QUANTIZATION OF HITCHIN'S INTEGRABLE SYSTEM AND HECKE EIGENSHEAVES

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0. Introduction

0.1. Let X be a connected smooth projective curve over $\mathbb{C}$ of genus $g > 1$, G a semisimple group over $\mathbb{C}$, $\mathfrak{g}$ the Lie algebra of G. Denote by $\mathrm{Bun}_G = \mathrm{Bun}_G(X)$ the moduli stack of G-bundles on X. In [Hit87] Hitchin defined a remarkable algebra $\mathfrak{z}^{cl} = \mathfrak{z}^{cl}(X)$ of Poisson-commuting functions on the cotangent stack of $\mathrm{Bun}_G(X)$.

0.2. In this note the following is shown:

(a) The Hitchin construction admits a natural quantization. Namely, we define a commutative ring $\mathfrak{z} = \mathfrak{z}(X)$ of twisted differential operators on Bun_G such that the symbols of operators from $\mathfrak{z}$ form exactly the ring $\mathfrak{z}^{cl}$ of Hitchin's Hamiltonians. Here "twisted" means that we consider the differential operators acting on a square root of the canonical bundle ω_{Bun_G}. The twist is essential: one knows that the only global untwisted differential operators on Bun_G are multiplications by locally constant functions.

(b) The spectrum of $\mathfrak{z}$ identifies canonically with the moduli of ${}^L\mathfrak{g}$-*opers*, which is a (Lagrangian) subspace of the moduli of irreducible $\left({}^LG\right)_{ad}$-local systems on X. Here ${}^LG$ is the Langlands dual of G, ${}^L\mathfrak{g}$ its Lie algebra, $\left({}^LG\right)_{ad}$ the adjoint group; for a brief comment on opers see 0.3.

(c) For an ${}^L\mathfrak{g}$-oper $\mathfrak{F}$ denote by $N_{\mathfrak{F}}$ the quotient of the sheaf of twisted differential operators modulo the left ideal generated by the maximal ideal $\mathfrak{m}_{\mathfrak{F}} \subset \mathfrak{z}$. This is a non-zero holonomic twisted $\mathcal{D}$-module on Bun_G.

(d) One assigns to an ${}^LG$-oper $\mathfrak{F}$ a usual (non-twisted) $\mathcal{D}$-module $M_{\mathfrak{F}}$ on Bun_G. If G is simply connected $M_{\mathfrak{F}}$ is isomorphic to $\omega_{\mathrm{Bun}_G}^{-1/2} \otimes N_{\mathfrak{F}}$ (in the simply connected case $\omega_{\mathrm{Bun}_G}^{1/2}$ is unique and on the other hand $N_{\mathfrak{F}}$ makes sense because there is no difference between ${}^LG$-opers and ${}^L\mathfrak{g}$-opers). In general $M_{\mathfrak{F}} := \lambda_{\mathfrak{F}}^{-1} \otimes N_{\bar{\mathcal{F}}}$ where $\bar{\mathcal{F}}$ is the ${}^L\mathfrak{g}$-oper corresponding to $\mathfrak{F}$ and $\lambda_{\mathfrak{F}}$ is a certain invertible sheaf on Bun_G equipped with a structure

of twisted $\mathcal{D}$-module (see 5.1.1). The isomorphism class of $\lambda_{\mathfrak{F}}$ depends only on the connected component of $\mathfrak{F}$ in the moduli of ${}^LG$-opers.

(e) *Main theorem:* $M_{\mathfrak{F}}$ is a Hecke eigensheaf with eigenvalue $\mathfrak{F}$ (see ???for the precise statement). In other words $M_{\mathfrak{F}}$ corresponds to the local system $\mathfrak{F}$ in the Langlands sense.

0.3. The notion of oper (not the name) is fairly well known (e.g., the corresponding local objects were studied in [DS85]). A G-oper on a smooth curve Y is a G-local system (=G-bundle with connection) equipped with some extra structure (see 3.1.3). If $G = SL_n$ (so we deal with local systems of vector spaces),the oper structure is a complete flag of sub-bundles that satisfies the Griffiths transversality condition and the appropriate non-degeneracy condition at every point of Y. A PSL_2-oper is the same as a projective connection on Y, i.e., a Sturm-Liouville operator on Y (see [Del70] ( )). By definition, a $\mathfrak{g}$-oper is an oper for the adjoint group G_{ad}.

If Y is complete and its genus is positive then a local system may carry at most one oper structure, so we may consider opers as special local systems.

0.4. The global constructions and statements from 0.2 have local counterparts which play a primary role. The local version of (a), (b) is a canonical isomorphism between the spectrum of the center of the critically twisted (completed) enveloping algebra of $\mathfrak{g}((t))$ and the moduli of ${}^L\mathfrak{g}$-opers on the punctured disc $\operatorname{Spec}\mathbb{C}((t))$. This isomorphism was established by Feigin and Frenkel [FF92] as a specialization of a remarkable symmetry between the W-algebras for $\mathfrak{g}$ and ${}^L\mathfrak{g}$. We do not know if this "doubly quantized" picture can be globalized. The local version of 0.2(c), (d) essentially amounts to another construction of the Feigin-Frenkel isomorphism based on the geometry of Bruhat-Tits affine Grassmannian. Here the key role belongs to a vanishing theorem for the cohomology of certain critically twisted $\mathcal{D}$-modules (a parallel result for "less than critical" twist was proved in [KT95]).

0.5. This note contains only sketches of proofs of principal results. A number of technical results is stated without the proofs. A detailed exposition will be given in subsequent publications.

0.6. We would like to mention that E. Witten independently found the idea of 0.2(a–d) and conjectured 0.2(e). As far as we know he did not publish anything on this subject.

0.7. A weaker version of the results of this paper was announced in [BD96].

0.8. The authors are grateful to P. Deligne, V. Ginzburg, B. Feigin, and E. Frenkel for stimulating discussions. We would also like to thank the Institute for Advanced Study (Princeton) for its hospitality. Our sincere gratitude is due to R. Becker, W. Snow, D. Phares, and S. Fryntova for careful typing of the manuscript.

1. Differential operators on a stack

1.1. **First definitions.** A general reference for stacks is [LMB93].

1.1.1. Let $\mathcal{Y}$ be a smooth equidimensional algebraic stack over $\mathbb{C}$. Denote by $\Theta_{\mathcal{Y}}$ the tangent sheaf; this is a coherent sheaf on $\mathcal{Y}$. The cotangent stack $T^*\mathcal{Y} = \operatorname{Spec}\operatorname{Sym}\Theta_{\mathcal{Y}}$ need not be smooth. Neither is it true in general that $\dim T^*\mathcal{Y} = 2\dim\mathcal{Y}$ (consider, e.g., the classifying stack of an infinite algebraic group or the quotient of sl_n modulo the adjoint action of SL_n). However one always has

$$\dim T^*\mathcal{Y} \geq 2\dim\mathcal{Y} \tag{1}$$

We say that $\mathcal{Y}$ is *good* if

$$\dim T^*\mathcal{Y} = 2\dim\mathcal{Y} \tag{2}$$

Then $T^*\mathcal{Y}$ is locally a complete intersection of pure dimension $2\dim\mathcal{Y}$. This is obvious if $\mathcal{Y} = K\backslash S$ for some smooth variety S with an action of an algebraic group K on it (in this case $T^*\mathcal{Y}$ is obtained from T^*S by Hamiltonian reduction; see 1.2.1), and the general case is quite similar.

It is easy to show that (2) is equivalent to the following condition:

$$\operatorname{codim}\{y \in \mathcal{Y} | \dim G_y = n\} \geq n \qquad \text{for all } n > 0\,. \tag{3}$$

Here G_y is the automorphism group of y (recall that a point of a stack may have non-trivial symmetries). $\mathcal{Y}$ is said to be *very good* if

$$\operatorname{codim}\{y \in \mathcal{Y} | \dim G_y = n\} > n \qquad \text{for all } n > 0\,. \tag{4}$$

It is easy to see that $\mathcal{Y}$ is very good if and only if $T^*\mathcal{Y}^0$ is dense in $T^*\mathcal{Y}$ where $\mathcal{Y}^0 := \{y \in \mathcal{Y} | \dim G_y = 0\}$ is the biggest Deligne-Mumford substack of $\mathcal{Y}$. In particular if $\mathcal{Y}$ is very good then $T^*\mathcal{Y}_i$ is irreducible for every connected component $\mathcal{Y}_i$ of $\mathcal{Y}$.

Remark "Good" actually means "good for lazybones" (see the remark at the end of 1.1.4).

1.1.2. Denote by $\mathcal{Y}_{sm}$ the smooth topology of $\mathcal{Y}$ (see [LMB93, Section 6]). An object of $\mathcal{Y}_{sm}$ is a smooth 1-morphism $\pi_S : S \to \mathcal{Y}$, S is a scheme. A morphism $(S, \pi_S) \to (S', \pi_{S'})$ is a pair (ϕ, α), $\phi : S \to S'$ is a *smooth* morphism of schemes, α is a 2-morphism $\pi_S \rightarrow\!\!\!\!\!\sim \pi_{S'}\phi$. We often abbreviate (S, π_S) to S.

For $S \in \mathcal{Y}_{sm}$ we have the relative tangent sheaf $\Theta_{S/\mathcal{Y}}$ which is a locally free $\mathcal{O}_S$-module. It fits into a canonical exact sequence

$$\Theta_{S/\mathcal{Y}} \to \Theta_S \to \pi_S^*\Theta_{\mathcal{Y}} \to 0\,.$$

Therefore $\pi_S^* \operatorname{Sym}\Theta_{\mathcal{Y}} = \operatorname{Sym}\Theta_S/I^{cl}$ where $I^{cl} := (\operatorname{Sym}\Theta_S)\Theta_{S/\mathcal{Y}}$. The algebra $\operatorname{Sym}\Theta_S$ considered as a sheaf on the étale topology of S carries the usual Poisson bracket $\{\}$. Let $\tilde{P} \subset \operatorname{Sym}\Theta_S$ be the $\{\}$-normalizer of the ideal I^{cl}. Set $(P_{\mathcal{Y}})_S := \tilde{P}/I^{cl}$, so $(P_{\mathcal{Y}})_S$ is the Hamiltonian reduction of $\operatorname{Sym}\Theta_S$ by $\Theta_{S/\mathcal{Y}}$. This is a sheaf of graded Poisson algebras on $S_{\acute{e}t}$. If $S \to S'$ is a morphism in $\mathcal{Y}_{sm}$ then $(P_{\mathcal{Y}})_S$ equals to the sheaf-theoretic inverse image of $(P_{\mathcal{Y}})_{S'}$. So when S varies $(P_{\mathcal{Y}})_S$ form a sheaf $P_{\mathcal{Y}}$ of Poisson algebras on $\mathcal{Y}_{sm}$ called the *algebra of symbols* of $\mathcal{Y}$. The embedding of commutative algebras $P_{\mathcal{Y}} \hookrightarrow \operatorname{Sym}\Theta_{\mathcal{Y}}$ induces an isomorphism between the spaces of global sections

$$\Gamma(\mathcal{Y}, P_{\mathcal{Y}}) \rightarrow\!\!\!\!\!\sim \Gamma(\mathcal{Y}, \operatorname{Sym}\Theta_{\mathcal{Y}}) = \Gamma(T^*\mathcal{Y}, \mathcal{O}) \tag{5}$$

1.1.3. For $S \in \mathcal{Y}_{sm}$ consider the sheaf of differential operators $\mathcal{D}_S$. This is a sheaf of associative algebras on $S_{\acute{e}t}$. Let $\tilde{D}_S \subset \mathcal{D}_S$ be the normalizer of the left ideal $I := \mathcal{D}_S\Theta_{S/\mathcal{Y}} \subset \mathcal{D}_S$. Set $(D_{\mathcal{Y}})_S := \tilde{D}_S/I$. This algebra acts on the $\mathcal{D}_S$-module $(\mathcal{D}_{\mathcal{Y}})_S := \mathcal{D}_S/I$ from the right; this action identifies $(D_{\mathcal{Y}})_S$ with the algebra opposite to $\underline{\operatorname{End}}_{\mathcal{D}_S}((\mathcal{D}_{\mathcal{Y}})_S)$.

For any morphism $(\phi, \alpha) : S \to S'$ in $\mathcal{Y}_{sm}$ we have the obvious isomorphism of $\mathcal{D}_S$-modules $\phi^*((\mathcal{D}_{\mathcal{Y}})_{S'}) \rightarrow\!\!\!\!\!\sim (\mathcal{D}_{\mathcal{Y}})_S$ which identifies $(D_{\mathcal{Y}})_S$ with the sheaf-theoretic inverse image of $(D_{\mathcal{Y}})_{S'}$. Therefore $(\mathcal{D}_{\mathcal{Y}})_S$ form an $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{D}_{\mathcal{Y}}$ (actually, it is a $\mathcal{D}$-module on $\mathcal{Y}$ in the sense of 1.1.5), and $(D_{\mathcal{Y}})_S$ form a sheaf of associative algebras $D_{\mathcal{Y}}$ on $\mathcal{Y}_{sm}$ called the *sheaf of differential operators on* $\mathcal{Y}$. The embedding of sheaves $D_{\mathcal{Y}} \hookrightarrow \mathcal{D}_{\mathcal{Y}}$ induces

an isomorphism between the spaces of global sections

$$\Gamma(\mathcal{Y}, D_{\mathcal{Y}}) \xrightarrow{\sim} \Gamma(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}}). \tag{6}$$

1.1.4. The $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{D}_{\mathcal{Y}}$ carries a natural filtration by degrees of the differential operators. The induced filtration on $D_{\mathcal{Y}}$ is an algebra filtration such that $\operatorname{gr} D_{\mathcal{Y}}$ is commutative; therefore $\operatorname{gr} D_{\mathcal{Y}}$ is a Poisson algebra in the usual way.

We have the obvious surjective morphism of graded $\mathcal{O}_{\mathcal{Y}}$-modules $\operatorname{Sym} \Theta_{\mathcal{Y}} \to \operatorname{gr} \mathcal{D}_{\mathcal{Y}}$. The condition (2) from 1.1.1 assures that this is an isomorphism. If this happens then the inverse isomorphism $\operatorname{gr} \mathcal{D}_{\mathcal{Y}} \xrightarrow{\sim} \operatorname{Sym} \Theta_{\mathcal{Y}}$ induces a canonical embedding of Poisson algebras

$$\sigma_{\mathcal{Y}} : \operatorname{gr} D_{\mathcal{Y}} \hookrightarrow P_{\mathcal{Y}} \tag{7}$$

called *the symbol map*.

Remark In the above exposition we made a shortcut using the technical condition (2). The true objects we should consider in 1.1.2–1.1.4 are complexes sitting in degrees ≤ 0 (now the symbol map is always defined); the naive objects we defined are their zero cohomology. The condition (2) implies the vanishing of the other cohomology, so we need not bother about the derived categories (see 7.3.3 for the definition of the "true" $\mathcal{D}_{\mathcal{Y}}$ for an arbitrary smooth stack $\mathcal{Y}$).

1.1.5. $\mathcal{D}$-modules are local objects for the smooth topology, so the notion of a $\mathcal{D}$-module on a smooth stack is clear [1]. Precisely, the categories $\mathcal{M}^{\ell}(S)$ of left $\mathcal{D}$-modules on S, $S \in \mathcal{Y}_{sm}$, form a sheaf $\underline{\mathcal{M}}^{\ell}$ of abelian categories on $\mathcal{Y}_{sm}$ (the pull-back functors are usual pull-backs of $\mathcal{D}$-modules; they are exact since the morphisms in $\mathcal{Y}_{sm}$ are smooth). The $\mathcal{D}$-modules on $\mathcal{Y}$ are Cartesian sections of $\underline{\mathcal{M}}^{\ell}$ over $\mathcal{Y}_{sm}$; they form an abelian category $\mathcal{M}^{\ell}(\mathcal{Y})$. In other words, a $\mathcal{D}$-module on $\mathcal{Y}$ is a quasicoherent $\mathcal{O}_{\mathcal{Y}}$-module M together with compatible $\mathcal{D}_S$-module structures on each $\mathcal{O}_S$-module M_S, $S \in \mathcal{Y}_{sm}$.

[1] The definition of the derived category of $\mathcal{D}$-modules is not so clear; see 7.3.

The usual tensor product makes $\mathcal{M}^\ell(\mathcal{Y})$ a tensor category. One defines coherent, holonomic, etc. $\mathcal{D}$-modules on $\mathcal{Y}$ in the obvious way. Note that a $\mathcal{D}$-module M on $\mathcal{Y}$ defines the sheaf of associative algebras $\underline{\mathrm{End}}M$ on $\mathcal{Y}_{sm}$, $\underline{\mathrm{End}}M(S) = \underline{\mathrm{End}}M_S$.

For example, in 1.1.3 we defined the $\mathcal{D}$-module $\mathcal{D}_\mathcal{Y}$ on $\mathcal{Y}$; the algebra $D_\mathcal{Y}$ is opposite to $\underline{\mathrm{End}}\mathcal{D}_\mathcal{Y}$.

1.1.6. Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ and $\lambda \in \mathbb{C}$. Any $S \in \mathcal{Y}_{sm}$ carries the line bundle $\pi_S^*\mathcal{L}$. Therefore we have the category $\mathcal{M}^\ell(S)_{\mathcal{L}^\lambda}$ of $\pi_S^*(\mathcal{L})^{\otimes\lambda}$-twisted left $\mathcal{D}$-modules (see, e.g., [BB93]). These categories form a sheaf $\underline{\mathcal{M}^\ell_{\mathcal{L}^\lambda}}$ of abelian categories on $\mathcal{Y}_{sm}$. The category $\mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda}$ of $\mathcal{L}^{\otimes\lambda}$-twisted $\mathcal{D}$-modules on $\mathcal{Y}$ is the category of Cartesian sections of $\underline{\mathcal{M}^\ell_{\mathcal{L}^\lambda}}$. There is a canonical fully faithful embedding $\mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda} \hookrightarrow \mathcal{M}^\ell(\mathcal{L}^\cdot)$ which identifies a $\mathcal{L}^{\otimes\lambda}$-twisted $\mathcal{D}$-module on $\mathcal{Y}$ with the λ-monodromic $\mathcal{D}$-module on $\mathcal{L}^\cdot$; here $\mathcal{L}^\cdot$ is the $\mathbb{G}_m$-torsor that corresponds to $\mathcal{L}$ (i.e., the space of $\mathcal{L}$ with zero section removed). See Section 2 from [BB93].

We leave it to the reader to define the distinguished object $\mathcal{D}_{\mathcal{Y},\mathcal{L}^\lambda} \in \mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda}$ and the sheaf $D_{\mathcal{Y},\mathcal{L}^\lambda}$ of filtered associative algebras on $\mathcal{Y}_{sm}$. All the facts from 1.1.3–1.1.5 render to the twisted situation without changes.

1.1.7. In Section 5 we will need the notion of $\mathcal{D}$-module on an arbitrary (not necessarily smooth) algebraic stack locally of finite type. In the case of schemes this notion is well known (see, e.g., [Sa91]). It is local with respect to the smooth topology, so the generalization for stacks is immediate.

1.2. **Some well-known constructions.**

1.2.1. Let K be an algebraic group acting on a smooth scheme S over $\mathbb{C}$. Consider the quotient stack $\mathcal{Y} = K \setminus S$. Then S is a covering of $\mathcal{Y}$ in $\mathcal{Y}_{sm}$, and $\mathcal{D}$-modules, line bundles and twisted $\mathcal{D}$-modules on $\mathcal{Y}$ are the same as the corresponding K-equivariant objects on S. The K-action on T^*S is Hamiltonian and $T^*\mathcal{Y}$ is obtained from T^*S by the Hamiltonian reduction (*i.e.*, $T^*\mathcal{Y} = K \setminus \mu^{-1}(0)$ where $\mu : T^*S \to \mathfrak{k}^*$ is the moment

map, $\mathfrak{k} := \mathrm{Lie}(K)$). The Poisson structure on $\Gamma(T^*\mathcal{Y}, \mathcal{O}_{T^*\mathcal{Y}})$ is obtained by identifying it with $\Gamma(\mathcal{Y}, P_{\mathcal{Y}})$ (see 1.1.2) which can be computed using the covering $S \to \mathcal{Y}$:

$$\Gamma(\mathcal{Y}, P_{\mathcal{Y}}) = \Gamma\big(S, \tilde{P}_S/I_S^{cl}\big)^{\pi_0(K)}. \tag{8}$$

Here $\tilde{P} \subset \mathrm{Sym}\,\Theta_S$ is the $\{\}$-normalizer of the ideal $I_S^{cl} := (\mathrm{Sym}\,\Theta_S)\,\mathfrak{k}$ (and $\mathfrak{k}$ is mapped to $\Theta_S \subset \mathrm{Sym}\,\Theta_S$). According to 1.1.3

$$\Gamma(\mathcal{Y}, D_{\mathcal{Y}}) = \Gamma\big(S, \tilde{D}_S/I_S\big)^{\pi_0(K)} \tag{9}$$

where $\tilde{D}_S \subset \mathcal{D}_S$ is the normalizer of $I_S := \mathcal{D}_S \cdot \mathfrak{k}$.

The following construction of symbols, differential operators, and $\mathcal{D}$-modules on $\mathcal{Y}$ is useful.

