BV} \to \mathcal{BV}^\sim$ yields an equivalence of the homotopy categories $\mathcal{H}o\mathcal{BV} \to \mathcal{H}o\mathcal{BV}^\sim$.

[265] Here $\cdot$ is the R-action on $\mathcal{L}^\flat$.

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