1.2.2. We start with a Harish-Chandra pair $(\mathfrak{g}, K)$ (so $\mathfrak{g}$ is a Lie algebra equipped with an action of K, called adjoint action, and an embedding of Lie algebras $\mathfrak{k} \hookrightarrow \mathfrak{g}$ compatible with the adjoint actions of K). Let $\tilde{P}_{(\mathfrak{g},K)} \subset \mathrm{Sym}\,\mathfrak{g}$ be the $\{\}$-normalizer of $I^{cl}_{(\mathfrak{g},K)} := (\mathrm{Sym}\,\mathfrak{g})\mathfrak{k}$ and $\tilde{D}_{(\mathfrak{g},K)} \subset U\mathfrak{g}$ be the normalizer of $I_{(\mathfrak{g},K)} := (U\mathfrak{g})\mathfrak{k}$. Set

$$P_{(\mathfrak{g},K)} := (\mathrm{Sym}(\mathfrak{g}/\mathfrak{k}))^K = \big(\tilde{P}_{(\mathfrak{g},K)}/I^{cl}_{(\mathfrak{g},k)}\big)^{\pi_0(K)} \tag{10}$$

$$D_{(\mathfrak{g},K)} := (U\mathfrak{g}/(U\mathfrak{g})\mathfrak{k})^K = \big(\tilde{D}_{(\mathfrak{g},K)}/I_{(\mathfrak{g},K)}\big)^{\pi_0(K)}. \tag{11}$$

Then $P_{(\mathfrak{g},K)}$ is a Poisson algebra and $D_{(\mathfrak{g},K)}$ is an associative algebra. The standard filtration on $U\mathfrak{g}$ induces a filtration on $D_{(\mathfrak{g},K)}$ such that $\mathrm{gr}\,D_{(\mathfrak{g},K)}$ is commutative. So $\mathrm{gr}\,D_{(\mathfrak{g},K)}$ is a Poisson algebra. One has the obvious embedding of Poisson algebras $\sigma = \sigma_{(\mathfrak{g},K)} : \mathrm{gr}\,D_{(\mathfrak{g},K)} \hookrightarrow P_{(\mathfrak{g},K)}$.

The *local quantization condition* for $(\mathfrak{g}, K)$ says that

$$\sigma_{(\mathfrak{g},K)} \text{ is an isomorphism.} \tag{12}$$

Remark Sometimes one checks this condition as follows. Consider the obvious morphisms

$$a^{cl} : ((\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}})^{\pi_0(K)} \to P_{(\mathfrak{g},K)}, \quad a : (\mathrm{Center}\,U\mathfrak{g})^{\pi_0(K)} \to D_{(\mathfrak{g},K)}. \tag{13}$$

If a^{cl} is surjective, then (12) is valid (because $\operatorname{gr}\operatorname{Center} U\mathfrak{g} = (\operatorname{Sym}\mathfrak{g})^{\mathfrak{g}}$). Actually, if a^{cl} is surjective, then a is also surjective and therefore $D_{(\mathfrak{g},K)}$ is commutative.

1.2.3. Assume now that we are in the situation of 1.2.1 and the K-action on S is extended to a $(\mathfrak{g},K)$-action (*i.e.*, we have a Lie algebra morphism $\mathfrak{g}\to\Theta_{S'}$ compatible with the K-action on S in the obvious sense). Comparing (8) with (10) and (9) with (11), one sees that the morphisms $\operatorname{Sym}\mathfrak{g}\to\operatorname{Sym}\Theta_S$ and $U\mathfrak{g}\to\mathcal{D}_S$ induce canonical morphisms

$$h^{cl}: P_{(\mathfrak{g},K)}\to\Gamma(\mathcal{Y},P_{\mathcal{Y}}),\quad h: D_{(\mathfrak{g},K)}\to\Gamma(\mathcal{Y},D_{\mathcal{Y}}) \tag{14}$$

of Poisson and, respectively, filtered associative algebras.

If $\mathcal{Y}$ is good in the sense of 1.1.1 then we have the symbol map $\sigma_{\mathcal{Y}}: \operatorname{gr} D_{\mathcal{Y}}\hookrightarrow P_{\mathcal{Y}}$, and the above morphisms are σ-compatible: $h^{cl}\sigma_{(\mathfrak{g},K)} = \sigma_{\mathcal{Y}}\operatorname{gr} h$.

The *global quantization condition* for our data says that

$$h \text{ is strictly compatible with filtrations.} \tag{15}$$

In other words, this means that the symbols of differential operators from $h\left(D_{(\mathfrak{g},K)}\right)$ lie in $h^{cl}\sigma_{(\mathfrak{g},K)}\,(\operatorname{gr} D_{(\mathfrak{g},K)})$. If both local and global quantization conditions meet then the algebra $h\left(D_{(\mathfrak{g},K)}\right)$ of differential operators is a quantization of the algebra $h^{cl}\left(P_{(\mathfrak{g},K)}\right)$ of symbols: the symbol map $\sigma_{\mathcal{Y}}$ induces an isomorphism $\operatorname{gr} h\left(D_{(\mathfrak{g},K)}\right)\approx h^{cl}\left(P_{(\mathfrak{g},K)}\right)$.

Remark The local and global quantization conditions are in a sense complementary: the local one tells that $D_{(\mathfrak{g},K)}$ is as large as possible, while the global one means that $h\left(D_{(\mathfrak{g},K)}\right)$ is as small as possible.

1.2.4. Denote by $\mathcal{M}(\mathfrak{g},K)$ the category of Harish-Chandra modules. One has the pair of adjoint functors (see, e.g., [BB93])

$$\Delta:\mathcal{M}(\mathfrak{g},K)\to\mathcal{M}^{\ell}(\mathcal{Y}),\quad \Gamma:\mathcal{M}^{\ell}(\mathcal{Y})\to\mathcal{M}(\mathfrak{g},K).$$

Namely, for a $\mathcal{D}$-module M on $\mathcal{Y}$ the Harish-Chandra module $\Gamma(M)$ is the space of sections $\Gamma(S, M_S)$ equipped with the obvious $(\mathfrak{g}, K)$-action (e.g., $\mathfrak{g}$ acts via $\mathfrak{g} \to \Theta_S \subset \mathcal{D}_S$) and for a $(\mathfrak{g}, K)$-module V the corresponding K-equivariant $\mathcal{D}$-module $\Delta(V)_S$ is $\mathcal{D}_S \bigotimes_{U\mathfrak{g}} V$.

For example, consider the "vacuum" Harish-Chandra module $Vac := U\mathfrak{g}/(U\mathfrak{g})\mathfrak{k}$. For any $V \in \mathcal{M}(\mathfrak{g}, K)$ one has $\mathrm{Hom}(Vac, V) = V^K$, so there is a canonical bijection $\mathrm{End}(Vac) \to Vac^K = D_{(\mathfrak{g},K)}$ (see (11)) which is actually an anti-isomorphism of algebras. One has the obvious isomorphism $\Delta(Vac) = \mathcal{D}_{\mathcal{Y}}$, and the map $\Delta : \mathrm{End}(Vac) \to \mathrm{End}(\mathcal{D}_{\mathcal{Y}}) = \Gamma(\mathcal{Y}, D_{\mathcal{Y}})^\circ$ coincides with the map h from (14).

1.2.5. The above constructions have twisted versions. Namely, assume we have a central extension $(\tilde{\mathfrak{g}}, K)$ of $(\mathfrak{g}, K)$ by $\mathbb{C}$, so $\mathbb{C} \subset \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}/\mathbb{C} = \mathfrak{g}$. Denote by $U'\mathfrak{g}$ the quotient of $U\tilde{\mathfrak{g}}$ modulo the ideal generated by the central element $\mathbf{1} - 1$, $\mathbf{1} \in \mathbb{C} \subset \tilde{\mathfrak{g}}$. This is a filtered associative algebra; one identifies $\mathrm{gr}\, U'\mathfrak{g}$ with $\mathrm{Sym}\,\mathfrak{g}$ (as Poisson algebras). We get the filtered associative algebra $D'_{(\mathfrak{g},K)} := (U'\mathfrak{g}/(U'\mathfrak{g})\mathfrak{k})^K$ equipped with the embedding $\sigma : \mathrm{gr}\, D'_{(\mathfrak{g},K)} \hookrightarrow P_{(\mathfrak{g},K)}$. The *twisted local quantization condition* says that σ is an isomorphism. Notice that the remark at the end of 1.2.2 is not valid in the twisted case because $\mathrm{gr}\,\mathrm{Center}\, U'\mathfrak{g}$ may not be equal to $(\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}}$.

Let $\mathcal{L}$ be a line bundle on S. Assume that the $(\mathfrak{g}, K)$-action on S lifts to a $(\tilde{\mathfrak{g}}, K)$-action on $\mathcal{L}$ such that $\mathbf{1}$ acts as multiplication by λ^{-1} for certain $\lambda \in \mathbb{C}^*$. Equivalently, we have a $(\tilde{\mathfrak{g}}, K)$-action on $\mathcal{L}^\cdot$ which extends the K-action, is compatible with the $\mathfrak{g}$-action on S, and $\mathbf{1}$ acts as $-\lambda^{-1}t\partial_t \in \Theta_{\mathcal{L}^\cdot}$. Set $D'_{\mathcal{Y}} = D_{\mathcal{Y},\mathcal{L}^\lambda}$. One has the morphism of filtered associative algebras $h : D'_{(\mathfrak{g},K)} \to \Gamma\left(\mathcal{Y}, D'_{\mathcal{Y}}\right)$ such that $\sigma\,\mathrm{gr}\, h = h^{cl}\sigma$. The *twisted global quantization condition* says that h is strictly compatible with filtrations.

Denote by $\mathcal{M}(\mathfrak{g}, K)'$ the full subcategory of $(\tilde{\mathfrak{g}}, K)\,\mathrm{mod}$ that consists of those Harish-Chandra modules on which $\mathbf{1}$ acts as identity. One has the adjoint functors Δ, Γ between $\mathcal{M}(\mathfrak{g}, K)'$ and $\mathcal{M}^\ell(\mathcal{Y})_{\mathcal{L}^\lambda}$ defined exactly as their untwisted version. Again for $Vac' := U'\mathfrak{g}/(U'\mathfrak{g})\mathfrak{k}$ one has $\Delta(Vac') =$

$\mathcal{D}_{\mathcal{Y},\mathcal{L}^\lambda}$; the algebra $\mathrm{End}(Vac')$ is opposite to $D'_{(\mathfrak{g},K)}$, and $\Delta\colon \mathrm{End}\,(Vac') \to \mathrm{End}\,\mathcal{D}_{\mathcal{Y},\mathcal{L}^\lambda} = \Gamma(\mathcal{Y}, D'_{\mathcal{Y}})$ coincides with h.

1.2.6. *An infinite-dimensional version.* Let K be an affine group scheme over $\mathbb{C}$ (so K is a projective limit of algebraic groups) which acts on a scheme S. Assume the following condition:

(16) There exists a Zariski open covering $\{U_i\}$ of S such that each U_i is K-invariant and for certain normal group subscheme $K_i \subset K$ with K/K_i of finite type U_i is a principal K_i-bundle over a smooth scheme T_i (so $T_i = K_i \backslash U_i$).

Then the *fpqc*-quotient $\mathcal{Y} = K \setminus S$ is a smooth algebraic stack (it is covered by open substacks $(K/K_i) \setminus T_i$).

Let us explain how to render 1.2.1–1.2.5 to our situation. Note that $\mathfrak{k} = \mathrm{Lie}\, K$ is a projective limit of finite dimensional Lie algebras, so it is a complete topological Lie algebra. Consider the sheaf $\Theta_S = \mathrm{Der}\,\mathcal{O}_S$ and the sheaf $\mathcal{D}_S \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_S)$ of Grothendieck's differential operators. These are the sheaves of complete topological Lie (respectively associative) algebras. Namely, for an affine open $U \subset S$ the bases of open subspaces in $\Gamma(U, \Theta_S)$ and $\Gamma(U, \mathcal{D}_S)$ are formed by the annihilators of finitely generated subalgebras of $\Gamma(U, \mathcal{O}_U)$. The topology on Θ_S defines the topology on $\mathrm{Sym}\,\Theta_S$; denote by $\overline{\mathrm{Sym}}\Theta_S$ the completed algebra. This is a sheaf of topological Poisson algebras. Let $I_S^{cl} \subset \overline{\mathrm{Sym}}\Theta_S$ be the closure of the ideal $(\overline{\mathrm{Sym}}\Theta_S)\mathfrak{k}$, and $\tilde{P}_S \subset \overline{\mathrm{Sym}}\Theta_S$ be its $\{\ \}$-normalizer. Similarly, let $I_S \subset \mathcal{D}_S$ be the closure of the ideal $\mathcal{D}_S \cdot \mathfrak{k}$ and $\tilde{D}_S$ be its normalizer. Then the formulas from (8), (9) remain valid.

In the definition of a Harish-Chandra pair $(\mathfrak{g}, K)$ we assume that for any $\mathrm{Ad}(K)$-invariant open subspace $\mathfrak{a} \subset k$ the action of K on $\mathfrak{g}/\mathfrak{a}$ is algebraic. Then $\mathfrak{g}$ is a complete topological Lie algebra (the topology on $\mathfrak{g}$ is such that $\mathfrak{k} \subset \mathfrak{g}$ is an open embedding). The algebras $\mathrm{Sym}\,\mathfrak{g}$, $U\mathfrak{g}$ carry natural topologies defined by the open ideals $(\mathrm{Sym}\,\mathfrak{g})\mathfrak{a}$, $(U\mathfrak{g})\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{g}$ is an

open subalgebra. Denote by $\overline{\mathrm{Sym}}\mathfrak{g}$, $\bar{U}\mathfrak{g}$ the corresponding completions. Let $I^{cl}_{(\mathfrak{g},K)} \subset \overline{\mathrm{Sym}\,\mathfrak{g}}$ be the closure of the ideal $(\overline{\mathrm{Sym}}\mathfrak{g})\mathfrak{k}$ and $\tilde{P}_{(\mathfrak{g},K)}$ be its $\{\ \}$-normalizer). Similarly, we have $I_{(\mathfrak{g},K)} \subset \tilde{D}_{(\mathfrak{g},K)} \subset \bar{U}_{\mathfrak{g}}$. Now we define $P_{(\mathfrak{g},K)}$, $D_{(\mathfrak{g},K)}$ by the formulas (10), (11). The rest of 1.2.2–1.2.5 remains valid, except the remark at the end of 1.2.2. It should be modified as follows.

1.2.7. The algebras $\overline{\mathrm{Sym}}\mathfrak{g}$ and $\bar{U}\mathfrak{g}$ carry the usual ring filtrations $\overline{\mathrm{Sym}}_n\mathfrak{g} = \bigoplus_{0\le i\le n} \overline{\mathrm{Sym}}^i\mathfrak{g}$ and $\bar{U}_i\mathfrak{g}$; however in the infinite dimensional situation the union of the terms of these filtrations does not coincide with the whole algebras. One has the usual isomorphism $\bar{\sigma}_{\mathfrak{g}} : \mathrm{gr}_i\,\bar{U}\mathfrak{g} \tilde{\to} \overline{\mathrm{Sym}}^i\mathfrak{g}$. The same facts are true for $\overline{\mathrm{Sym}}\Theta_S$ and $\mathcal{D}_S$.

The morphisms a^{cl}, a from the end of 1.2.2 extend in the obvious way to the morphisms

$$\bar{a}^{cl} : \big((\overline{\mathrm{Sym}}\mathfrak{g})^{\mathfrak{g}}\big)^{\pi_0(K)} \to P_{(\mathfrak{g},K)}, \quad \bar{a} : \big(\mathrm{Center}\,\bar{U}\mathfrak{g}\big)^{\pi_0(K)} \to D_{(\mathfrak{g},K)}. \tag{17}$$

The local quantization condition (12) from 1.2.2 and the surjectivity of $\bar{a}$ follow from the surjectivity of $\bar{a}^{cl}\bar{\sigma}_{\mathfrak{g}} : \mathrm{gr}\,\big(\mathrm{Center}\,\bar{U}\mathfrak{g}\big)^{\pi_0(K)} \to P_{(\mathfrak{g},K)}$. The same is true in the twisted situation. Note that the equality $\mathrm{gr}\,\mathrm{Center}\,\bar{U}\mathfrak{g} = (\mathrm{Sym}\,\mathfrak{g})^{\mathfrak{g}}$ is not necessarily valid (even in the non-twisted case!).

2. Quantization of Hitchin's Hamiltonians

2.1. **Geometry of $\mathbf{Bun_G}$.** We follow the notation of 0.1; in particular G is semisimple and X is a smooth projective curve of genus $g > 1$.

2.1.1. One knows that Bun_G is a smooth algebraic stack of pure dimension $(g-1)\dim G$. The set of connected components of Bun_G can be canonically identified (via the "first Chern class" map) with $H^2(X, \pi_1^{\mathrm{et}}(G)) = \pi_1(G)$. Here $\pi_1^{\mathrm{et}}(G)$ is the fundamental group in Grothendieck's sense and $\pi_1(G)$ is the quotient of the group of coweights of G modulo the subgroup of coroots; they differ by a Tate twist: $\pi_1^{\mathrm{et}}(G) = \pi_1(G)(1)$.

For $\mathcal{F} \in \mathrm{Bun}_G$ the fiber at $\mathcal{F}$ of the tangent sheaf $\Theta = \Theta_{\mathrm{Bun}_G}$ is $H^1(X, \mathfrak{g}_{\mathcal{F}})$. Let us explain that for a G-module W we denote by $W_{\mathcal{F}}$ the $\mathcal{F}$-twist of W, which is a vector bundle on X; we consider $\mathfrak{g}$ as a G-module via the adjoint action.

By definition, the canonical line bundle $\omega = \omega_{\mathrm{Bun}_G}$ is the determinant of the cotangent complex of Bun_G (see [LMB93]). The fiber of this complex over $\mathcal{F} \in \mathrm{Bun}_G$ is dual to $R\Gamma(X, \mathfrak{g}_{\mathcal{F}})[1]$ (see [LMB93]), so the fiber of ω over $\mathcal{F}$ is $\det R\Gamma(X, \mathfrak{g}_{\mathcal{F}})$.[2]

2.1.2. *Proposition.* Bun_G *is very good in the sense of* 1.1.1.

A proof will be given in 2.10.5. Actually, we will use the fact that Bun_G is good. According to 1.1 we have the sheaf of Poisson algebras $P = P_{\mathrm{Bun}_G}$ and the sheaves of twisted differential operators $D^\lambda = D_{\mathrm{Bun}_G, \omega^\lambda}$. One knows that for $\lambda \neq 1/2$ the only global sections of D^λ are locally constant functions. In Sections 2 and 3 we will deal with $D' := D^{1/2}$; we refer to its sections as simply twisted differential operators.

2.2. **Hitchin's construction I.**

[2] The authors shouldn't forget to check that [LMB93] really contains what is claimed here!!

2.2.1. Set $C = C_{\mathfrak{g}} := \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g})^G$; this is the affine scheme quotient of $\mathfrak{g}^*$ with respect to the coadjoint action. C carries a canonical action of the multiplicative group $\mathbb{G}_m$ that comes from the homotheties on $\mathfrak{g}^*$. A (non-canonical) choice of homogeneous generators $p_i \in (\mathrm{Sym}\,\mathfrak{g})^G$ of degrees d_i, $i \in I$, identifies C with the coordinate space $\mathbb{C}^I$, an element $\lambda \in \mathbb{G}_m$ acts by the diagonal matrix $\left(\lambda^{d_i}\right)$.

2.2.2. Denote by C_{ω_X} the ω_X-twist of C with respect to the above $\mathbb{G}_m$-action (we consider the canonical bundle ω_X as a $\mathbb{G}_m$-torsor over X). This is a bundle over X; the above p_i identify C_{ω_X} with $\prod_I \omega_X^{\otimes d_i}$. Set

$$\mathrm{Hitch}(X) = \mathrm{Hitch}_{\mathfrak{g}}(X) := \Gamma(X, C_{\omega_X}).$$

In other words, $\mathrm{Hitch}(X) = \mathrm{Mor}\left((\mathrm{Sym}^{\cdot}\,\mathfrak{g})^G, \Gamma\left(X, \omega_X^{\otimes \cdot}\right)\right)$ (the morphisms of graded algebras). We consider $\mathrm{Hitch}(X)$ as an algebraic variety equipped with a $\mathbb{G}_m$-action; it is non-canonically isomorphic to the vector space $\prod_I \Gamma\left(X, \omega_X^{\otimes d_i}\right)$. There is a unique point $0 \in \mathrm{Hitch}(X)$ which is fixed by the action of $\mathbb{G}_m$. Denote by $\mathfrak{z}^{cl}(X) = \mathfrak{z}^{cl}_{\mathfrak{g}}(X)$ the ring of functions on $\mathrm{Hitch}(X)$; this is a graded commutative algebra. More precisely, the grading on $\mathfrak{z}^{cl}(X)$ corresponds to the $\mathbb{G}_m$-action on $\mathfrak{z}^{cl}(X)$ opposite to that induced by the $\mathbb{G}_m$-action on C; so the grading on $\mathfrak{z}^{cl}(X)$ is positive.

2.2.3. By Serre duality and 2.1.1 the cotangent space $T^*_{\mathcal{F}}\mathrm{Bun}_G$ at $\mathcal{F} \in \mathrm{Bun}_G$ coincides with $\Gamma(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X)$. The G-invariant projection $\mathfrak{g}^* \to C$ yields the morphism $\mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X \to C_{\omega_X}$ and the map $p_{\mathcal{F}} : T^*_{\mathcal{F}}\mathrm{Bun}_G \to \mathrm{Hitch}(X)$. When $\mathcal{F}$ varies we get a morphism

$$p : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X)$$

or, equivalently, a morphism of graded commutative algebras

$$h^{cl}_X : \mathfrak{z}^{cl}(X) \to \Gamma\left(T^*\mathrm{Bun}_G, \mathcal{O}\right) = \Gamma\left(\mathrm{Bun}_G, P\right).$$

p is called *Hitchin's fibration*.

We denote by $\mathrm{Bun}_G{}^\gamma$ the connected component of Bun_G corresponding to $\gamma \in \pi_1(G)$ (see 2.1.1) and by p^γ the restriction of p to $T^*\mathrm{Bun}_G^\gamma$.

2.2.4. *Theorem.* ([Hit87], [Fal93], [Gi97]).

(i) The image of h_X^{cl} consists of Poisson-commuting functions.

(ii) $\dim \mathrm{Hitch}(X) = \dim \mathrm{Bun}_G = (g-1)\cdot \dim \mathfrak{g}$.

(iii) p is flat and its fibers have pure dimension $\dim \mathrm{Bun}_G$. For each $\gamma \in \pi_1(X)$, p^γ is surjective.

(iv) There exists a non-empty open $U \subset \mathrm{Hitch}(X)$ such that for any $\gamma \in \pi_1(G)$ the morphism $(p^\gamma)^{-1}(U) \to U$ is proper and smooth, and its fibers are connected. Actually, the fiber of p^γ over $u \in U$ is isomorphic to the product of some abelian variety A_u by the classifying stack of the center $Z \subset G$.

(v) For each $\gamma \in \pi_1(X)$ the morphism $\mathfrak{z}^{cl}(X) \to \Gamma(\mathrm{Bun}_G^\gamma, P)$ is an isomorphism. □

Remarks

(i) Needless to say the main contribution to Theorem 2.2.4 is that of Hitchin [Hit87].

(ii) Theorem 2.2.4 implies that p is a Lagrangian fibration or, if you prefer, the Hamiltonians from $h_X^{cl}(\mathfrak{z}^{cl}(X))$ define a completely integrable system on $T^*\mathrm{Bun}_G$. We are not afraid to use these words in the context of stacks because the notion of Lagrangian fibration is birational and since Bun_G is very good in the sense of 1.1.1 $T^*\mathrm{Bun}_G$ has an open dense Deligne-Mumford substack $T^*\mathrm{Bun}_G^0$ which is symplectic in the obvious sense (here Bun_G^0 is the stack of G-bundles with a finite automorphism group).

(iii) Hitchin gave in [Hit87] a complex-analytical proof of statement (i). We will give an algebraic proof of (i) in 2.4.3.

(iv) Hitchin's proof of (ii) is easy: according to 2.2.2 $\dim \mathrm{Hitch}(X) = \sum_i \dim \Gamma(X, {\omega_X}^{\otimes d_i})$, $\dim \Gamma(X, {\omega_X}^{\otimes d_i}) = (g-1)(2d_i - 1)$ since $g > 1$, and finally $(g-1)\sum_i (2d_i - 1) = (g-1)\dim \mathfrak{g} = \dim \mathrm{Bun}_G$.

(v) Statement (iv) for classical groups G was proved by Hitchin [Hit87]. In the general case it was proved by Faltings (Theorem III.2 from [Fal93]).

(vi) Statement (v) follows from (iii) and (iv).

(vii) Some comments on the proof of (iii) will be given in 2.10.

2.2.5. Our aim is to solve the following quantization problem: construct a filtered commutative algebra $\mathfrak{z}(X)$ equipped with an isomorphism $\sigma_{\mathfrak{z}(X)} : \mathrm{gr}\, \mathfrak{z}(X) \widetilde{\to} \mathfrak{z}^{cl}(X)$ and a morphism of filtered algebras $h_X : \mathfrak{z}(X) \to \Gamma(\mathrm{Bun}_G, D')$ compatible with the symbol maps, i.e., such that $\sigma_{\mathrm{Bun}_G} \circ \mathrm{gr}\, h_X = h_X^{cl} \circ \sigma_{\mathfrak{z}(X)}$ (see 1.1.4 and 1.1.6 for the definition of σ_{Bun_G}). Note that 2.2.4(v) implies then that for any $\gamma \in \pi_1(X)$ the map $h_X^\gamma : \mathfrak{z}(X) \to \Gamma\left(\mathrm{Bun}_G^\gamma, D'\right)$ is an isomorphism. Therefore if G is simply connected then such a construction is unique, and it reduces to the claims that $\Gamma(\mathrm{Bun}_G, D')$ is a commutative algebra, and any global function on $T^*\mathrm{Bun}_G$ is a symbol of a global twisted differential operator.

We do not know how to solve this problem directly by global considerations. We will follow the quantization scheme from 1.2 starting from a local version of Hitchin's picture. Two constructions of the same solution to the above quantization problem will be given. The first one (see 2.5.5) is easier to formulate, the second one (see 2.7.4) has the advantage of being entirely canonical. To prove that the first construction really gives a solution we use the second one. It is the second construction that will provide an identification of $\mathrm{Spec}\, \mathfrak{z}(X)$ with a certain subspace of the stack of $({}^L G)_{\mathrm{ad}}$-local systems on X (see 3.3.2).

2.3. **Geometry of Bun_G II.** Let us recall how Bun_G fits into the framework of 1.2.6.

2.3.1. Fix a point $x \in X$. Denote by O_x the completed local ring of x and by K_x its field of fractions. Let $\mathfrak{m}_x \subset O_X$ be the maximal ideal. Set $O_x^{(n)} := \mathcal{O}_X/\mathfrak{m}_x^n$ (so $O_x = \varprojlim O_x^{(n)}$). The group $G(O_x^{(n)})$ is the group of $\mathbb{C}$-points of an affine algebraic group which we denote also as $G\big(O_x^{(n)}\big)$ by abuse of notation; $G\big(O_x^{(n)}\big)$ is the quotient of $G\big(O_x^{(n+1)}\big)$. So $G(O_x) = \varprojlim G\big(O_x^{(n)}\big)$ is an affine group scheme.

Denote by $\mathrm{Bun}_{G,nx}$ the stack of G-bundles on X trivialized over $\operatorname{Spec} O_x^{(n)}$ (notice that the divisor nx is the same as the subscheme $\operatorname{Spec} O_x^{(n)} \subset X$). This is a $G\big(O_x^{(n)}\big)$-torsor over Bun_G. We denote a point of $\mathrm{Bun}_{G,nx}$ as $\big(\mathcal{F}, \alpha^{(n)}\big)$. We have the obvious affine projections $\mathrm{Bun}_{G,(n+1)x} \to \mathrm{Bun}_{G,nx}$. Set $\mathrm{Bun}_{G,\underline{x}} := \varprojlim \mathrm{Bun}_{G,nx}$; this is a $G(O_x)$-torsor over Bun_G.

2.3.2. *Proposition.* $\mathrm{Bun}_{G,\underline{x}}$ *is a scheme. The* $G(O_x)$*-action on* $\mathrm{Bun}_{G,\underline{x}}$ *satisfies condition* (16) *from* 1.2.6. □

2.3.3. It is well known that the $G(O_x)$-action on $\mathrm{Bun}_{G,\underline{x}}$ extends canonically to an action of the group ind-scheme $G(K_x)$ (see 7.11.1 for the definition of ind-scheme and 7.11.2 (iv) for the definition of the ind-scheme $G(K_x)$). Since $\operatorname{Lie} G(K_x) = \mathfrak{g} \otimes K_x$ we have, in particular, the action of the Harish-Chandra pair $(\mathfrak{g} \otimes K_x, G(O_x))$ on $\mathrm{Bun}_{G,\underline{x}}$.

Let us recall the definition of the $G(K_x)$-action. According to 7.11.2 (iv) one has to define a $G(R\widehat{\otimes}K_x)$-action on $\mathrm{Bun}_{G,\underline{x}}(R)$ for any $\mathbb{C}$-algebra R. To this end we use the following theorem, which is essentially due to A.Beauville and Y.Laszlo. Set $X' := X \setminus \{x\}$.

2.3.4. *Theorem.* A G-bundle $\mathcal{F}$ on $X \otimes R$ is the same as a triple $(\mathcal{F}_1, \mathcal{F}_2, \varphi)$ where $\mathcal{F}_1$ is a G-bundle on $X' \otimes R$, $\mathcal{F}_2$ is a G-bundle on $\operatorname{Spec}(R\widehat{\otimes}O_x)$, and φ is an isomorphism between the pullbacks of $\mathcal{F}_1$ and $\mathcal{F}_2$ to $\operatorname{Spec}(R\widehat{\otimes}K_x)$. More precisely, the functor from the category (=groupoid) of G-bundles $\mathcal{F}$ on $X \otimes R$ to the category of triples $(\mathcal{F}_1, \mathcal{F}_2, \varphi)$ as above defined by $\mathcal{F}_1 := \mathcal{F}|_{X' \otimes R}$, $\mathcal{F}_2 :=$ the pullback of $\mathcal{F}$ to $\operatorname{Spec}(R\widehat{\otimes}O_x)$, $\varphi :=$ id, is an equivalence.

According to the theorem an R-point of $\mathrm{Bun}_{G,\underline{x}}$ is the same as a G-bundle on $X' \otimes R$ with a trivialization of its pullback to $\mathrm{Spec}(R\widehat{\otimes}K_x)$. So $G(R\widehat{\otimes}K_x)$ acts on $\mathrm{Bun}_{G,\underline{x}}(R)$ by changing the trivialization. Thus we get the action of $G(K_x)$ on $\mathrm{Bun}_{G,\underline{x}}$.

The proof of Theorem 2.3.4 is based on the following theorem, which is a particular case of the main result of [BLa95].

2.3.5. *Theorem.* (Beauville-Laszlo). The category of flat quasi-coherent $\mathcal{O}_{X\otimes R}$-modules $\mathcal{M}$ is equivalent to the category of triples $(\mathcal{M}_1, \mathcal{M}_2, \varphi)$ where $\mathcal{M}_1$ is a flat quasi-coherent $\mathcal{O}$-module on $X' \otimes R$, $\mathcal{M}_2$ is a flat quasi-coherent $\mathcal{O}$-module on $\mathrm{Spec}(R\widehat{\otimes}O_x)$, and φ is an isomorphism between the pullbacks of $\mathcal{M}_1$ and $\mathcal{M}_2$ to $\mathrm{Spec}(R\widehat{\otimes}K_x)$ (the functor from the first category to the second one is defined as in Theorem 2.3.4). $\mathcal{M}$ is locally free of finite rank if and only if the corresponding $\mathcal{M}_1$ and $\mathcal{M}_2$ have this property.

Remark. If R is noetherian and the sheaves are coherent then there is a much more general "glueing theorem" due to M.Artin (Theorem 2.6 from [Ar]). But since subschemes of $G(K_x)$ are usually of infinite type we use the Beauville-Laszlo theorem, which holds without noetherian assumptions.

To deduce Theorem 2.3.4 from 2.3.5 it suffices to interpret a G-bundle as a tensor functor $\{G\text{-modules}\}\to\{\text{vector bundles}\}$. Or one can interpret a G-bundle on $X \otimes R$ as a principle G-bundle, i.e., a flat affine morphism $\pi : \mathcal{F} \to X \otimes R$ with an action of G on $\mathcal{F}$ satisfying certain properties; then one can rewrite these data in terms of the sheaf $\mathcal{M} := \pi_*\mathcal{O}_\mathcal{F}$ and apply Theorem 2.3.5.

2.3.6. *Remark.* Here is a direct description of the action of $\mathfrak{g} \otimes K_x$ on $\mathrm{Bun}_{G,\underline{x}}$ induced by the action of $G(K_x)$ (we will not use it in the future ???). Take $(\mathcal{F}, \bar{\alpha}) \in \mathrm{Bun}_{G,\underline{x}}$, $\bar{\alpha} = \varprojlim \alpha^{(n)}$. The tangent space to $\mathrm{Bun}_{G,nx}$ at $\left(\mathcal{F}, \alpha^{(n)}\right)$ is $H^1(X, \mathfrak{g}_\mathcal{F}(-nx))$, so the fiber of $\Theta_{\mathrm{Bun}_{G,\underline{x}}}$ at $(\mathcal{F}, \bar{\alpha})$ equals $\varprojlim H^1\left(X, \mathfrak{g}_\mathcal{F}(-nx)\right) = H^1_c(X \setminus \{x\}, \mathfrak{g}_\mathcal{F})$. We have the usual surjection $\mathfrak{g}_\mathcal{F} \bigotimes_{\mathcal{O}_X} K_x \twoheadrightarrow H^1_c(X \setminus \{x\}, \mathfrak{g}_\mathcal{F})$. Use $\bar{\alpha}$ to identify $\mathfrak{g}_\mathcal{F} \bigotimes_{\mathcal{O}_X} K_x$ with $\mathfrak{g} \otimes K_x$.

When $(\mathcal{F}, \bar{\alpha})$ varies one gets the map $\mathfrak{g} \otimes K_x \to \Theta_{\mathrm{Bun}_G, \underline{x}}$. Our $\mathfrak{g} \otimes K_x$-action is minus this map (???).

2.3.7. *Remark.* Let $D \subset X \otimes R$ be a closed subscheme finite over $\operatorname{Spec} R$ which can be locally defined by one equation (i.e., D is an effective relative Cartier divisor). Denote by $\tilde{D}$ the formal neighbourhood of D and let A be the coordinate ring of $\tilde{D}$ (so $\tilde{D}$ is an affine formal scheme and $\operatorname{Spec} A$ is a true scheme). Then Theorems 2.3.4 and 2.3.5 remain valid if $X' \otimes R$ is replaced by $(X \otimes R) \setminus D$, $R\widehat{\otimes}O_x$ by A, and $\operatorname{Spec}(R\widehat{\otimes}K_x)$ by $(\operatorname{Spec} A) \setminus D$. This follows from the main theorem of [BLa95] if the normal bundle of D is trivial: indeed, in this case one can construct an affine neighbourhood $U \supset D$ such that inside U the subscheme D is defined by a global equation $f = 0$, $f \in H^0(U, \mathcal{O}_U)$ (this is the situation considered in [BLa95]).[3] For the purposes of this work the case where the normal bundle of D is trivial is enough. To treat the general case one needs a globalized version of the main theorem of [BLa95] (see 2.12). Among other things, one has to extend the morphism $\tilde{D} \to X \otimes R$ to a morphism $\operatorname{Spec} A \to X \otimes R$ (clearly such an extension is unique, but its existence has to be proved); see 2.12.

2.4. **Hitchin's construction II.**

2.4.1. Set $\omega_{O_x} := \lim_{\leftarrow} \omega_{O_x{}^{(n)}}$ where $\omega_{O_x{}^{(n)}}$ is the module of differentials of $O_x{}^{(n)} = O_x/m_x^n$. Denote by $\mathrm{Hitch}_x^{(n)}$ the scheme of sections of C_{ω_X} over $\operatorname{Spec} O_x^{(n)}$. This is an affine scheme with $\mathbb{G}_m$-action non-canonically isomorphic to the vector space $M/m_x^n M$, $M := \prod \omega_{O_x}^{\otimes d_i}$. Set

$$\mathrm{Hitch}_x = \mathrm{Hitch}_{\mathfrak{g}}(O_x) := \varprojlim \mathrm{Hitch}_x^{(n)}\,.$$

This is an affine scheme with $\mathbb{G}_m$-action non-canonically isomorphic to $M = \prod \omega_{O_x}^{\otimes d_i}$. So Hitch_x is the scheme of sections of C_{ω_X} over $\operatorname{Spec} O_x$.

[3] To construct U and f notice that for n big enough there exists $\varphi_n \in H^0(X \otimes R, \mathcal{O}_{X\otimes R}(nD))$ such that $\mathcal{O}_{X\otimes R}(nD)/\mathcal{O}_{X\otimes R}((n-1)D)$ is generated by φ_n; then put $U := (X \otimes R) \setminus \{\text{the set of zeros of } \varphi_n\varphi_{n+1}\}$, $f := \varphi_n/\varphi_{n+1}$ (this construction works if the map $D \to \operatorname{Spec} R$ is surjective, which is a harmless assumption).

Denote by $\mathfrak{z}_x^{cl} = \mathfrak{z}_{\mathfrak{g}}^{cl}(O_x)$ the graded Poisson algebra $P_{(\mathfrak{g}\otimes K_x, G(O_x))} = \mathrm{Sym}(\mathfrak{g}\otimes K_x/O_x)^{G(O_x)}$ from 1.2.2. We will construct a canonical $\mathbb{G}_m$-equivariant isomorphism $\operatorname{Spec}\mathfrak{z}_x^{cl} \xrightarrow{\sim} \mathrm{Hitch}_x$ (the $\mathbb{G}_m$-action on $\mathfrak{z}_x^{cl}$ is opposite to that induced by the grading; cf. the end of 2.2.2).

The residue pairing identifies $(K_x/O_x)^*$ with ω_{O_x}, so $\operatorname{Spec}\mathrm{Sym}(\mathfrak{g}\otimes K_x/O_x) = \mathfrak{g}^*\otimes\omega_{O_x}$. The projection $\mathfrak{g}^* \to C$ yields a morphism of affine schemes $\mathfrak{g}^*\otimes\omega_{O_x} \to \mathrm{Hitch}_x$. It is $G(O_x)$-invariant, so it induces a morphism $\operatorname{Spec}\mathfrak{z}_x^{cl} \to \mathrm{Hitch}_x$. To show that this is an isomorphism we have to prove that every $G(O_x)$-invariant regular function on $\mathfrak{g}^*\otimes\omega_{O_x}$ comes from a unique regular function on Hitch_x. Clearly one can replace $\mathfrak{g}^*\otimes\omega_{O_x}$ by $\mathfrak{g}^*\otimes O_x = \mathrm{Paths}(\mathfrak{g}^*)$ and Hitch_x by $\mathrm{Paths}(C)$ (for a scheme Y we denote by $\mathrm{Paths}(Y)$ the scheme of morphisms $\operatorname{Spec}O_x \to Y$). Regular elements of $\mathfrak{g}^*$ form an open subset $\mathfrak{g}^*_{\mathrm{reg}}$ such that $\mathrm{codim}(\mathfrak{g}^*\backslash\mathfrak{g}^*_{\mathrm{reg}}) > 1$. So one can replace $\mathrm{Paths}(\mathfrak{g}^*)$ by $\mathrm{Paths}(\mathfrak{g}^*_{\mathrm{reg}})$. Since the morphism $\mathfrak{g}^*_{\mathrm{reg}} \to C$ is smooth and surjective, and the action of G on its fibers is transitive, we are done.

2.4.2. According to 1.2.2 $\mathfrak{z}_x^{cl} = P_{(\mathfrak{g}\otimes K_x, G(O_x))}$ is a Poisson algebra. Actually the Poisson bracket on $\mathfrak{z}_x^{cl}$ is zero because the morphism $\overline{a}^{cl} : (\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x))^{\mathfrak{g}\otimes K_x} \longrightarrow \mathfrak{z}_x^{cl}$ from 1.2.7 is surjective (this follows, e.g., from the description of $\mathfrak{z}_x^{cl}$ given in 2.4.1) and $(\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x))^{\mathfrak{g}\otimes K_x}$ is the Poisson center of $\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x)$.

Remark (which may be skipped by the reader). Actually for *any* algebraic group G the natural morphism $\overline{a}^{cl} : (\overline{\mathrm{Sym}}(\mathfrak{g}\otimes K_x))^{G(K_x)} \to \mathfrak{z}_x^{cl} = \mathfrak{z}_{\mathfrak{g}}^{cl}(O_x)$ is surjective and therefore the Poisson bracket on $\mathfrak{z}_x^{cl}$ is zero. The following proof is the "classical limit" of Feigin-Frenkel's arguments from [FF92], p. 200–202. Identify O_x and K_x with $O := \mathbb{C}[[t]]$ and $K := \mathbb{C}[[t]]$. Let f be a $G(O)$-invariant regular function on $\mathfrak{g}^*\otimes O$. We have to extend it to a $G(K)$-invariant regular function $\tilde{f}$ on the ind-scheme $\mathfrak{g}^*\otimes K := \varinjlim \mathfrak{g}^*\otimes t^{-n}O$ (actually $\mathfrak{g}^*$ can be replaced by any finite dimensional G-module). For

$\varphi \in \mathfrak{g}^*((t))$ define $h_\varphi \in \mathbb{C}((\zeta))$ by

$$h_\varphi(\zeta) = f\left(\sum_{k=0}^{N} \varphi^{(k)}(\zeta) t^k/k!\right)$$

where N is big enough (h_φ is well-defined because there is an m such that f comes from a function on $\mathfrak{g}^* \otimes (O/t^m O)$). Write $h_\varphi(\zeta)$ as $\sum_n h_n(\varphi)\zeta^n$. The functions $h_n : \mathfrak{g}^* \otimes K \to \mathbb{C}$ are $G(K)$-invariant. Set $\tilde{f} := h_0$.

2.4.3. According to 2.3 and 1.2.6 we have the morphism

$$h_x^{cl} : \mathfrak{z}_x^{cl} \to \Gamma(\mathrm{Bun}_G, P).$$

analogous to the morphism h^{cl} from 1.2.3. To compare it with h_X^{cl} consider the closed embedding of affine schemes $\mathrm{Hitch}(X) \hookrightarrow \mathrm{Hitch}_x$ which assigns to a global section of C_{ω_X} its restriction to the formal neighbourhood of x. Let $\theta_x^{cl} : \mathfrak{z}_x^{cl} \twoheadrightarrow \mathfrak{z}^{cl}(X)$ be the corresponding surjective morphism of graded algebras. It is easy to see that

$$h_x^{cl} = h_X^{cl} \theta_x^{cl}.$$

Since the Poisson bracket on $\mathfrak{z}_x^{cl}$ is zero (see 2.4.2) and h_x^{cl} is a Poisson algebra morphism the Poisson bracket on $\mathrm{Im}\, h_x^{cl} = \mathrm{Im}\, h_X^{cl}$ is also zero. So we have proved 2.2.4(i).

2.5. **Quantization I.**

2.5.1. Let $\widetilde{\mathfrak{g} \otimes K_x}$ be the Kac-Moody central extension of $\mathfrak{g} \otimes K_x$ by $\mathbb{C}$ defined by the cocycle $(u, v) \mapsto \mathrm{Res}_x\, c(du, v)$, $u, v \in \mathfrak{g} \otimes K_x$, where

$$c(a, b) := -\frac{1}{2}\,\mathrm{Tr}(\mathrm{ad}_a \cdot \mathrm{ad}_b)\,, \quad a, b \in \mathfrak{g}\,. \tag{18}$$

As a vector space $\widetilde{\mathfrak{g} \otimes K_x}$ equals $\mathfrak{g} \otimes K_x \oplus \mathbb{C} \cdot \mathbf{1}$. We define the *adjoint action*[4] of $G(K_x)$ on $\widetilde{\mathfrak{g} \otimes K_x}$ by assigning to $g \in G(K_x)$ the following automorphism

[4] As soon as we have a central extension of $G(K_x)$ with Lie algebra $\widetilde{\mathfrak{g} \otimes K_x}$ the action (19) becomes the true adjoint action (an automorphism of $\widetilde{\mathfrak{g} \otimes K_x}$ that acts identically on $\mathbb{C} \cdot \mathbf{1}$ and $\mathfrak{g} \otimes K_x$ is identical because $\mathrm{Hom}(\mathfrak{g} \otimes K_x, \mathbb{C}) = 0$).

of $\widetilde{\mathfrak{g}\otimes K_x}$:

$$\mathbf{1}\mapsto\mathbf{1},\quad u\mapsto gug^{-1}+\operatorname{Res}_x c(u,g^{-1}dg)\cdot\mathbf{1}\text{ for }u\in\mathfrak{g}\otimes K_x \tag{19}$$

In particular we have the Harish-Chandra pair $\left(\widetilde{\mathfrak{g}\otimes K_x},G(O_x)\right)$, which is a central extension of $(\mathfrak{g}\otimes K_x,G(O_x))$ by $\mathbb{C}$. Set

$$\mathfrak{z}x=\mathfrak{z}_{\mathfrak{g}}(O_x):=D'_{(\mathfrak{g}\otimes K_x,G(O_x))}\,,$$

where D' has the same meaning as in 1.2.5.

2.5.2. *Theorem.* **(**[FF92]**).**

(i) *The algebra $\mathfrak{z}_x$ is commutative.*

(ii) *The pair $\left(\widetilde{\mathfrak{g}\otimes K_x},G(O_x)\right)$ satisfies the twisted local quantization condition (see 1.2.5). That is, the canonical morphism $\sigma_{\mathfrak{z}_x}:\operatorname{gr}\mathfrak{z}_x\to\mathfrak{z}_x^{cl}$ is an isomorphism.* □

Remark Statement (i) of the theorem is proved in [FF92] for any algebraic group G and any central extension of $\mathfrak{g}\otimes K_x$ defined by a symmetric invariant bilinear form on $\mathfrak{g}$. Moreover, it is proved in [FF92] that the $\pi_0(G(K_x))$-invariant part of the center of the completed twisted universal enveloping algebra $\overline{U}'(\mathfrak{g}\otimes K_x)$ maps *onto* $\mathfrak{z}_x$. A version of Feigin–Frenkel's proof of (i) will be given in 2.9.3–2.9.5. We have already explained the "classical limit" of their proof in the Remark at the end of 2.4.2.

2.5.3. The line bundle ω_{Bun_G} defines a $G(O_x)$-equivariant bundle on $\mathrm{Bun}_{G,\underline{x}}$. The $(\mathfrak{g}\otimes K_x,G(O_x))$-action on $\mathrm{Bun}_{G,\underline{x}}$ lifts canonically to a $\left(\widetilde{\mathfrak{g}\otimes K_x},G(O_x)\right)$-action on this line bundle, so that $\mathbf{1}$ acts as multiplication by 2. Indeed, according to 2.1.1 $\omega_{\mathrm{Bun}_G}=f^*(\det R\Gamma)$ where $f:\mathrm{Bun}_G\to\mathrm{Bun}_{SL(\mathfrak{g})}$ is induced by the adjoint representation $G\to SL(\mathfrak{g})$ and $\det R\Gamma$ is the determinant line bundle on $\mathrm{Bun}_{SL(\mathfrak{g})}$. On the other hand, it is well known (see,e.g., [BLa94]) that the pullback of $\det R\Gamma$ to $\mathrm{Bun}_{SL_n,\underline{x}}$ is equipped with the action of the Kac–Moody extension of $sl_n(K_x)$ of level -1.

Remark. In fact, the action of this extension integrates to an action of a certain central extension of $SL_n(K_x)$ (see, e.g., [BLa94]). Therefore one gets a canonical central extension

$$0 \to \mathbb{G}_m \to \widehat{G(K_x)} \to G(K_x) \to 0 \tag{20}$$

that acts on the pullback of ω_{Bun_G} to $\mathrm{Bun}_{G,\underline{x}}$ so that $\lambda \in \mathbb{G}_m$ acts as multiplication by λ. The extension $0 \to \mathbb{C} \to \widetilde{\mathfrak{g} \otimes K_x} \to \mathfrak{g} \otimes K_x \to 0$ is one half of the Lie algebra extension corresponding to (20). In Chapter 4 we will introduce a square root[5] of ω_{Bun_G} (the *Pfaffian* bundle) and a central extension

$$0 \to \mathbb{G}_m \to \widetilde{G(K_x)} \to G(K_x) \to 0 \tag{21}$$

(see 4.4.8), which is a square root of (20). These square roots are more important for us than ω_{Bun_G} and (20), so we will not give a precise definition of $\widetilde{G(K_x)}$.

2.5.4. According to 2.5.3 and 1.2.5 we have a canonical morphism of filtered algebras

$$h_x : \mathfrak{z}_x \to \Gamma\left(\mathrm{Bun}_G, D'\right).$$

In 2.7.5 we will prove the following theorem.

2.5.5. *Theorem. Our data satisfy the twisted global quantization condition (see 1.2.5).* □

As explained in 1.2.3 since the local and global quantization conditions are satisfied we obtain a solution $\mathfrak{z}^{(x)}(X)$ to the quantization problem from 2.2.5: set $\mathfrak{z}^{(x)}(X) = h_x(\mathfrak{z}_x)$ and equip $\mathfrak{z}^{(x)}(X)$ with the filtration induced from that on $\Gamma(\mathrm{Bun}_G, D')$ (2.5.5 means that it is also induced from the filtration on $\mathfrak{z}_x$); then the symbol map identifies $\mathrm{gr}\,\mathfrak{z}^{(x)}(X)$ with $h_x^{cl}(\mathfrak{z}_x^{cl})$ and according to 2.4.3 $h_x^{cl}(\mathfrak{z}_x^{cl}) = h_X^{cl}(\mathfrak{z}_x^{cl}(X)) \simeq \mathfrak{z}^{cl}(X)$.

[5]This square root and the extension (21) depend on the choice of a square root of ω_X.

The proof of Theorem 2.5.5 is based on the second construction of the solution to the quantization problem from 2.2.5; it also shows that $\mathfrak{z}^{(x)}(X)$ does not depend on x.

Remark If G is simply connected then 2.5.5 follows immediately from 2.2.4(v).

2.6. $\mathcal{D}_X$-scheme generalities.

2.6.1. Let X be any smooth connected algebraic variety. A $\mathcal{D}_X$-*scheme* is an X-scheme equipped with a flat connection along X. $\mathcal{D}_X$-schemes affine over X are spectra of commutative $\mathcal{D}_X$-*algebras* (= quasicoherent $\mathcal{O}_X$-algebras equipped with a flat connection). The fiber of an $\mathcal{O}_X$-algebra $\mathcal{A}$ at $x \in X$ is denoted by $\mathcal{A}_x$; in particular this applies to $\mathcal{D}_X$-algebras. For a $\mathbb{C}$-algebra C denote by C_X the corresponding "constant" $\mathcal{D}_X$-algebra (i.e., C_X is $C \otimes \mathcal{O}_X$ equipped with the obvious connection).

2.6.2. *Proposition.* Assume that X is complete.

(i) The functor $C \rightsquigarrow C_X$ admits a left adjoint functor: for a $\mathcal{D}_X$-algebra $\mathcal{A}$ there is a $\mathbb{C}$-algebra $H_\nabla(X, \mathcal{A})$ such that

$$\mathrm{Hom}(\mathcal{A}, C_X) = \mathrm{Hom}(H_\nabla(X, A), C) \tag{22}$$

for any $\mathbb{C}$-algebra C.

(ii) The canonical projection $\theta_{\mathcal{A}} : \mathcal{A} \to H_\nabla(X, \mathcal{A})_X$ is surjective. So $H_\nabla(X, \mathcal{A})_X$ is the maximal "constant" quotient $\mathcal{D}_X$-algebra of $\mathcal{A}$. In particular for any $x \in X$ the morphism $\theta_{\mathcal{A}_x} : \mathcal{A}_x \to (H_\nabla(X, \mathcal{A})_X)_x = H_\nabla(X, \mathcal{A})$ is surjective.

Remarks. (i) Here algebras are not supposed to be commutative, associative, etc. We will need the proposition for commutative $\mathcal{A}$.

(ii) Suppose that $\mathcal{A}$ is commutative (abbreviation for "commutative associative unital"). Then $H_\nabla(X, \mathcal{A})$ is commutative according to statement (ii) of the proposition. If C is also assumed commutative then (22) just means that $\mathrm{Spec}\, H_\nabla(X, \mathcal{A})$ *is the scheme of horizontal sections of* $\mathrm{Spec}\, \mathcal{A}$.

From the geometrical point of view it is clear that such a scheme exists and is affine: all the sections of $\operatorname{Spec}\mathcal{A}$ form an affine scheme S (here we use the completeness of X; otherwise S would be an ind-scheme, see the next Remark) and horizontal sections form a closed subscheme of S. The surjectivity of $\theta_{\mathcal{A}_x}$ and $\theta_{\mathcal{A}}$ means that the morphisms $\operatorname{Spec}H_\nabla(X,\mathcal{A}) \to \operatorname{Spec}\mathcal{A}_x$ and $X \times \operatorname{Spec}H_\nabla(X,\mathcal{A}) \to \operatorname{Spec}\mathcal{A}$ are closed embeddings.

(iii) If X is arbitrary (not necessary complete) then $H_\nabla(X,\mathcal{A})$ defined by (22) is representable by a projective limit of algebras with respect to a directed family of surjections. So if $\mathcal{A}$ is commutative then the space of horizontal sections of $\operatorname{Spec}\mathcal{A}$ is an ind-affine ind-scheme[6].

Proof. (a) Denote by $\mathcal{M}(X)$ the category of $\mathcal{D}_X$-modules and by $\mathcal{M}_{\mathrm{const}}(X)$ the full subcategory of constant $\mathcal{D}_X$-modules, i.e., $\mathcal{D}_X$-modules isomorphic to $V \otimes \mathcal{O}_X$ for some vector space V (actually the functor $V \mapsto V \otimes \mathcal{O}_X$ is an equivalence between the category of vector spaces and $\mathcal{M}_{\mathrm{const}}(X)$). We claim that the embedding $\mathcal{M}_{\mathrm{const}}(X) \to \mathcal{M}(X)$ has a left adjoint functor, i.e., for $\mathcal{F} \in \mathcal{M}(X)$ there is an $\mathcal{F}_\nabla \in \mathcal{M}_{\mathrm{const}}(X)$ such that $\operatorname{Hom}(\mathcal{F},\mathcal{E}) = \operatorname{Hom}(\mathcal{F}_\nabla,\mathcal{E})$ for $\mathcal{E} \in \mathcal{M}_{\mathrm{const}}(X)$. It is enough to construct $\mathcal{F}_\nabla$ for coherent $\mathcal{F}$. In this case $\mathcal{F}_\nabla := (\operatorname{Hom}_{\mathcal{D}_X}(\mathcal{F},\mathcal{O}_X))^* \otimes \mathcal{O}_X$ (here we use that $\dim \operatorname{Hom}_{\mathcal{D}_X}(\mathcal{F},\mathcal{O}_X) < \infty$ because X is complete).

(b) Since $\mathcal{O}_X$ is an irreducible $\mathcal{D}_X$-module a $\mathcal{D}_X$-submodule of a constant $\mathcal{D}_X$-module is constant. So the natural morphism $\mathcal{F} \to \mathcal{F}_\nabla$ is surjective.

(c) If $\mathcal{A}$ is a $\mathcal{D}_X$-algebra and $\mathcal{I}$ is the ideal of $\mathcal{A}$ generated by $\operatorname{Ker}(\mathcal{A} \to \mathcal{A}_\nabla)$ then $\mathcal{A}/\mathcal{I}$ is a quotient of the constant $\mathcal{D}_X$-module $\mathcal{A}_\nabla$. So $\mathcal{A}/\mathcal{I}$ is constant, i.e., $\mathcal{A}/\mathcal{I} = H_\nabla(X,\mathcal{A}) \otimes \mathcal{O}_X$ for some vector space $H_\nabla(X,\mathcal{A})$. $\mathcal{A}/\mathcal{I}$ is a $\mathcal{D}_X$-algebra, so $H_\nabla(X,\mathcal{A})$ is an algebra. Clearly it satisfies (22). □

[6]This is also clear from the geometric viewpoint. Indeed, horizontal sections form a closed subspace in the space S_X of all sections. If X is affine S_X is certainly an ind-scheme. In the general case X can be covered by open affine subschemes $U_1,...,U_n$; then S_X is a closed subspace of the product of S_{U_i}'s.

Remark. The geometrically oriented reader can consider the above Remark (ii) as a proof of the proposition for commutative algebras. However in 2.7.4 we will apply (22) in the situation where $\mathcal{A}$ is commutative while $C = \Gamma(\mathrm{Bun}_G, D')$ is not obviously commutative. Then it is enough to notice that the image of a morphism $\mathcal{A} \to C \otimes \mathcal{O}_X$ is of the form $C' \otimes \mathcal{O}_X$ (see part (b) of the proof of the proposition) and C' is commutative since $\mathcal{A}$ is. One can also apply (22) for $C :=$ the subalgebra of $\Gamma(\mathrm{Bun}_G, D')$ generated by the images of the morphisms $h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')$ for all $x \in X$ (this C is "obviously" commutative; see 2.9.1). Actually one can show that $\Gamma(\mathrm{Bun}_G, D')$ is commutative using 2.2.4(v) (it follows from 2.2.4(v) that for any connected component $\mathrm{Bun}_G^\gamma \subset \mathrm{Bun}_G$ and any $x \in X$ the morphism $\mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G^\gamma, D')$ induced by h_x is surjective).

2.6.3. In this subsection all algebras are assumed commutative. The forgetful functor $\{\mathcal{D}_X\text{-algebras}\} \to \{\mathcal{O}_X\text{-algebras}\}$ has an obvious left adjoint functor $\mathcal{J}$ ($\mathcal{J}\mathcal{A}$ is the $\mathcal{D}_X$-algebra generated by the $\mathcal{O}_X$-algebra $\mathcal{A}$). We claim that $\operatorname{Spec}\mathcal{J}\mathcal{A}$ is nothing but *the scheme of ∞-jets of sections of* $\operatorname{Spec}\mathcal{A}$. In particular this means that there is a canonical one-to-one correspondence between $\mathbb{C}$-points of $\operatorname{Spec}(\mathcal{J}\mathcal{A})_x$ and sections $\operatorname{Spec} O_x \to \operatorname{Spec}\mathcal{A}$ (where O_x is the formal completion of the local ring at x). More precisely, we have to construct a functorial bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \hat{\mathcal{B}}) \tag{23}$$

where $\mathcal{B}$ is a (quasicoherent) $\mathcal{O}_X$-algebra and $\hat{\mathcal{B}}$ is the completion of $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B}$ with respect to the ideal $\mathrm{Ker}(\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B} \to \mathcal{B})$. Here $\hat{\mathcal{B}}$ is equipped with the $\mathcal{O}_X$-algebra structure coming from the morphism $\mathcal{O}_X \to \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B}$ defined by $a \mapsto a \otimes 1$. Let us temporarily drop the quasicoherence assumption in the definition of $\mathcal{D}_X$-algebra. Then $\hat{\mathcal{B}}$ is a $\mathcal{D}_X$-algebra (the connection on $\hat{\mathcal{B}}$ comes from the connection on $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{B}$ such that sections of $1 \otimes \mathcal{B}$ are horizontal). So $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \hat{\mathcal{B}}) = \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{J}\mathcal{A}, \hat{\mathcal{B}})$ and to construct (23) it is

enough to construct a functorial bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{R},\mathcal{B}) \longleftarrow \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{R},\hat{\mathcal{B}}) \tag{24}$$

for any $\mathcal{D}_X$-algebra $\mathcal{R}$ and $\mathcal{O}_X$-algebra $\mathcal{B}$ (i.e., to show that the functor $\mathcal{B}\mapsto\hat{\mathcal{B}}$ is *right* adjoint to the forgetful functor $\{\mathcal{D}_X\text{-algebras}\}\to\{\mathcal{O}_X\text{-algebras}\}$). The mapping (24) comes from the obvious morphism $\hat{\mathcal{B}}\to\mathcal{B}$. The reader can easily prove that (24) is bijective.

For a $\mathcal{D}_X$-algebra $\mathcal{A}$ and a $\mathbb{C}$-algebra C we have

$$\mathrm{Hom}_{\mathcal{D}_X\text{-alg}}(\mathcal{J}\mathcal{A}, C\otimes\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, C\otimes\mathcal{O}_X)$$

This means that *the canonical morphism* $\operatorname{Spec}\mathcal{J}\mathcal{A}\to\operatorname{Spec}\mathcal{A}$ *identifies the ind-scheme of horizontal sections of* $\operatorname{Spec}\mathcal{J}\mathcal{A}$ *with that of (all) sections of* $\operatorname{Spec}\mathcal{A}$. If X is complete then, by 2.6.2, these spaces are actually schemes.

Finally let us mention that the results of this subsection can be globalized in the obvious way. The forgetful functor $\{\mathcal{D}_X\text{-schemes}\}\to\{X\text{-schemes}\}$ has a right adjoint functor $\mathcal{J}:\{X\text{-schemes}\}\to\{\mathcal{D}_X\text{-schemes}\}$. For an X-scheme Y, $\mathcal{J}Y$ is the scheme of ∞-jets of sections of Y. For an $\mathcal{O}_X$-algebra $\mathcal{A}$ we have $\mathcal{J}\operatorname{Spec}\mathcal{A}=\operatorname{Spec}\mathcal{J}\mathcal{A}$. The canonical morphism $\mathcal{J}Y\to Y$ identifies the space[7] of horizontal sections of $\mathcal{J}Y$ with the space of (all) sections of Y. If X is complete and Y is quasiprojective then our space is a scheme.

2.6.4. Let $(\mathfrak{l},P)$ be a Harish-Chandra pair in the sense of 1.2.6 (so P can be any affine group scheme; we do not assume that it is of finite type[8]).

Definition. An $(\mathfrak{l},P)$*-structure on* X is a morphism $\pi: X^\wedge\to X$ together with an action of $(\mathfrak{l},P)$ on $X^\wedge$ such that

(i) $X^\wedge$ is a P-torsor over X.

(ii) The action of $\mathfrak{l}$ on $X^\wedge$ is formally free and transitive, i.e., it yields an isomorphism $\mathfrak{l}\widehat{\otimes}\mathcal{O}_{X^\wedge}\rightleftarrows\Theta_{X^\wedge}$.

[7] In the most general situation "space" means "functor $\{\mathbb{C}\text{-algebras}\}\to\{\text{Sets}\}$".

[8] As follows from the definition below $\operatorname{Lie}P$ has finite codimension in $\mathfrak{l}$ (equal to $\dim X$).

Remark. Let L be the group ind-scheme with $\operatorname{Lie} L = \mathfrak{l}$, $L_{\mathrm{red}} = P$ (see 7.11.2(v)). Consider the homogenuous space $P \setminus L = \operatorname{Spf} O$ where $O = O_{(\mathfrak{l},P)} = (U\mathfrak{l}/(U\mathfrak{l})\mathfrak{p})^*$. Take $x \in X$ and choose $x^\wedge \in \pi^{-1}(x)$. The map $L \to X^\wedge$, $l \mapsto lx^\wedge$, yields a morphism $\alpha_{x^\wedge} : \operatorname{Spf} O \to X$, which identifies $\operatorname{Spf} O$ with the formal neighbourhood of x. For $l \in L$, $a \in \operatorname{Spf} O$ one has $\alpha_{lx^\wedge}(a) = \alpha_{x^\wedge}(al)$. Note that if the action of P on O is faithful then $x^\wedge$ is uniquely defined by $\alpha_{x^\wedge}$.

2.6.5. *Example.* Set $O = O_n := \mathbb{C}[[t_1, .., t_n]]$. The group of automorphisms of the $\mathbb{C}$-algebra O is naturally the group of $\mathbb{C}$-points of an affine group scheme $\operatorname{Aut}^0 O$ over $\mathbb{C}$. Denote by $\operatorname{Aut} O$ the group ind-scheme such that, for any $\mathbb{C}$-algebra R, $(\operatorname{Aut} O)(R)$ is the automorphism group of the topological R-algebra $R\widehat{\otimes}O = R[[t_1, .., t_n]]$. So $\operatorname{Aut}^0 O$ is the group subscheme of $\operatorname{Aut} O$; in fact, $\operatorname{Aut}^0 O = (\operatorname{Aut} O)_{\mathrm{red}}$. One has $\operatorname{Lie} \operatorname{Aut} O = \operatorname{Der} O$, $\operatorname{Lie} \operatorname{Aut}^0 O = \operatorname{Der}^0 O := \mathfrak{m}_O \cdot \operatorname{Der} O$. Therefore $\operatorname{Aut} O$ is the group ind-scheme that corresponds to the Harish-Chandra pair $\operatorname{Aut}^{HC} O := (\operatorname{Der} O, \operatorname{Aut}^0 O)$. By abuse of notation we will write $\operatorname{Aut} O$ instead of $\operatorname{Aut}^{HC} O$.

As explained by Gelfand and Kazhdan (see [GK], [GKF], and [BR]) any smooth variety X of dimension n carries a canonical[9] $\operatorname{Aut} O$-structure. The space $X^\wedge = X^\wedge_{can}$ is the space of "formal coordinate systems" on X. In other words, a $\mathbb{C}$-point of $X^\wedge$ is a morphism $\operatorname{Spec} O \to X$ with non-vanishing differential and an R-point of $X^\wedge$ is an R-morphism $\alpha : \operatorname{Spec}(R\widehat{\otimes}O) \to X \otimes R$ whose differential does not vanish over any point of $\operatorname{Spec} R$. The group ind-scheme $\operatorname{Aut} O$ acts on $X^\wedge$ in the obvious way, and we have the projection $\pi : X^\wedge \to X$, $\alpha \mapsto \alpha(0)$. It is easy to see that $X^\wedge$ (together with these structures) is an $\operatorname{Aut} O$-structure on X.

We will use the canonical $\operatorname{Aut} O_n$-structure in the case $n = 1$, i.e., when X is a curve, so $O = \mathbb{C}[[t]]$. Here the group $\operatorname{Aut} O$ looks as follows. There is an epimorphism $\operatorname{Aut}^0 O \to \operatorname{Aut}(tO/t^2O) = \mathbb{G}_m$, which

[9] In fact, an $\operatorname{Aut} O$-structure on X is unique up to unique isomorphism (this follows from the Remark in 2.6.4).

we call the *standard character* of $\mathrm{Aut}^0 O$; its kernel is pro-unipotent. For a $\mathbb{C}$-algebra R an automorphism of $R[[t]]$ is defined by $t \mapsto \sum_i c_i t^i$ where $c_1 \in R^*$ and c_0 is nilpotent. So $\mathrm{Aut}\, O$ is the union of schemes $\mathrm{Spec}\, \mathbb{C}[c_0, c_1, c_1^{-1}, c_2, c_3, \ldots]/(c_0^k)$, $k \in \mathbb{N}$. $\mathrm{Aut}^0 O$ is the group subscheme of $\mathrm{Aut}\, O$ defined by $c_0 = 0$.

Some other examples of $(\mathfrak{l}, P)$-structures may be found in **??**.

2.6.6. Let X be a variety equipped with an $(\mathfrak{l}, P)$-structure $X^\wedge$ (we will apply the constructions below in the situation where X is a curve, $\mathfrak{l} = \mathrm{Der}\, O$, $P = \mathrm{Aut}^0 O$ (or a certain covering of $\mathrm{Aut}^0 O$), $O := \mathbb{C}[[t]]$). Denote by $\mathcal{M}(X, \mathcal{O})$ the category of $\mathcal{O}$-modules on X, and by $\mathcal{M}^l(X)$ that of left $\mathcal{D}$-modules. For $F_X \in \mathcal{M}(X, \mathcal{O})$ its pull-back $F_{X^\wedge}$ to $X^\wedge$ is a P-equivariant $\mathcal{O}$-module on $X^\wedge$. If F_X is actually a left $\mathcal{D}_X$-module then $F_{X^\wedge}$ is in addition $\mathfrak{l}$-equivariant (since, by 2.6.4(ii), an $\mathfrak{l}$-action on an $\mathcal{O}_{X^\wedge}$-module is the same as a flat connection). The functors $\mathcal{M}(X, \mathcal{O}) \to \{P\text{-equivariant } \mathcal{O}\text{-modules on } X^\wedge\}$, $\mathcal{M}^l(X) \to \{(\mathfrak{l}, P)\text{-equivariant } \mathcal{O}\text{-modules on } X^\wedge\}$ are equivalences of tensor categories.

One has the faithful exact tensor functors

$$\mathcal{M}(P) \longrightarrow \mathcal{M}(X, \mathcal{O}), \quad \mathcal{M}(\mathfrak{l}, P) \longrightarrow \mathcal{M}^l(X) \tag{25}$$

which send a representation V to the $\mathcal{O}_X$- or $\mathcal{D}_X$-module V_X such that $V_{X^\wedge}$ equals to $V \otimes \mathcal{O}_{X^\wedge}$ (the tensor product of P- or $(\mathfrak{l}, P)$-modules). In other words, the $\mathcal{O}_X$-module V_X is the twist of V by the P-torsor $X^\wedge$. Therefore any algebra A with P-action yields an $\mathcal{O}_X$-algebra A_X; if A actually carries a $(\mathfrak{l}, P)$-action then A_X is a $\mathcal{D}_X$-algebra. Similarly, any scheme H with P-action (a P-scheme for short) yields an X-scheme H_X. If H is actually a $(\mathfrak{l}, P)$-scheme then H_X is a $\mathcal{D}_X$-scheme. One has $(\mathrm{Spec}\, A)_X = \mathrm{Spec}(A_X)$.

Remarks. (i) The functor $\mathcal{M}(\mathfrak{l}, P) \longrightarrow \mathcal{M}^l(X)$ coincides with the localization functor Δ for the $(\mathfrak{l}, P)$-action on $X^\wedge$ (see 1.2.4).

(ii) The functors (25) admit right adjoints which assign to an $\mathcal{O}_X$- or $\mathcal{D}_X$-module F_X the vector space $\Gamma(X^\wedge, F_{X^\wedge})$ equipped with the obvious P-

or $(\mathfrak{l}, P)$-module structure. Same adjointness holds if you consider algebras instead of modules.

(iii) Let C be a $\mathbb{C}$-algebra; consider C as an $(\mathfrak{l}, P)$-algebra with trivial $\operatorname{Aut} O$-action. Then C_X is the "constant" $\mathcal{D}_X$-algebra from 2.6.1.

2.6.7. The forgetful functor $\{(\mathfrak{l}, P)\text{-algebras}\} \to \{P\text{-algebras}\}$ admits a left adjoint (induction) functor $\mathcal{J}$. For a P-algebra A one has a canonical isomorphism

$$(\mathcal{J}A)_X = \mathcal{J}(A_X). \tag{26}$$

Indeed, the natural $\mathcal{O}_X$-algebra morphism $A_X \to (\mathcal{J}A)_X$ induces a $\mathcal{D}_X$-algebra morphism $\mathcal{J}(A_X) \to (\mathcal{J}A)_X$. To show that it is an isomorphism use the adjointness properties of $\mathcal{J}$ and $A \mapsto A_X$ (see 2.6.3 and Remark (ii) of 2.6.6).

Here is a geometric version of the above statements. The forgetful functor $\{(\mathfrak{l}, P)\text{-schemes}\} \to \{P\text{-schemes}\}$ admits a right adjoint functor[10] $\mathcal{J}$. For a P-algebra A one has $\mathcal{J}(\operatorname{Spec} A) = \operatorname{Spec} \mathcal{J}(A)$. For any P-scheme H one has $(\mathcal{J}H)_X = \mathcal{J}(H_X)$.

2.7. **Quantization II.** From now on $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$.

2.7.1. Consider first the "classical" picture. The schemes Hitch_x, $x \in X$, are fibers of the $\mathcal{D}_X$-scheme $\operatorname{Hitch} = \mathcal{J}C_{\omega_X}$ affine over X; denote by $\mathfrak{z}^{cl}$ the corresponding $\mathcal{D}_X$-algebra. By 2.6.3 the projection $\operatorname{Hitch} \to C_{\omega_X}$ identifies the scheme of horizontal sections of Hitch with $\operatorname{Hitch}(X)$. In other words

$$\mathfrak{z}^{cl}(X) = H_\nabla\Big(X, \mathfrak{z}^{cl}\Big),$$

and the projections $\theta_x^{cl} : \mathfrak{z}_x^{cl} \to \mathfrak{z}^{cl}(X)$ from 2.4.3 are just the canonical morphisms $\theta_{\mathfrak{z}_x^{cl}}$ from Proposition 2.6.2(ii).

[10]For affine schemes this is just a reformulation of the above statement for P-algebras. The general situation does not reduce immediately to the affine case (a P-scheme may not admit a covering by P-invariant affine subschemes), but the affine case is enough for our purposes.

Consider C as an $\mathrm{Aut}^0 O$-scheme via the standard character $\mathrm{Aut}^0 O \to \mathbb{G}_m$ (see 2.6.5). The X-scheme C_{ω_X} coincides with the $X^\wedge$-twist of C. Therefore the isomorphism (26) induces a canonical isomorphism

$$\mathfrak{z}^{cl} = \mathfrak{z}_{\mathfrak{g}}^{cl}(O)_X$$

where $\mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is the $\mathrm{Aut}\, O$-algebra $\mathcal{J}(\mathrm{Sym}\,\mathfrak{g})^G$, and the $\mathrm{Aut}^0 O$-action on $(\mathrm{Sym}\,\mathfrak{g})^G$ comes from the $\mathbb{G}_m$-action opposite to that induced by the grading of $(\mathrm{Sym}\,\mathfrak{g})^G$ (cf. the end of 2.2.2).

2.7.2. Let us pass to the "quantum" situation. Set $\mathfrak{z}_{\mathfrak{g}}(O) := D'_{(\mathfrak{g}\otimes K, G(O))}$. This is a commutative algebra (see 2.5.2(i)). $\mathrm{Aut}\, O$ acts on $\mathfrak{z}_{\mathfrak{g}}(O)$ since $\mathfrak{z}_{\mathfrak{g}}(O)$ is the endomorphism algebra of the twisted vacuum module Vac' (see 1.2.5) and $\mathrm{Aut}\, O$ acts on Vac'. (The latter action is characterized by two properties: it is compatible with the natural action of $\mathrm{Aut}\, O$ on $\widetilde{\mathfrak{g}\otimes K}$ and the vacuum vector is invariant; the action of $\mathrm{Aut}\, O$ on $\widetilde{\mathfrak{g}\otimes K}$ is understood in the topological sense, i.e., $\mathrm{Aut}(O\widehat{\otimes}R)$ acts on $\widetilde{\mathfrak{g}\otimes K}\widehat{\otimes}R$ for any commutative $\mathbb{C}$-algebra R.) Consider the $\mathcal{D}_X$-algebra

$$\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}} := \mathfrak{z}_{\mathfrak{g}}(O)_X$$

corresponding to the commutative $(\mathrm{Aut}\, O)$-algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ (see 2.6.5, 2.6.6). Its fibers are the algebras $\mathfrak{z}_x$ from 2.5.1. A standard argument shows that when $x \in X$ varies the morphisms h_x from 2.5.4 define a morphism of $\mathcal{O}_X$-algebras $h : \mathfrak{z} \to \Gamma(\mathrm{Bun}_G, D')_X$.

2.7.3. *Horizontality Theorem.* h is horizontal, i.e., it is a morphism of $\mathcal{D}_X$-algebras.

For a proof see 2.8.

2.7.4. Set

$$\mathfrak{z}(X) = \mathfrak{z}_{\mathfrak{g}}(X) := H_\nabla(X, \mathfrak{z})\,. \tag{27}$$

According to 2.6.2(i) the $\mathcal{D}_X$-algebra morphism h induces a $\mathbb{C}$-algebra morphism

$$h_X : \mathfrak{z}(X) \to \Gamma(\mathrm{Bun}_G, D')$$

We are going to show that $(\mathfrak{z}(X), h_X)$ is a solution to the quantization problem from 2.2.5. Before doing this we have to define the filtration on $\mathfrak{z}(X)$ and the isomorphism $\sigma_{\mathfrak{z}(X)} : \operatorname{gr}\mathfrak{z}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}(X)$.

The canonical filtration on $\mathfrak{z}_{\mathfrak{g}}(O)$ is $\operatorname{Aut} O$-invariant and the isomorphism $\sigma_{\mathfrak{z}(O)} : \operatorname{gr}\mathfrak{z}_{\mathfrak{g}}(O) \rightrightarrows \mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ (see 2.5.2(ii)) is compatible with $\operatorname{Aut} O$-actions. Therefore $\mathfrak{z}$ carries a horizontal filtration and we have the isomorphism of $\mathcal{D}_X$-algebras

$$\sigma_{\mathfrak{z}} : \operatorname{gr}\mathfrak{z} \rightrightarrows \mathfrak{z}^{cl}$$

which reduces to the isomorphism $\sigma_{\mathfrak{z}_x}$ from 2.5.2(ii) at each fiber. The image of this filtration by $\theta_{\mathfrak{z}} : \mathfrak{z} \twoheadrightarrow H_\nabla(X, \mathfrak{z})_X = \mathfrak{z}(X)_X$ is a horizontal filtration on $\mathfrak{z}(X)_X$ which is the same as a filtration on $\mathfrak{z}(X)$. Consider the surjective morphism of graded $\mathcal{D}_X$-algebras $(\operatorname{gr}\theta_{\mathfrak{z}})\sigma_{\mathfrak{z}}^{-1} : \mathfrak{z}^{cl} \twoheadrightarrow \operatorname{gr}\mathfrak{z}(X)_X$. By adjunction (see (22)) it defines the surjective morphism of graded $\mathbb{C}$-algebras $j : \mathfrak{z}^{cl}(X) = H_\nabla\left(X, \mathfrak{z}^{cl}\right) \twoheadrightarrow \operatorname{gr}\mathfrak{z}(X)$.

Note that h_X is compatible with filtrations, and we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{z}^{cl}(X) & \overset{h^{cl}_X}{\hookrightarrow} & \Gamma(\mathrm{Bun}_G, P) \\ j \downarrow & & \downarrow \sigma_{\mathrm{Bun}_G} \\ \operatorname{gr}\mathfrak{z}(X) & \xrightarrow{\operatorname{gr} h_X} & \operatorname{gr}\Gamma(\mathrm{Bun}_G, D') \end{array} \tag{28}$$

Therefore j is an isomorphism and $\operatorname{gr} h_X$ (hence h_X) is injective. Define $\sigma_{\mathfrak{z}(X)} : \operatorname{gr}\mathfrak{z}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}(X)$ by $\sigma_{\mathfrak{z}(X)} := j^{-1}$. The triple $(\mathfrak{z}(X), h_X, \sigma_{\mathfrak{z}(X)})$ is a solution to the quantization problem from 2.2.5.

2.7.5. Let us prove Theorem 2.5.5 and compare $\mathfrak{z}^{(x)}(X)$ from 2.5.5 with $\mathfrak{z}(X)$. Clearly $h_x = h_X \cdot \theta_{\mathfrak{z}_x}$ where $\theta_{\mathfrak{z}_x} : \mathfrak{z}_x \to \mathfrak{z}(X)$ was defined in Proposition 2.6.2(ii). $\theta_{\mathfrak{z}_x}$ is surjective (see 2.6.2(ii)) and strictly compatible

with filtrations (see the definition of the filtration on $\mathfrak{z}(X)$ in 2.7.4). h_X is injective and strictly compatible with filtrations (see the end of 2.7.4). So h_x is strictly compatible with filtrations (which is precisely Theorem 2.5.5) and h_X induces an isomorphism between the filtered algebras $\mathfrak{z}(X)$ and $\mathfrak{z}^{(x)}(X) := h_x(\mathfrak{z}_x)$.

2.8. **Horizontality.** In this subsection we introduce $\mathcal{D}_X$-structure on some natural moduli schemes and prove the horizontality theorem 2.7.3 modulo certain details explained in 4.4.14. The reader may skip this subsection for the moment.

In 2.8.1–2.8.2 we sketch a proof of Theorem 2.7.3. The method of 2.8.2 is slightly modified in 2.8.3. In 2.8.4–2.8.5 we explain some details and refer to 4.4.14 for the rest of them. In 2.8.6 we consider very briefly the ramified situation.

2.8.1. Let us construct the morphism h from Theorem 2.7.3.

Recall that the construction of h_x from 2.5.3–2.5.4 involves the scheme $\mathrm{Bun}_{G,\underline{x}}$, i.e., the moduli scheme of G-bundles on X trivialized over the formal neighbourhood of x. It also involves the action of the Harish-Chandra pair $(\mathfrak{g}\otimes K_x, G(O_x))$ on $\mathrm{Bun}_{G,\underline{x}}$ and its lifting to the action of $(\widetilde{\mathfrak{g}\otimes K_x}, G(O_x))$ on the line bundle $\pi_x^*\omega_{\mathrm{Bun}_G}$ where π_x is the natural morphism $\mathrm{Bun}_{G,\underline{x}} \to \mathrm{Bun}_G$. These actions come from the action of the group ind-scheme $G(K_x)$ on $\mathrm{Bun}_{G,\underline{x}}$ and its lifting to the action of a certain central extension[11] $\widehat{G(K_x)}$ on $\pi_x^*\omega_{\mathrm{Bun}_G}$.

To construct h one has to organize the above objects depending on x into families. One defines in the obvious way a scheme M over X whose fiber over x equals $\mathrm{Bun}_{G,\underline{x}}$. One defines a group scheme $J(G)$ over X and a group ind-scheme $J^{\mathrm{mer}}(G)$ over X whose fibers over x are respectively $G(O_x)$ and $G(K_x)$. $J(G)$ is the scheme of jets of functions $X \to G$ and $J^{\mathrm{mer}}(G)$ is the ind-scheme of "meromorphic jets". $J^{\mathrm{mer}}(G)$ acts on M. Finally one

[11] This extension was mentioned (rather than defined) in the Remark from 2.5.3. This is enough for the sketch we are giving.

defines a central extension $\widehat{J}^{\mathrm{mer}}(G)$ and its action on $\pi^*\omega_{\mathrm{Bun}_G}$ where π is the natural morphism $M \to \mathrm{Bun}_G$. These data being defined the construction of $h : \mathfrak{z} \to \Gamma(\mathrm{Bun}_G, D')_X$ is quite similar to that of h_x (see 2.5.3–2.5.4).

2.8.2. The crucial observation is that *there are canonical connections along X on $J(G)$, $J^{\mathrm{mer}}(G)$, $\widehat{J}^{\mathrm{mer}}(G)$, M and $\pi^*\omega_{\mathrm{Bun}_G}$ such that the action of $J^{\mathrm{mer}}(G)$ on M and the action of $\widehat{J}^{\mathrm{mer}}(G)$ on $\pi^*\omega_{\mathrm{Bun}_G}$ are horizontal.* This implies the horizontality of h.

For an X-scheme Y we denote by $\mathcal{J}Y$ the scheme of jets of sections $X \to Y$. It is well known (and more or less explained in 2.6.3) that $\mathcal{J}Y$ has a canonical connection along X (i.e., $\mathcal{J}Y$ is a $\mathcal{D}_X$-scheme in the sense of 2.6.1). In particular this applies to $J(G) = \mathcal{J}(G \times X)$. If $\mathcal{F}$ is a principal G-bundle over X then the fiber of $\pi : M \to \mathrm{Bun}_G$ over $\mathcal{F}$ equals $\mathcal{J}\mathcal{F}$, so it is equipped with a connection along X. One can show that these connections come from a connection along X on M.

To define the connection on M as well as the other connections it is convenient to use Grothendieck's approach [Gr68]. According to [Gr68] a connection (=integrable connection = "stratification") along X on an X-scheme Z is a collection of bijections $\varphi_{\alpha\beta} : \mathrm{Mor}_\alpha(S, Z) \xrightarrow{\sim} \mathrm{Mor}_\beta(S, Z)$ for every scheme S and every pair of infinitely close "points" $\alpha, \beta : S \to X$ (here $\mathrm{Mor}_\alpha(S, Z)$ is the preimage of α in $\mathrm{Mor}(S, Z)$ and "infinitely close" means that the restrictions of α and β to S_{red} coincide); the bijections $\varphi_{\alpha\beta}$ are required to be functorial with respect to S and to satisfy the equation $\varphi_{\beta\gamma}\varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$.

For instance, if Z is the jet scheme of a scheme Y over X then $\mathrm{Mor}_\alpha(S, Z) := \mathrm{Mor}_X(S'_\alpha, Y)$ where S'_α is the formal neighbourhood of the graph $\Gamma_\alpha \subset S \times X$ and the morphism $S'_\alpha \to X$ is induced by the projection $\mathrm{pr}_X : S \times X \to X$. It is easy to show that if α and β are infinitely close then $S'_\alpha = S'_\beta$, so we obtain a connection along X on Z. One can show that it coincides with the connection defined in 2.6.3.

The connections along X on $J^{\mathrm{mer}}(G)$, $\widehat{J}^{\mathrm{mer}}(G)$, and M are defined in the similar way. The horizontality of the action of $J^{\mathrm{mer}}(G)$ on M and the action of $\widehat{J}^{\mathrm{mer}}(G)$ on $\pi^*\omega_{\mathrm{Bun}_G}$ easily follows from the definitions.

2.8.3. The method described in 2.8.2 can be modified as follows. Recall that $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$; $\operatorname{Aut} O$ and $X^\wedge$ were defined in 2.6.5. Set $M^\wedge = M \times_X X^\wedge$. So $M^\wedge$ is the moduli space of quadruples $(x, t_x, \mathcal{F}, \gamma_x)$ where $x \in X$, t_x is a formal parameter at x, $\mathcal{F}$ is a G-torsor on X, γ_x is a section of $\mathcal{F}$ over the formal neighbourhood of x. The group ind-scheme $G(K)$ acts on the fiber of $M^\wedge$ over any $\widehat{x} \in X^\wedge$ (indeed, this fiber coincides with $\mathrm{Bun}_{G,\underline{x}}$ where x is the image of $\widehat{x}$ in X, so $G(K_x)$ acts on the fiber; on the other hand the formal parameter at x corresponding to $\widehat{x}$ defines an isomorphism $K_x \xrightarrow{\sim} K$). Actually $G(K)$ acts on $M^\wedge$ (see 2.8.4) and the central extension $\widehat{G(K)}$ acts on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ where $\widehat{\pi}$ is the natural morphism $M^\wedge \to \mathrm{Bun}_G$. This action induces a morphism $\widehat{h} : \mathfrak{z}_{\mathfrak{g}}(O) \to \Gamma(X^\wedge, \mathcal{O}_{X^\wedge}) \otimes \Gamma(\mathrm{Bun}_G, D')$ (see 2.7.2 for the definition of $\mathfrak{z}_{\mathfrak{g}}(O)$).

On the other hand the action of $\operatorname{Aut} O$ on $X^\wedge$ from 2.6.5 lifts canonically to its action on $M^\wedge$ (see 2.8.4) and the sheaf $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$. The actions of $\operatorname{Aut} O$ and $\widehat{G(K)}$ on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ are compatible in the obvious sense. Therefore $\widehat{h}$ is $\operatorname{Aut} O$-equivariant. So $\widehat{h}$ induces a horizontal morphism $h : \mathfrak{z} = \mathfrak{z}_{\mathfrak{g}}(O)_X \to \Gamma(\mathrm{Bun}_G, D')_X$.

2.8.4. To turn the sketch from 2.8.3 into a proof of Theorem 2.7.3 we first of all give a precise definition of the action of the semidirect product $\operatorname{Aut} O \ltimes G(K)$ on $M^\wedge$. Let R be a $\mathbb{C}$-algebra. By definition, an R-point of $M^\wedge$ is a triple $(\alpha, \mathcal{F}, \gamma)$ where $\alpha : \operatorname{Spec} R\widehat{\otimes}O \to X \otimes R$ is an R-morphism whose differential does not vanish over any point of $\operatorname{Spec} R$, $\mathcal{F}$ is a G-torsor on $X \otimes R$, and γ is a section of $\alpha^*\mathcal{F}$. Let Γ_α denote the graph of the composition $\operatorname{Spec} R \to \operatorname{Spec} R\widehat{\otimes}O \xrightarrow{\alpha} X \otimes R$ and α' the morphism $\operatorname{Spec} R\widehat{\otimes}K \to (X \otimes R) \setminus \Gamma_\alpha$ induced by α. According to

Beauville and Laszlo[12] (see 2.3.7 and 2.3.4) R-points of $M^\wedge$ are in one-to-one correspondence with triples $(\alpha, \mathcal{F}', \gamma')$ where α is as above, $\mathcal{F}'$ is a G-torsor on $(X \otimes R) \setminus \Gamma_\alpha$, and γ' is a section of $\alpha'^*\mathcal{F}'$ (of course, $\mathcal{F}'$ is the restriction of $\mathcal{F}$, γ' is the restriction of γ). This interpretation shows that $G(R\widehat{\otimes}K)$ and $\mathrm{Aut}(R\widehat{\otimes}O)$ act on $M^\wedge(R)$: the action of $G(R\widehat{\otimes}K)$ changes γ' and the action of $\mathrm{Aut}(R\widehat{\otimes}O)$ changes α (if α is replaced by $\alpha\circ\varphi$, $\varphi \in \mathrm{Aut}\,\mathrm{Spec}\, R\widehat{\otimes}O$, then Γ_α changes as a subscheme of $X \otimes R$ but not as a subset, so $(X \otimes R) \setminus \Gamma_\alpha$ remains unchanged). Thus we obtain the action of $\mathrm{Aut}\, O \ltimes G(K)$ on $M^\wedge$ mentioned in 2.8.3.

2.8.5. According to 2.8.4 $\mathrm{Aut}\, O$ acts on $M^\wedge$ considered as a scheme over Bun_G. So $\mathrm{Aut}\, O$ acts on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$. In 2.5.3 we mentioned the canonical action of $\widehat{G(K_x)}$ on the pullback of ω_{Bun_G} to $\mathrm{Bun}_{G,\underline{x}}$. So $\widehat{G(K)}$ acts on the restriction of $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$ to the fiber of $M^\wedge$ over any $\widehat{x} \in X^\wedge$. As explained in 2.8.3, to finish the proof of 2.7.3 it suffices to show that

(i) the actions of $\widehat{G(K)}$ corresponding to various $\widehat{x} \in X^\wedge$ come from an (obviously unique) action of $\widehat{G(K)}$ on $\widehat{\pi}^*\omega_{\mathrm{Bun}_G}$,

(ii) this action is compatible with that of $\mathrm{Aut}\, O$.

To prove (i) and (ii) it is necessary (and almost sufficient) to define the central extension $\widehat{G(K_x)}$ and its action on the pullback of ω_{Bun_G} to $\mathrm{Bun}_{G,\underline{x}}$. The interested reader can do it using, e.g., [BLa94].

Instead of proving (i) and (ii) we will prove in 4.4.14 a similar statement for a square root of ω_{Bun_G} (because we need the square roots of ω_{Bun_G} to formulate and prove Theorem 5.4.5, which is the main result of this work). More precisely, for any square root $\mathcal{L}$ of ω_X one defines a line bundle $\lambda'_{\mathcal{L}}$ on Bun_G, which is essentially a square root of ω_{Bun_G} (see 4.4.1). One constructs a central extension[13] $\widetilde{G(K_x)}_{\mathcal{L}}$ acting on the pullback of $\lambda'_{\mathcal{L}}$ to $\mathrm{Bun}_{G,\underline{x}}$ (see 4.4.7 – 4.4.8). The morphism $h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')$ from 2.5.4 can be

[12]The normal bundle of $\Gamma_\alpha \subset X \otimes R$ is trivial, so according to 2.3.7 one can apply the main theorem of [BLa95] rather than its globalized version.

[13]In fact, this extension is a square root of $\widehat{G(K_x)}$.

naturally defined using this action (see 4.4.12 – 4.4.13). Finally, in 4.4.14 we prove the analog of the above statements (i) and (ii) for $\lambda'_{\mathcal{L}}$, which implies the horizontality theorem 2.7.3.

2.8.6. Let $\Delta \subset X$ be a finite subscheme. Denote by $\mathrm{Bun}_{G,\Delta}$ the stack of G-bundles on X trivialized over Δ. Denote by D' the sheaf $D_{\mathcal{Y},\mathcal{L}^\lambda}$ from 1.1.6 for $\mathcal{Y} = \mathrm{Bun}_{G,\Delta}$, $\mathcal{L}$ = the pullback of ω_{Bun_G}, $\lambda = 1/2$. Just as in the case $\Delta = \emptyset$ one defines a horizontal morphism $h : \mathfrak{z}_{X\backslash\Delta} \to \Gamma(\mathrm{Bun}_{G,\Delta}, D') \otimes \mathcal{O}_{X\backslash\Delta}$ where $\mathfrak{z}_{X\backslash\Delta}$ is the restriction of $\mathfrak{z}$ to $X\backslash\Delta$. h induces an injection $\Gamma(N, \mathcal{O}_N) \to \Gamma(\mathrm{Bun}_{G,\Delta}, D')$ where $N = N_\Delta(G)$ is a closed subscheme of the ind-scheme $N'_\Delta(G)$ of horizontal sections of $\mathrm{Spec}\, \mathfrak{z}_{X\backslash\Delta}$.

Problem. Describe $N_\Delta(G)$ explicitly.

We are going to indicate the geometric objects used in the solution of the problem. Since we do not explain the details of the solution one can read the rest of this subsection without knowing the answer to the problem, which can be found in 3.8.2.

For $n \in \mathbb{Z}_+$ denote by $M_{\Delta,n}$ the stack of triples consisting of a point $x \in X$, a G-bundle $\mathcal{F}$ on X, and a trivialization of $\mathcal{F}$ over $\Delta + nx$ (here we identify finite subshemes of X with effective divisors, so $\Delta + nx$ makes sense). $M_{\Delta,n}$ is an algebraic stack and $M_\Delta := \varprojlim_n M_{\Delta,n}$ is a scheme over X.

Remark. Let $M_{\Delta,x}$ be the fiber of M_Δ over $x \in X$. If $x \in X\backslash\Delta$ then $M_{\Delta,x}$ is the moduli scheme of G-bundles trivialized over Δ and the formal neighbourhood of x. If $x \in \Delta$ then $M_{\Delta,x} = M_{\Delta\backslash\{x\},x}$.

Consider the "congruence subgroup" scheme $\underline{G}_\Delta$ defined as follows: $\underline{G}_\Delta$ is a scheme flat over X such that for any scheme S flat over X

$$\mathrm{Mor}_X(S, \underline{G}_\Delta) = \{f : S \to G \text{ such that } f|_{\Delta_S} = 1\}$$

where Δ_S is the preimage of Δ in S. $\underline{G}_\Delta$ is a group scheme over X. A G-bundle on X trivialized over Δ is the same as a $\underline{G}_\Delta$-bundle (this becomes

clear if G-bundles and $\underline{G}_\Delta$-bundles are considered as torsors for the étale topology). So $\mathrm{Bun}_{G,\Delta}$ is the stack of $\underline{G}_\Delta$-bundles.

One can show that if $D \subset X$ is a finite subscheme and $\Delta + D$ is understood in the sense of divisors then for every scheme S flat over X

$$\mathrm{Mor}_X(S, \underline{G}_{\Delta+D}) = \{f \in \mathrm{Mor}_X(S, \underline{G}_\Delta) \text{ such that } f|_{D_S} = 1\}$$

Therefore a G-bundle on X trivialized over $\Delta + D$ is the same as a $\underline{G}_\Delta$-bundle trivialized over D. So M_Δ is the moduli scheme of triples consisting of a point $x \in X$, a $\underline{G}_\Delta$-bundle on X, and its trivialization over the formal neighbourhood of x. Now one can easily define a canonical action of $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ on M_Δ where $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ is the group ind-scheme of "meromorphic jets" of sections $X \to \underline{G}_\Delta$. $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ and M_Δ are equipped with connections along X and the above action is horizontal. And so on...

Remarks

(i) If $\Delta \neq \emptyset$ the method of 2.8.3 does not allow to avoid using group ind-schemes over X.

(ii) There are pitfalls connected with infinite dimensional schemes and ind-schemes like M_Δ or $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$. Here is an example. The morphism $\underline{G}_\Delta \to \underline{G} := \underline{G}_\emptyset = G \times X$ induces $f : \mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta) \to \mathcal{J}^{\mathrm{mer}}(\underline{G})$. This f induces an isomorphism of the fibers over any point $x \in X$ (the fiber of $\mathcal{J}^{\mathrm{mer}}(\underline{G}_\Delta)$ over x is $G(K_x)$, it does not depend on Δ). But if $\Delta \neq \emptyset$ then f is not an isomorphism, nor even a monomorphism.

2.9. **Commutativity of $\mathfrak{z}_\mathfrak{g}(O)$.** The algebras $\mathfrak{z}_\mathfrak{g}(O)$ and $\mathfrak{z}_x = \mathfrak{z}_\mathfrak{g}(O_x)$ were defined in 2.5.1 and 2.7.2 (of course they are isomorphic). Feigin and Frenkel proved in [FF92] that $\mathfrak{z}_\mathfrak{g}(O)$ is commutative. In this subsection we give two proofs of the commutativity of $\mathfrak{z}_\mathfrak{g}(O)$: the global one (see 2.9.1–2.9.2) and the local one (see 2.9.3–2.9.5). The latter is in fact a version of the original proof from [FF92].

The reader may skip this subsection for the moment. We will not use 2.9.1–2.9.2 in the rest of the paper.

2.9.1. Let us prove that

$$[h_x(\mathfrak{z}_x), h_y(\mathfrak{z}_y)] = 0 \tag{29}$$

(see 2.5.4 for the definition of $h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')$). Since $\mathfrak{z}_x$ is the fiber at x of the $\mathcal{O}_X$-algebra $\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}}(O)_X$ and h_x comes from the $\mathcal{O}_X$-algebra morphism $h : \mathfrak{z} \to \mathcal{O}_X \otimes \Gamma(\mathrm{Bun}_G, D')$ it is enough to prove (29) for $x \neq y$. Denote by $\mathrm{Bun}_{G,\underline{x},\underline{y}}$ the moduli scheme of G-bundles on X trivialized over the formal neighbourhoods of x and y. $G(K_x) \times G(K_y)$ acts on $\mathrm{Bun}_{G,\underline{x},\underline{y}}$. In particular the Harish-Chandra pair $((\mathfrak{g} \otimes K_x) \times (\mathfrak{g} \otimes K_y), G(O_x) \times G(O_y))$ acts on $\mathrm{Bun}_{G,\underline{x},\underline{y}}$. This action lifts canonically to an action of $((\widetilde{\mathfrak{g} \otimes K_x}) \times (\widetilde{\mathfrak{g} \otimes K_y}), G(O_x) \times G(O_y))$ on the pullback of ω_{Bun_G} to $\mathrm{Bun}_{G,\underline{x},\underline{y}}$ such that $\mathbf{1}_x \in \mathfrak{g} \otimes K_x$ and $\mathbf{1}_y \in \mathfrak{g} \otimes K_y$ act as multiplication by 2 and $G(O_x) \times G(O_y)$ acts in the obvious way. The action of $G(O_x) \times G(O_y)$ on $\mathrm{Bun}_{G,\underline{x},\underline{y}}$ satisfies condition (16) from 1.2.6 and the quotient stack equals Bun_G. So according to 1.2.5 we have a canonical morphism $h_{x,y} : \mathfrak{z}_x \otimes \mathfrak{z}_y \to \Gamma(\mathrm{Bun}_G, D')$. Its restrictions to $\mathfrak{z}_x$ and $\mathfrak{z}_y$ are equal to h_x and h_y. So (29) is obvious.

2.9.2. Let us prove the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$. Suppose that $a \in [\mathfrak{z}_{\mathfrak{g}}(O), \mathfrak{z}_{\mathfrak{g}}(O)]$, $a \neq 0$. If $x = y$ then (29) means that $h_x(\mathfrak{z}_x)$ is commutative. So for any X, $x \in X$, and $f : O \xrightarrow{\sim} O_x$ one has $h_x(f_*(a)) = 0$. Let $\bar{a} \in \mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ be the principal symbol of a. Then for any X, x, f as above one has $h^{cl}_x(f_*(\bar{a})) = 0$ (see 2.4.3 for the definition and geometric description of $h^{cl}_x : \mathfrak{z}^{cl}_x \to \Gamma(\mathrm{Bun}_G, P) = \Gamma(T^*\mathrm{Bun}_G, \mathcal{O})$. This means that $\bar{a}$ considered as a polynomial function on $\mathfrak{g}^* \otimes \omega_O$ (see 2.4.1) has the following property: for any X, x as above, any G-bundle $\mathcal{F}$ on X trivialized over the formal neighbourhood of x, and any isomorphism $O_x \xrightarrow{\sim} O$ the restriction of $\bar{a}$ to the image of the map $H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X) \to \mathfrak{g}^* \otimes \omega_{O_x} \xrightarrow{\sim} \mathfrak{g}^* \otimes \omega_O$ is zero. There is an n such that $\bar{a}$ comes from a function on $\mathfrak{g}^* \otimes (\omega_O/\mathfrak{m}^n\omega_O)$ where $\mathfrak{m}$ is the maximal ideal of O. Choose X and x so that the mapping $H^0(X, \omega_X) \to \omega_{O_x}/\mathfrak{m}^n_x\omega_{O_x}$ is surjective and let $\mathcal{F}$ be the trivial bundle. Then

the map $H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X) \to \mathfrak{g}^* \otimes (\omega_{O_x}/\mathfrak{m}_x^n \omega_{O_x})$ is surjective and therefore $\bar{a} = 0$, i.e., a contradiction.

Remark. Let $\mathrm{Bun}_{G,\Delta}$ be the stack of G-bundles on X trivialized over a finite subscheme $\Delta \subset X$. To deduce from (29) the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$ one can use the natural homomorphism from $\mathfrak{z}_x$, $x \notin \Delta$, to the ring of twisted differential operators on $\mathrm{Bun}_{G,\Delta}$. Then instead of choosing (X, x) as in the above proof one can fix (X, x) and take Δ big enough.

2.9.3. Denote by $\mathfrak{Z}$ the center of the completed twisted universal enveloping algebra $\overline{U}'(\mathfrak{g} \otimes K)$, $K := \mathbb{C}((t)) \supset \mathbb{C}[[t]] = O$. In [FF92] Feigin and Frenkel deduce the commutativity of $\mathfrak{z}_{\mathfrak{g}}(O)$ from the surjectivity of the natural homomorphism $f : \mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O)$. We will present a proof of the surjectivity of f which can be considered as a geometric version of the one from [FF92] and also as a "quantization" of the remark at the end of 2.4.2. The relation with [FF92] and 2.4.2 will be explained in 2.9.7 and 2.9.8.

Remark. In the definition of the central extension of $\mathfrak{g} \otimes K$ (see 2.5.1) and therefore in the definition of $\mathfrak{Z}$ and $\mathfrak{z}_{\mathfrak{g}}(O)$ we used the "critical" bilinear form c defined by (18). In the proof of the surjectivity of f one can assume that c is *any* invariant symmetric bilinear form on $\mathfrak{g}$ and $\mathfrak{g}$ is *any* finite dimensional Lie algebra. On the other hand it is known that if $\mathfrak{g}$ is simple and c is non-critical then the corresponding algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ is trivial (see ???).

2.9.4. We need the interpretation of $\overline{U}' := \overline{U}'(\mathfrak{g} \otimes K)$ from [BD94]. Denote by U' the non-completed twisted universal enveloping algebra of $\mathfrak{g} \otimes K$. For $n \geq 0$ let I_n be the left ideal of U' generated by $\mathfrak{g} \otimes \mathfrak{m}^n \subset \mathfrak{g} \otimes O \subset U'$. By definition, $\overline{U}' := \varprojlim_n U'/I_n$. Let U'_k be the standard filtration of U' and $\overline{U}'_k$ the closure of U'_k in $\overline{U}'$, i.e., $\overline{U}'_k := \varprojlim_n U'_k/I_{n,k}$, $I_{n,k} := I_n \cap U'_k$. The main theorem of [BD94] identifies the dual space $(U'_k/I_{n,k})^*$ with a certain topological vector space $\Omega_{n,k}$. So

$$U'_k/I_{n,k} = (\Omega_{n,k})^*\,, \qquad \overline{U}'_k = (\Omega_k)^* \tag{30}$$

where $\Omega_k = \varinjlim_n \Omega_{n,k}$ and $*$ denotes the topological dual.

To define $\Omega_{n,k}$ we need some notation. Denote by O_r (resp. ω_r^O) the completed tensor product of r copies of O (resp. of ω_O). Set $\omega_r^K = \omega_r^O \bigotimes_{O_r} K_r$ where K_r is the field of fractions of O_r. We identify O_r with $\mathbb{C}[[t_1, \dots, t_r]]$ and write elements of ω_r^K as $f(t_1, \dots, t_r)\, dt_1 \dots dt_r$ where f belongs to the field of fractions of $\mathbb{C}[[t_1, \dots, t_r]]$.

Definition. $\Omega_{n,k}$ is the set of $(k+1)$-tuples $(w_0, \dots, w_k)$, $w_r \in (\mathfrak{g}^*)^{\otimes r} \otimes \omega_r^K$, such that

1) w_r is invariant with respect to the action of the symmetric group S_r (S_r acts both on $(\mathfrak{g}^*)^{\otimes r}$ and ω_r^K);
2) w_r has poles of order $\leq n$ at the hyperplanes $t_i = 0$, $1 \leq i \leq r$, poles of order ≤ 2 at the hyperplanes $t_i = t_j$, $1 \leq i < j \leq r$, and no other poles;
3) if $w_r = f_r(t_1, \dots, t_r)\, dt_1 \dots dt_r$, $r \geq 2$, then

$$\begin{aligned} f_r(t_1 \dots, t_r) \;&=\; \frac{f_{r-2}(t_1, \dots, t_{r-2}) \otimes c}{(t_{r-1} - t_r)^2} \\ &+\; \frac{\varphi^*(f_{r-1}(t_1, \dots, t_{r-1}))}{t_{r-1} - t_r} + \dots \end{aligned} \tag{31}$$

Here $c \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ is the bilinear form used in the definition of the central extension of $\mathfrak{g} \otimes K$, $\varphi^* : (\mathfrak{g}^*)^{\otimes(r-1)} \to (\mathfrak{g}^*)^{\otimes r}$ is dual to the mapping $\varphi : \mathfrak{g}^{\otimes r} \to \mathfrak{g}^{\otimes(r-1)}$ given by $\varphi(a_1 \otimes \dots \otimes a_r) = a_1 \otimes \dots \otimes a_{r-2} \otimes [a_{r-1}, a_r]$ and the dots in (31) denote an expression which does not have a pole at the generic point of the hyperplane $t_{r-1} = t_r$.

The topology on $\Omega_{n,k}$ is induced by the embedding $\Omega_{n,k} \hookrightarrow \prod_{0 \leq r \leq k} (\mathfrak{g}^*)^{\otimes r} \otimes \Omega_r^O$ given by $(w_0, \dots, w_k) \mapsto (\eta_0, \dots, \eta_k)$, $\eta_r = \prod_i t_i^n \cdot \prod_{i<j} (t_i - t_j)^2 \cdot w_r$.

Let us explain that in (31) we consider f_r as a function with values in $(\mathfrak{g}^*)^{\otimes r}$.

We will not need the explicit formula from [BD94] for the isomorphism (30). Let us only mention that according to Proposition 5 from [BD94] the

adjoint action of $\mathfrak{g}\otimes K$ on $\overline{U}'_k$ induces via (30) the following action of $\mathfrak{g}\otimes K$ on Ω_k : $a\in\mathfrak{g}\otimes K$ sends $(w_0,\dots,w_k)\in\Omega_k$ to $(0,w''_1,\dots,w''_k)$ where

$$\begin{aligned} w''_r &= \frac{1}{(r-1)\,!}\ \mathrm{Sym}\ w'_r\,,\\ w'_r(t_1,\dots,t_r) &:= (\mathrm{id}\otimes\dots\otimes\mathrm{id}\otimes\mathrm{ad}_{a(t_r)})w_r(t_1,\dots,t_r)\\ &- \ w_{r-1}(t_1,\dots,t_{r-1})\otimes c\cdot da(t_r)\,. \end{aligned}$$

Here Sym denotes the symmetrization operator (without the factor $1/r!$), $\mathrm{ad}_{a(t_r)}:\mathfrak{g}^*\to\mathfrak{g}^*$ is the operator corresponding to $a(t_r)$ in the coadjoint representation, and $c:\mathfrak{g}\to\mathfrak{g}^*$ is the bilinear form of $\mathfrak{g}$.

Remark. Suppose that $c=0$ and $\mathfrak{g}$ is commutative. Then $U'_k/I_{n,k}=\bigoplus_{r=0}^k\mathrm{Sym}^r(\mathfrak{g}\otimes K/\mathfrak{m}^n)$ and $\Omega_{n,k}=\bigoplus_{r=0}^k\overline{\mathrm{Sym}}^r(\mathfrak{g}^*\otimes\mathfrak{m}^{-n}\omega_O)$ where $\overline{\mathrm{Sym}}^r$ denotes the completed symmetric power. The isomorphism $U'_k/I_{n,k}\xrightarrow{\sim}(\Omega_{n,k})^*$ is the identification of $\mathrm{Sym}(\mathfrak{g}\otimes K/\mathfrak{m}^n)$ with the space of polynomial functions on $\mathfrak{g}^*\otimes\mathfrak{m}^{-n}\omega_O$ used in 2.4.1 and 2.4.2.

2.9.5. According to 2.9.4 to prove the surjectivity of $f:\mathfrak{Z}\to\mathfrak{z}_{\mathfrak{g}}(O)$ it is enough to show that any $(\mathfrak{g}\otimes O)$-invariant continuous linear functional $l:\Omega_{0,k}\to\mathbb{C}$ can be extended to a $(\mathfrak{g}\otimes K)$-invariant continuous linear functional $\Omega_k\to\mathbb{C}$. Consider the continuous linear operator

$$T:\Omega_k\to\mathbb{C}((\zeta))\widehat{\otimes}\Omega_{0,k}=\Big\{\sum_{n=-\infty}^{\infty}a_n\zeta^n|a_n\in\Omega_{0,k}\,,\ a_n\to 0\text{ for }n\to-\infty\Big\}$$

defined by

$$T(w_0,\dots,w_k)=(\widehat{w}_0,\dots,\widehat{w}_k)\,,\quad \widehat{w}_r=w_r(\zeta+t_1,\dots,\zeta+t_r) \tag{32}$$

where $w_r(\zeta+t_1,\dots,\zeta+t_r)$ is considered as an element of

$$\Delta^{-1}\mathbb{C}((\zeta))[[t_1,\dots,t_n]]\,dt_1\dots dt_n=\mathbb{C}((\zeta))\widehat{\otimes}\Delta^{-1}\mathbb{C}[[t_1,\dots,t_n]]\,dt_1\dots dt_n\,,$$

$$\Delta:=\prod_{1\le i<j\le r}(t_i-t_j)^2\,.$$

If $l\in(\Omega_{0,k})^*$ let $\bar{l}:\Omega_k\to\mathbb{C}((\zeta))$ be the composition of $T:\Omega_k\to\mathbb{C}((\zeta))\widehat{\otimes}\Omega_{0,k}$ and $\mathrm{id}\otimes l:\mathbb{C}((\zeta))\widehat{\otimes}\Omega_{0,k}\to\mathbb{C}((\zeta))$. Write $\bar{l}$ as $\sum_i l_i\zeta^i$, $l_i\in(\Omega_k)^*$.

If l is $\mathfrak{g}\otimes O$-invariant then the functionals l_i are $\mathfrak{g}\otimes K$-invariant. Besides $l_0|_{\Omega_{0,k}} = l$.

Remark. Let G be an algebraic group such that $\operatorname{Lie} G = \mathfrak{g}$. Then $G(K)$ acts on our central extension of $\mathfrak{g}\otimes K$ (see (19)), so it acts on $\overline{U}'_k$; moreover, $G(O)$ acts on $U'_k/I_{n,k}$. Therefore $G(K)$ acts on Ω_k and $G(O)$ acts on $\Omega_{n,k}$. In the above situation if l is $G(O)$-invariant then the functionals l_i are $G(K)$-invariant (see formula (24) from [BD94] for the action of $G(K)$ on Ω_k). Notice that if G is connected $G(K)$ is not necessarily connected, so $G(K)$-invariance does not follow immediately from $(\mathfrak{g}\otimes K)$-invariance.

2.9.6. Since $\bar{l}$ is continuous $l_i \to 0$ for $i \to -\infty$ (i.e., for every n we have $l_{-i}(\Omega_{n,k}) = 0$ if i is big enough). So the map $l \mapsto \bar{l}$ can be considered as a map from $U'_k/I_{0,k}$ to $W_k := \{\sum_{i=-\infty}^{\infty} a_i\zeta^i | a_i \in \overline{U}'_k\,, a_i \to 0 \text{ for } i \to -\infty\}$. These maps define an operator

$$\Phi:\ Vac' \to W := \bigcup_k W_k \tag{33}$$

where $Vac' = U'/I_0$ is the twisted vacuum module. As explained in 2.9.5, Φ induces a map

$$\mathfrak{z}_{\mathfrak{g}}(O) \to \mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta)) := \{\sum_{i=-\infty}^{\infty} a_i\zeta^i | a_i \in \mathfrak{Z}\,, a_i \to 0 \text{ for } i \to -\infty\}\,. \tag{34}$$

One can prove that (34) is a ring homomorphism (see ???). It is easy to see that the composition of (34) and the projection $\mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta)) \to \mathfrak{z}_{\mathfrak{g}}(O)((\zeta))$ maps $\mathfrak{z}_{\mathfrak{g}}(O)$ to $\mathfrak{z}_{\mathfrak{g}}(O)[[\zeta]]$ and the composition $\mathfrak{z}_{\mathfrak{g}}(O) \to \mathfrak{z}_{\mathfrak{g}}(O)[[\zeta]] \xrightarrow{\zeta=0} \mathfrak{z}_{\mathfrak{g}}(O)$ is the identity.

Remark. Let G be a connected algebraic group such that $\operatorname{Lie} G = \mathfrak{g}$. Then all elements of the image of (34) are $G(K)$-invariant (see the remark from 2.9.5).

2.9.7. One can show that (33) coincides with the operator $F:\ Vac' \to W$ constructed by Feigin and Frenkel (see the proof of Lemma 1 from [FF92]) and therefore 2.9.5 is just a version of a part of [FF92].

The definition of F from [FF92] can be reformulated as follows. Set $W_k^+ := \overline{U}'_k((\zeta))$, $W_k^- := \{\sum_i a_i\zeta^i \in W_k | a_{-i} = 0 \text{ for } i \text{ big enough}\}$. Define $W^\pm \subset W$ by $W^\pm = \bigcup_k W_k^\pm$. W^+ and W^- have natural algebra structures and W has a natural structure of (W^+, W^-)-bimodule (W is a left W^+-module and a right W^--module). Consider the linear maps $\varphi^\pm : \widetilde{\mathfrak{g} \otimes K} \to W^\pm$ such that

$$\varphi^+(\mathbf{1}) = 1\,, \qquad \varphi^-(\mathbf{1}) = 0$$

and for $a \in \mathfrak{g}((t)) = \mathfrak{g} \otimes K \subset \widetilde{\mathfrak{g} \otimes K}$

$$\varphi^+(a) = a(t-\zeta) \in \mathfrak{g}((t))((\zeta))\,, \qquad \varphi^-(a) = a(t-\zeta) \in \mathfrak{g}((\zeta))((t))\,.$$

It is easy to show that $\varphi^\pm$ are Lie algebra homomorphisms. Consider the $\widetilde{\mathfrak{g} \otimes K}$-module structure on W defined by $a \circ w := \varphi_+(a)w - w\varphi_-(a)$, $a \in \widetilde{\mathfrak{g} \otimes K}$, $w \in W$. Then $F : Vac' \to W$ is the $\widetilde{\mathfrak{g} \otimes K}$-module homomorphism that maps the vacuum vector from Vac' to $1 \in W$.

2.9.8. Let us explain the relation between (34) and its classical analog from 2.4.2.

$\overline{U}'$ is equipped with the standard filtration $\overline{U}'_k$ (see 2.9.4). It induces the filtration $\mathfrak{Z}_k := \mathfrak{Z} \cap \overline{U}_k$. We identify $\operatorname{gr}_k \overline{U}' := \overline{U}'_k / \overline{U}'_{k-1}$ with the completion of $\operatorname{Sym}^k(\mathfrak{g} \otimes K)$, i.e., the space of homogeneous polynomial functions $\mathfrak{g}^* \otimes \omega_K \to \mathbb{C}$ of degree k where $\omega_K := \omega_O \otimes_O K$ (a function f on $\mathfrak{g}^* \otimes \omega_K$ is said to be polynomial if for every n its restriction to $\mathfrak{g}^* \otimes \mathfrak{m}^{-n}$ is polynomial, i.e., comes from a polynomial function on $\mathfrak{g}^* \otimes (\mathfrak{m}^{-n}/\mathfrak{m}^N)$ for some N depending on n). Denote by $\mathfrak{Z}^{cl}$ the algebra of $\mathfrak{g} \otimes K$-invariant polynomial functions on $\mathfrak{g}^* \otimes \omega_K$. Clearly the image of $\operatorname{gr} \mathfrak{Z}$ in $\operatorname{gr} \overline{U}'$ is contained in $\mathfrak{Z}^{cl}$.

The filtration of $\mathfrak{Z}$ induces a filtration of $\mathfrak{Z} \widehat{\otimes} \mathbb{C}((\zeta))$ and the map (34) is compatible with the filtrations. We claim that the following diagram is

commutative:

$$\begin{array}{ccc} \mathrm{gr}_k\,\mathfrak{z}_{\mathfrak{g}}(O) & \longrightarrow & \mathrm{gr}_k\,\mathfrak{Z}\widehat{\otimes}\mathbb{C}((\zeta)) \\ \sigma\big\downarrow & & \big\downarrow \\ \mathfrak{z}^{cl}_{\mathfrak{g}}(O) & \xrightarrow{\nu} & \mathfrak{Z}^{cl}\widehat{\otimes}\mathbb{C}((\zeta)) \end{array} \tag{35}$$

Here the upper arrow is induced by (34), $\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ was defined in 2.4.1, σ is the symbol map from 1.2.5, and ν is defined by

(36)

$$\nu(f) := h(\zeta)\,, \quad (h(\zeta))(\varphi) := f(\varphi(\zeta+t))$$
$$f \in \mathfrak{z}^{cl}_{\mathfrak{g}}(O)\,,\ \varphi \in \mathfrak{g}^* \otimes \omega_K\,,\ \varphi(\zeta+t) \in \mathfrak{g}^*((\zeta))[[t]]dt = (\mathfrak{g}^* \otimes \omega_O)\widehat{\otimes}\mathbb{C}((\zeta))\,.$$

Here $\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ is identified with the algebra of $\mathfrak{g} \otimes O$-invariant polynomial functions on $\mathfrak{g}^* \otimes \omega_O$ (cf.2.4.1). The map ν was considered in the Remark from 2.4.2.

The commutativity of (35) follows from the commutativity of the diagram

$$\begin{array}{ccc} (U'_k/I_{n,k})^* & \xrightarrow{\sim} & \Omega_{n,k} \\ \sigma^*\big\uparrow & & \big\uparrow \\ (\mathrm{Sym}^k(\mathfrak{g} \otimes K/\mathfrak{g} \otimes \mathfrak{m}^n))^* & \xrightarrow{\sim} & ((\mathfrak{m}^{-n}\omega_O)^{\otimes k})^{S_k} \end{array} \tag{37}$$

Here the upper arrow is dual to (30), $\sigma : U'_k/I_{n,k} \to \mathrm{Sym}^k(\mathfrak{g} \otimes K/\mathfrak{g} \otimes \mathfrak{m}^n)$ is the symbol map, and the right vertical arrow is defined by $w \mapsto (0, \ldots, 0, w)$. The commutativity of (37) is an immediate consequence of the definition of (30); see [BD94].

2.10. **Geometry of $T^*\mathbf{Bun}_G$.** This subsection should be considered as an appendix; the reader may certainly skip it.

Set $\mathrm{Nilp} = \mathrm{Nilp}(G) := p^{-1}(0)$ where $p : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X)$ is the Hitchin fibration (see 2.2.3). Nilp was introduced in [La87] and [La88] under the name of *global nilpotent cone* (if $\mathcal{F}$ is a G-bundle on X and $\eta \in T^*_{\mathcal{F}}\mathrm{Bun}_G = H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X)$ then $(\mathcal{F}, \eta) \in \mathrm{Nilp}$ if and only if the image of η in $H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \omega_X)$ is nilpotent).

In 2.10.1 we show that Proposition 2.2.4 (iii) easily follows from the equality

$$\dim \mathrm{Nilp} = \dim \mathrm{Bun}_G \,. \tag{38}$$

We also deduce from (38) that Bun_G is good in the sense of 1.1.1. The equality (38) was proved by Faltings and Ginzburg; in the particular case $G = PSL_n$ it had been proved by Laumon. In 2.10.2 we give some comments on their proofs. In 2.10.3 we discuss the set of irreducible components of Nilp. In 2.10.4 we show that Nilp is equidimensional even if the genus of X equals 0 or 1 (if $g > 1$ this follows from 2.2.4 (iii)). In 2.10.5 we prove that Bun_G is very good in the sense of 1.1.1.

We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using an invariant scalar product on $\mathfrak{g}$.

2.10.1. Assuming (38) we are going to prove 2.2.4 (iii) and show that Bun_G is good in the sense of 1.1.1. Let $U \subset T^*\mathrm{Bun}_G$ be the biggest open substack such that $\dim U \leq 2 \dim \mathrm{Bun}_G$. (38) means that the fiber of $p : T^*\mathrm{Bun}_G \to \mathrm{Hitch}(X)$ over 0 has dimension $\dim \mathrm{Bun}_G$. Since $\dim \mathrm{Hitch}(X) = \dim \mathrm{Bun}_G$ this implies that $U \supset p^{-1}(0)$. U is invariant with respect to the natural action of $\mathbb{G}_m$ on $T^*\mathrm{Bun}_G$. Therefore $U = T^*\mathrm{Bun}_G$. So $\dim T^*\mathrm{Bun}_G \leq 2 \dim \mathrm{Bun}_G$. According to 1.1.1 this means that Bun_G is good and $T^*\mathrm{Bun}_G$ is a locally complete intersection of pure dimension $2 \dim \mathrm{Bun}_G$.

For an open $V \subset T^*\mathrm{Bun}_G$ the following properties are equivalent: 1) the restriction of p to V is flat, 2) the fibers of this restriction have dimension $\dim \mathrm{Bun}_G$. Let $V_{\max}$ be the maximal V with these properties. $V_{\max}$ is $\mathbb{G}_m$-invariant and according to (38) $V_{\max} \supset p^{-1}(0)$. So $V_{\max} = T^*\mathrm{Bun}_G$ and we have proved the first statement of 2.2.4 (iii). It implies that the image of p^γ is open. On the other hand it is $\mathbb{G}_m$-invariant and contains 0. So p^γ is surjective. QED.

Since Nilp contains the zero section of $T^*\mathrm{Bun}_G$ (38) follows from the inequality $\dim \mathrm{Nilp} \leq \dim \mathrm{Bun}_G$, which was obtained in [La88], [Fal93], [Gi97] as a corollary of the following theorem.

2.10.2. *Theorem.* **([La88], [Fal93], [Gi97]).** Nilp is isotropic.

Remarks

(i) Let us explain that a subscheme N of a smooth symplectic variety M is said to be *isotropic* if any smooth subvariety of N is isotropic. One can show that N is isotropic if and only if the set of nonsingular points of N_{red} is isotropic. N is said to be *Lagrangian* if it is isotropic and $\dim_x N = \frac{1}{2}\dim_x M$ for all $x \in N$. If $\mathcal{Y}$ is a smooth algebraic stack then a substack $\mathcal{N} \subset T^*\mathcal{Y}$ is said to be isotropic (resp. Lagrangian) if $\mathcal{N} \times_{\mathcal{Y}} S \subset (T^*\mathcal{Y}) \times_{\mathcal{Y}} S \subset T^*S$ is isotropic (resp. Lagrangian) for some presentation[14]

$S \to \mathcal{Y}$ (then it is true for all presentations $S \to \mathcal{Y}$).

(ii) The proofs of Theorem 2.10.2 given in [Fal93] and [Gi97] do not use the assumption $g > 1$ where g is the genus of X. If $g > 1$ then Faltings and Ginzburg show that Nilp is Lagrangian. Their argument was explained in 2.10.1: (38) implies that Nilp has *pure* dimension $\dim \mathrm{Bun}_G$. In 2.10.1 we used the equality $\dim \mathrm{Hitch}(X) = \dim \mathrm{Bun}_G$, which holds only if $g > 1$. In fact Nilp is Lagrangian even if $g = 0, 1$ (see 2.10.4).

(iii) Since $\mathrm{Nilp} \subset T^*\mathrm{Bun}_G$ is Lagrangian and $\mathbb{G}_m$-invariant it is a union of conormal bundles to certain reduced irreducible closed substacks of Bun_G. For $G = PSL_n$ a description of some of these substacks was obtained by Laumon (see §§3.8–3.9 from [La88]).

(iv) Ginzburg's proof of Theorem 2.10.2 is based on the following interpretation of Nilp in terms of $\pi : \mathrm{Bun}_B \to \mathrm{Bun}_G$ where B is a Borel subgroup of G: if $\mathcal{F} \in \mathrm{Bun}_G$, $\eta \in T^*_{\mathcal{F}}\mathrm{Bun}_G$ then $(\mathcal{F}, \eta) \in \mathrm{Nilp}$ if and only if

[14] A presentation of $\mathcal{Y}$ is a smooth surjective morphism $S \to \mathcal{Y}$ where S is a scheme.

there is an $\mathcal{E} \in \pi^{-1}(\mathcal{F})$ such that the image of η in $T^*_{\mathcal{E}} \operatorname{Bun}_B$ equals 0. This interpretation enables Ginzburg to prove Theorem 2.10.2 using a simple and general argument from symplectic geometry (see §§6.5 from [Gi97]). Falting's proof of Theorem 2.10.2 is also very nice and short (see the first two paragraphs of the proof of Theorem II.5 from [Fal93]).

(v) The proof of Theorem 2.10.2 for $G = PSL_n$ given in [La88] does not work in the general case because it uses the following property of $\mathfrak{g} = sl_n$: for every nilpotent $A \in \mathfrak{g}$ there is a parabolic subgroup $P \subset G$ such that A belongs to the Lie algebra of the unipotent radical $U \subset P$, the P-orbit of A is open in $\operatorname{Lie} U$, and the centralizer of A in G is contained in P. This property holds for $\mathfrak{g} = sl_n$ (e.g., one can take for P the stabilizer of the flag $0 \subset \operatorname{Ker} A \subset \operatorname{Ker} A^2 \subset \dots$) but not for an arbitrary semisimple $\mathfrak{g}$ (e.g., it does not hold if $\mathfrak{g} = sp_4$ and $A \in sp_4$ is a nilpotent operator of rank 1).

2.10.3. In this subsection we "describe" the set of irreducible components of Nilp.

Recall that Nilp is the stack of pairs $(\mathcal{F}, \eta)$ where $\mathcal{F}$ is a G-bundle on X and $\eta \in H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \omega_X) = H^0(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \omega_X)$ is nilpotent. For a nilpotent conjugacy class $C \subset \mathfrak{g}$ we have the locally closed substack Nilp_C parametrizing pairs $(\mathcal{F}, \eta)$ such that $\eta(x) \in C$ for generic $x \in X$.

Fix some $e \in C$ and include it into an sl_2-triple $\{e, f, h\}$. Let $\mathfrak{g}^k$ be the decreasing filtration of $\mathfrak{g}$ such that $[h, \mathfrak{g}^k] \subset \mathfrak{g}^k$ and ad_h acts on $\mathfrak{g}^k/\mathfrak{g}^{k+1}$ as multiplication by k. $\mathfrak{g}^k$ depend on e but not on h and f. Set $\mathfrak{p} = \mathfrak{p}_e := \mathfrak{g}^0$. $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$. Let $P \subset G$ be the corresponding subgroup. We have the map $C \to G/P$ that associates to $a \in C$ the parabolic subalgebra $\mathfrak{p}_a$. Its fiber $\{a \in C | \mathfrak{p}_a = \mathfrak{p}\}$ (i.e., the P-orbit of $e \in C$) equals $\mathfrak{g}^2 \cap C$; this is an open subset of $\mathfrak{g}^2$. An element of $\mathfrak{g}^2$ is said to be *generic* if it belongs to $\mathfrak{g}^2 \cap C$.

Let $(\mathcal{F},\eta)\in \mathrm{Nilp}_C$, $U:=\{x\in X|\eta(x)\in C\}$. The image of $\eta\in\Gamma(U,C_{\mathcal{F}}\otimes\omega_X)$ in $\Gamma(U,(G/P)_{\mathcal{F}})$ extends to a section of $(G/P)_{\mathcal{F}}$ over X. So we obtain a P-structure on $\mathcal{F}$. In terms of this P-structure $\eta\in H^0(X,\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X)$ and $\eta(x)$ is generic for $x\in U$.

Denote by Y_C the stack of pairs $(\mathcal{F},\eta)$ where $\mathcal{F}$ is a P-bundle on X and $\eta\in H^0(X,\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X)$ is such that $\eta(x)$ is generic for almost all $x\in X$. For a P-bundle $\mathcal{F}$ let $\deg\mathcal{F}\in\mathrm{Hom}(P,\mathbb{G}_m)^*$ be the functional that associates to $\varphi: P\to\mathbb{G}_m$ the degree of the push-forward of $\mathcal{F}$ by φ. Y_C is the disjoint union of open substacks Y_C^u, $u\in\mathrm{Hom}(P,\mathbb{G}_m)^*$, parametrizing pairs $(\mathcal{F},\eta)\in Y_C$ such that $\deg\mathcal{F}=u$. It is easy to show that for each $u\in\mathrm{Hom}(P,\mathbb{G}_m)^*$ the natural morphism $Y_C^u\to\mathrm{Nilp}_C$ is a locally closed embedding and the substacks $Y_C^u\subset\mathrm{Nilp}_C$ form a stratification of Nilp_C.

Lemma.

1) Y_C^u is a smooth equidimensional stack. $\dim Y_C^u\leq\dim\mathrm{Bun}_G$.
2) Let Y_C^* be the union of connected components of Y_C of dimension $\dim\mathrm{Bun}_G$. Then Y_C^* is the stack of pairs $(\mathcal{F},\eta)\in Y_C$ such that $\mathrm{ad}_\eta:(\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{F}}\to(\mathfrak{g}^1/\mathfrak{g}^2)_{\mathcal{F}}\otimes\omega_X$ is an isomorphism.

Remark. (38) follows from the lemma.

Proof. The deformation theory of $(\mathcal{F},\eta)\in Y_C^u$ is controled by the hypercohomology of the complex $C^\cdot$ where $C^0=\mathfrak{p}_{\mathcal{F}}=\mathfrak{g}^0_{\mathcal{F}}$, $C^1=\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X$, $C^i=0$ for $i\neq 0,1$, and the differential $d: C^0\to C^1$ equals ad_η. Since $\mathrm{Coker}\,d$ has finite support $\mathbb{H}^2(X,C^\cdot)=0$. So Y_C is smooth and

$$\begin{aligned}\dim_{(\mathcal{F},\eta)}Y_C&=\chi(\mathfrak{g}^2_{\mathcal{F}}\otimes\omega_X)-\chi(\mathfrak{g}^0_{\mathcal{F}})=-\chi(\mathfrak{g}_{\mathcal{F}}/\mathfrak{g}^{-1}_{\mathcal{F}})-\chi(\mathfrak{g}^0_{\mathcal{F}})\\&=-\chi(\mathfrak{g}_{\mathcal{F}})+\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})=\dim\mathrm{Bun}_G+\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})\,.\end{aligned}$$

Clearly $\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})$ depends only on $u=\deg\mathcal{F}$. The morphism $\mathrm{ad}_\eta:\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}}\to(\mathfrak{g}^1/\mathfrak{g}^2)_{\mathcal{F}}\otimes\omega_X$ is injective and its cokernel $\mathcal{A}$ has finite support. So $2\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})=\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})-\chi((\mathfrak{g}^1/\mathfrak{g}^2)_{\mathcal{F}}\otimes\omega_X)=-\chi(\mathcal{A})\leq 0$ and $\chi(\mathfrak{g}^{-1}_{\mathcal{F}}/\mathfrak{g}^0_{\mathcal{F}})=0$ if and only if $\mathcal{A}=0$. □

Since Nilp has pure dimension $\dim \mathrm{Bun}_G$ the lemma implies that the irreducible components of Nilp are parametrized by $\bigsqcup_C \pi_0(Y_C^*)$.

$\pi_0(Y_C^*)$ can be identified with π_0 of a simpler stack M_C defined as follows. Set $L = P/U$ where U is the unipotent radical of P. L acts on $V := \mathfrak{g}^2/\mathfrak{g}^3$. Denote by D_i the set of $a \in V$ such that the determinant of $(\mathrm{ad}_a)^i : \mathfrak{g}^{-i}/\mathfrak{g}^{-i+1} \to \mathfrak{g}^i/\mathfrak{g}^{i+1}$ equals 0. $D_i \subset V$ is an L-invariant closed subset of pure codimension 1. An element of $\mathfrak{g}^2$ is generic if and only if its image in V does not belong to D_2. Therefore $D_i \subset D_2$ for all i. Denote by M_C the stack of pairs $(\mathcal{F}, \eta)$ where $\mathcal{F}$ is an L-bundle on X and $\eta \in H^0(X, V_{\mathcal{F}} \otimes \omega_X)$ is such that $\eta(x) \notin D_1$ for all $x \in X$ and $\eta(x) \notin D_2$ for generic $x \in X$. It is easy to see that the natural morphism $Y_C^* \to M_C$ induces a bijection $\pi_0(Y_C^*) \to \pi_0(M_C)$.

So irreducible components of Nilp are parametrized by $\bigsqcup_C \pi_0(M_C)$. Hopefully $\pi_0(M_C)$ can be described in terms of "standard" objects associated to C and $X \ldots$

Remark. If $G = PSL_n$ then Nilp_C has pure dimension $\dim \mathrm{Bun}_G$ for every nilpotent conjugacy class $C \subset sl_n$ (see [La88]). This is not true, e.g., if $G = Sp_4$ and C is the set of nilpotent matrices from sp_4 of rank 1. Indeed, let $(\mathcal{F}, \eta) \in Y_C$ be such that $\eta \in H^0(X, \mathfrak{g}^2_{\mathcal{F}} \otimes \omega_X)$ has only simple zeros. Then it is easy to show that the morphism $Y_C \to \mathrm{Nilp}_C$ is an open embedding in a neighbourhood of $(\mathcal{F}, \eta)$. On the other hand it follows from the above lemma that if η has a zero then the dimension of Y_C at $(\mathcal{F}, \eta)$ is less than Bun_G.

2.10.4. *Theorem.* Nilp is Lagrangian.

In this theorem we do not assume that $g > 1$.

Proof. As explained in Remark (ii) from 2.10.2 we only have to show that Nilp has pure dimension $\dim \mathrm{Bun}_G$ for $g \leq 1$.

1) Let $g = 0$. Then $\mathrm{Nilp} = T^*\mathrm{Bun}_G$. A quasicompact open substack of Bun_G can be represented as $H\backslash M$ where M is a smooth variety and H is an

algebraic group acting on M. Then $T^*(H\backslash M) = H\backslash N$ where $N \subset T^*M$ is the union of the conormal bundles of the orbits of H. Each conormal bundle has pure dimension $\dim M$ and since $g = 0$ the number of H-orbits is finite.

Remark. Essentially the same argument shows that for any smooth algebraic stack $\mathcal{Y}$ the dimension of $T^*\mathcal{Y}$ at each point is $\geq \dim \mathcal{Y}$. If $g = 0$ and $\mathcal{Y} = \mathrm{Bun}_G$ then $T^*\mathcal{Y} = \mathrm{Nilp}$ and $\dim T^*\mathcal{Y} = \dim \mathcal{Y}$ according to Theorem 2.10.2. So we have again proved Theorem 2.10.4 for $g = 0$.

2) Let $g = 1$. It is convenient to assume G reductive but not necessarily semisimple (this is not really essential because Theorem 2.10.4 for reductive G easily follows from the semisimple case).

Before proceeding to the proof let us recall the notions of semistability and Shatz stratification. Fix a Borel subgroup $B \subset G$ and denote by H its maximal abelian quotient. Let $P \subset G$ be a parabolic subgroup containing B, L the maximal reductive quotient of P, Z the center of L. Let Γ (resp. Δ) be the set of simple roots of G (resp. L). The embedding $Z \hookrightarrow L$ induces an isomorphism $\mathrm{Hom}(Z, \mathbb{G}_m) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}(L, \mathbb{G}_m) \otimes \mathbb{Q}$. Denote by p the composition $\mathrm{Hom}(H, \mathbb{G}_m) \to \mathrm{Hom}(Z, \mathbb{G}_m) \to \mathrm{Hom}(L, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q}$. We say that $l \in \mathrm{Hom}(P, \mathbb{G}_m)^*$ is *strictly dominant* if $l(p(\alpha)) > 0$ for $\alpha \in \Gamma\backslash\Delta$.

For a P-bundle $\mathcal{F}$ let $\deg \mathcal{F} \in \mathrm{Hom}(P, \mathbb{G}_m)^*$ be the functional that associates to $\varphi : P \to \mathbb{G}_m$ the degree of the push-forward of $\mathcal{F}$ by φ. A G-bundle is said to be *semistable* if it does not come from a P-bundle of strictly dominant degree for any $P \neq G$. Semistable G-bundles form an open substack $\mathrm{Bun}_G{}^{ss} \subset \mathrm{Bun}_G$. Semistable G-bundles of fixed degree $d \in \mathrm{Hom}(G, \mathbb{G}_m)$ form an open substack $\mathrm{Bun}_G{}^{ss,d} \subset \mathrm{Bun}_G{}^{ss}$. If $P \subset G$ is a parabolic subgroup containing B and $d \in \mathrm{Hom}(P, \mathbb{G}_m)^*$ is strictly dominant denote by Shatz_P^d the stack of P-bundles $\mathcal{F}$ of degree d such that the corresponding L-bundle is semistable. It is known that the natural morphism $\mathrm{Shatz}_P^d \to \mathrm{Bun}_G$ is a locally closed embedding and the substacks

Shatz_P^d for all P, d form a stratification of Bun_G, which is called the *Shatz stratification*.

Denote by $\mathrm{Nilp}_P^d(G)$ (resp. $\mathrm{Nilp}^{ss}(G)$, $\mathrm{Nilp}^{ss,d}(G)$) the fibered product of $\mathrm{Nilp} = \mathrm{Nilp}(G)$ and Shatz_P^d (resp. ${\mathrm{Bun}_G}^{ss}$, ${\mathrm{Bun}_G}^{ss,d}$) over Bun_G. To show that $\mathrm{Nilp}(G)$ has pure dimension $\dim \mathrm{Bun}_G = 0$ it is enough to show that $\mathrm{Nilp}_P^d(G)$ has pure dimension 0 for each P and d. Let L be the maximal reductive quotient of P, $\mathfrak{p} := \mathrm{Lie}\, P$, $\mathfrak{l} := \mathrm{Lie}\, L$. If $\mathcal{F}$ is a P-bundle of strictly dominant degree such that the corresponding L-bundle $\overline{\mathcal{F}}$ is semistable then $H^0(X, \mathfrak{g}_{\mathcal{F}}) = H^0(X, \mathfrak{p}_{\mathcal{F}})$, so we have the natural map $\eta \mapsto \bar{\eta}$ from $H^0(X, \mathfrak{g}_{\mathcal{F}})$ to $H^0(X, \mathfrak{l}_{\mathcal{F}})$. Define $\pi : \mathrm{Nilp}_P^d(G) \to \mathrm{Nilp}^{ss,d}(L)$ by $(\mathcal{F}, \eta) \mapsto (\overline{\mathcal{F}}, \bar{\eta})$, $\eta \in H^0(X, \mathfrak{g}_{\mathcal{F}} \otimes \omega_X) = H^0(X, \mathfrak{g}_{\mathcal{F}})$ (ω_X is trivial because $g = 1$). Using again that $g = 1$ one shows that π is smooth and its fibers are 0-dimensional stacks. So it is enough to show that $\mathrm{Nilp}^{ss}(L)$ is of pure dimension 0.

A point of $\mathrm{Nilp}^{ss}(L)$ is a pair consisting of a semistable L-bundle $\mathcal{F}$ and a nilpotent $\eta \in H^0(X, \mathfrak{l}_{\mathcal{F}})$. Since $\mathfrak{l}_{\mathcal{F}}$ is a semistable vector bundle $\mathrm{ad}_\eta : \mathfrak{l}_{\mathcal{F}} \to \mathfrak{l}_{\mathcal{F}}$ has constant rank. So the conjugacy class of $\eta(x)$ does not depend on $x \in X$. For a nilpotent conjugacy class $C \subset \mathfrak{l}$ denote by $\mathrm{Nilp}_C^{ss}(L)$ the locally closed substack of $\mathrm{Nilp}^{ss}(L)$ parametrizing pairs $(\mathcal{F}, \eta)$ such that $\eta(x) \in C$. It is enough to show that $\mathrm{Nilp}_C^{ss}(L)$ has pure dimension 0 for each C. Let $Z(A) \subset L$ be the centralizer of some $A \in C$, $\mathfrak{z}(A) := \mathrm{Lie}\, Z(A)$. If $(\mathcal{F}, \eta) \in \mathrm{Nilp}_C^{ss}(L)$ then $\eta \in \Gamma(X, C_{\mathcal{F}}) = \Gamma(X, (G/Z(A))_{\mathcal{F}})$ defines a $Z(A)$-structure on $\mathcal{F}$. Thus we obtain an open embedding $\mathrm{Nilp}_C^{ss}(L) \hookrightarrow \mathrm{Bun}_{Z(A)}$. Finally $\mathrm{Bun}_{Z(A)}$ has pure dimension 0 because for any $Z(A)$-bundle $\mathcal{E}$ one has $\chi(\mathfrak{z}(A)_{\mathcal{E}}) = \deg \mathfrak{z}(A)_{\mathcal{E}} = 0$ (notice that since $G/Z(A) = C$ has a G-invariant symplectic structure the adjoint representation of $Z(A)$ has trivial determinant and therefore $\mathfrak{z}(A)_{\mathcal{E}}$ is trivial). □

2.10.5. *Proof of Proposition 2.1.2.* We must prove that (4) holds for $\mathcal{Y} = \mathrm{Bun}_G$, i.e., $\mathrm{codim}\{\mathcal{F} \in \mathrm{Bun}_G | \dim H^0(X, \mathfrak{g}_{\mathcal{F}}) = n\} > n$ for all $n > 0$. This

is equivalent to proving that

$$\dim(A(G)\backslash A^0(G)) < \dim \mathrm{Bun}_G \tag{39}$$

where $A(G)$ is the stack of pairs $(\mathcal{F}, s)$, $\mathcal{F} \in \mathrm{Bun}_G$, $s \in H^0(X, \mathfrak{g}_\mathcal{F})$, and $A^0(G) \subset A(G)$ is the closed substack defined by the equation $s = 0$. Set $C := \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g}^*)^G$. This is the affine scheme quotient of $\mathfrak{g}$ with respect to the adjoint action of G; in fact $C = W\backslash\mathfrak{h}$ where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. The morphism $\mathfrak{g} \to C$ induces a map $H^0(X, \mathfrak{g}_\mathcal{F}) \to \mathrm{Mor}(X, C) = C$. So we have a canonical morphism $f : A(G) \to C = W\backslash\mathfrak{h}$. For $h \in \mathfrak{h}$ set $A_h(G) = f^{-1}(\bar{h})$ where $\bar{h} \in W\backslash\mathfrak{h}$ is the image of h. Set $G^h := \{g \in G | ghg^{-1} = h\}$, $\mathfrak{g}^h := \mathrm{Lie}\, G^h = \{a \in \mathfrak{g} | [a, h] = 0\}$. Denote by $\mathfrak{z}_h$ the center of $\mathfrak{g}^h$. Since $h \in \mathfrak{z}_h$ and there is a finite number of subalgebras of $\mathfrak{g}$ of the form $\mathfrak{z}_h$ (39) follows from the inequality $\dim(A_h(G)\backslash A^0(G)) < \dim \mathrm{Bun}_G - \dim \mathfrak{z}_h$. So it is enough to prove that

$$\dim A_h(G) < \dim \mathrm{Bun}_G - \dim \mathfrak{z}_h \quad \text{for} \quad h \neq 0 \tag{40}$$

$$\dim(A_0(G)\backslash A^0(G)) < \dim \mathrm{Bun}_G\,. \tag{41}$$

Denote by Z_h the center of G^h. Let us show that (40) follows from the inequality (41) with G replaced by G^h/Z_h. Indeed, we have the natural isomorphisms $A_0(G^h) \xrightarrow{\sim} A_h(G^h) \xrightarrow{\sim} A_h(G)$ and the obvious morphism $\varphi : A_0(G^h) \to A_0(G^h/Z_h)$. A non-empty fiber of φ is isomorphic to Bun_{Z_h}, so $\dim A_h(G) \leq \dim \mathrm{Bun}_{Z_h} + \dim A_0(G^h/Z_h)$. Since $\dim \mathrm{Bun}_{Z_h} = (g-1)\cdot \dim \mathfrak{z}_h$ and (41) implies that $\dim A_0(G^h/Z_h) = (g-1)\cdot \dim(\mathfrak{g}^h/\mathfrak{z}_h)$ we have $\dim A_h(G) \leq (g-1)\cdot \dim \mathfrak{g}^h = \dim \mathrm{Bun}_G - (g-1)\cdot \dim(\mathfrak{g}/\mathfrak{g}^h) \leq \dim \mathrm{Bun}_G - \dim(\mathfrak{g}/\mathfrak{g}^h)$. Finally $\dim(\mathfrak{g}/\mathfrak{g}^h) \geq 2\cdot \dim \mathfrak{z}_h > \dim \mathfrak{z}_h$ if $h \neq 0$.

To prove (41) we will show that if $Y \subset A_0(G)$ is a locally closed reduced irreducible substack then $\dim Y \leq \dim \mathrm{Bun}_G$ and $\dim Y = \dim \mathrm{Bun}_G$ only if $Y \subset A^0(G)$. For $\xi \in H^0(X, \omega_X)$ consider the morphism $m_\xi : A_0(G) \to \mathrm{Nilp}$ defined by $(\mathcal{F}, s) \mapsto (\mathcal{F}, s\xi)$, $\mathcal{F} \in \mathrm{Bun}_G$, $s \in H^0(X, \mathfrak{g}_\mathcal{F})$. The morphisms

m_ξ define $m : A_0(G) \times H^0(X, \omega_X) \to \mathrm{Nilp}$. The image of m is contained in some locally closed reduced irreducible substack $Z \subset \mathrm{Nilp}$. If $\xi \neq 0$ then m_ξ induces an embedding $Y \hookrightarrow Z_\xi$ where Z_ξ is the closed substack of Z consisting of pairs $(\mathcal{F}, \eta) \in H^0(X, \mathfrak{g}_\mathcal{F} \otimes \omega_X)$ such that the restriction of η to the subscheme $D_\xi := \{x \in X | \xi(x) = 0\}$ is zero. So $\dim Y \leq \dim Z_\xi \leq \dim Z \leq \dim \mathrm{Nilp} = \dim \mathrm{Bun}_G$. If $\dim Y = \dim \mathrm{Bun}_G$ then $Z_\xi = Z$ for all nonzero $\xi \in H^0(X, \omega_X)$. This means that $\eta = 0$ for all $(\mathcal{F}, \eta) \in Z$ and therefore $s = 0$ for all $(\mathcal{F}, s) \in Y$, i.e., $Y \subset A^0(G)$. □

2.11. **On the stack of local systems.** Denote by $\mathcal{LS}_G$ the stack of G-local systems on X (a G-local system is a G-bundle with a connection). Kapranov [Kap97] explained that $\mathcal{LS}_G$ has a derived version $R\mathcal{LS}_G$, which is a DG stack. Using the results of 2.10 we will show that if $g > 1$ and G is semisimple then $R\mathcal{LS}_G = \mathcal{LS}_G$. We also describe the set of irreducible components of $\mathcal{LS}_G$. This section may be skipped by the reader; its results are not used in the rest of the work.

2.11.1. Fix $x \in X$. Denote by $\mathcal{LS}_G^x$ the stack of G-biundles $\mathcal{F}$ on X equipped with a connection ∇ having a simple pole at x. Denote by $\mathcal{E}$ the restriction to $\mathcal{LS}_G^x = \mathcal{LS}_G^x \times \{x\}$ of the universal G-bundle on $\mathcal{LS}_G^x \times X$. The residue of ∇ at x is a section $R \in \Gamma(\mathcal{LS}_G^x, \mathfrak{g}_\mathcal{E})$, and $\mathcal{LS}_G$ is the closed substack of $\mathcal{LS}_G^x$ defined by the equation $R = 0$. Consider the open substack $\widetilde{\mathcal{LS}}_G^x \subset \mathcal{LS}_G^x$ parametrizing pairs $(\mathcal{F}, \nabla)$ such that $\nabla : H^1(X, \mathfrak{g}_\mathcal{F}) \to H^1(X, \mathfrak{g}_\mathcal{F} \otimes \omega_X(x))$ is surjective. It is easy to see that $\widetilde{\mathcal{LS}}_G^x$ is a smooth stack of pure dimension $(2g-1) \cdot \dim G$ and $\mathcal{LS}_G \subset \widetilde{\mathcal{LS}}_G^x$.

Consider $\mathfrak{g}_\mathcal{E}$ as a stack over $\mathcal{LS}_G^x$. The sections $R, 0 \in \Gamma(\mathcal{LS}_G^x, \mathfrak{g}_\mathcal{E})$ define two closed substacks of $\mathfrak{g}_\mathcal{E}$, and $R\mathcal{LS}$ is their intersection in the derived sense while $\mathcal{LS}_G$ is their usual intersection. So the following conditions are equivalent:

1) $R\mathcal{LS}_G = \mathcal{LS}_G$;
2) $\mathcal{LS}_G$ is a locally complete intersection of pure dimension $(2g-2) \cdot \dim G$;

3) $\dim \mathcal{LS}_G \leq (2g-2)\cdot \dim G$.

The following proposition shows that these conditions are satisfied if $g>1$ and G is semisimple.

2.11.2. *Proposition.* Suppose that $g>1$ and G is reductive. Then $\mathcal{LS}_G$ is a locally complete intersection of pure dimension $(2g-2)\cdot\dim G + l$ where l is the dimension of the center of G.

Proof. Let R have the same meaning as in 2.11.1. Clearly $R \in \Gamma(\mathcal{LS}^x_G, [\mathfrak{g},\mathfrak{g}]\mathcal{E})$, so it suffices to show that

$$\dim \mathcal{LS}_G \leq (2g-2)\cdot \dim G + l. \tag{42}$$

Denote by G_{ad} the quotient of G by its center. Consider the projection $p : \mathcal{LS}_G \to \mathrm{Bun}_{G_{\mathrm{ad}}}$. If the fiber of p over a G_{ad}-bundle $\mathcal{F}$ is not empty then its dimension equals $\dim T^*_{\mathcal{F}} \mathrm{Bun}_{G_{\mathrm{ad}}} + l(2g-1)$, so $\dim \mathcal{LS}_G \leq \dim T^* \mathrm{Bun}_{G_{\mathrm{ad}}} + l(2g-1)$. Finally $\dim T^* \mathrm{Bun}_{G_{\mathrm{ad}}} \leq \dim G_{\mathrm{ad}}\cdot(2g-2)$ because $\mathrm{Bun}_{G_{\mathrm{ad}}}$ is good in the sense of 1.1.1 (we proved this in 2.10.1). □

2.11.3. Let $\mathrm{Bun}'_G \subset \mathrm{Bun}_G$ denote the preimage of the connected component of $\mathrm{Bun}_{G/[G,G]}$ containing the trivial bundle. The image of $\mathcal{LS}_G \to \mathrm{Bun}_G$ is contained in Bun'_G.

2.11.4. *Proposition.* Suppose that $g>1$ and G is reductive. Then the preimage in $\mathcal{LS}_G$ of every connected component of Bun'_G is non-empty and irreducible.

So irreducible components of $\mathcal{LS}_G$ are parametrized by

$$\mathrm{Ker}(\pi_1(G) \to \pi_1(G/[G,G])) = \pi_1([G,G]).$$

Proof. Consider the open substack $\mathrm{Bun}^0_{G_{\mathrm{ad}}} \subset \mathrm{Bun}_{G_{\mathrm{ad}}}$ parametrizing G_{ad}-bundles $\mathcal{F}$ such that $H^0(X, (\mathfrak{g}_{\mathrm{ad}})_{\mathcal{F}}) = 0$ (this is the biggest Deligne-Mumford substack of $\mathrm{Bun}_{G_{\mathrm{ad}}}$). Denote by $\mathrm{Bun}^{0\prime}_G$ the preimage of $\mathrm{Bun}^0_{G_{\mathrm{ad}}}$ in Bun'_G. Let $\mathcal{LS}^0_G$ denote the preimage of $\mathrm{Bun}^0_{G_{\mathrm{ad}}}$ in $\mathcal{LS}_G$. In 2.10.5 we proved that $\mathrm{Bun}_{G_{\mathrm{ad}}}$ is very good in the sense of 1.1.1, so $\dim(T^* \mathrm{Bun}_{G_{\mathrm{ad}}} \setminus T^* \mathrm{Bun}^0_{G_{\mathrm{ad}}}) <$

$\dim T^*\operatorname{Bun}_{G_{\mathrm{ad}}}$. The argument used in the proof of (42) shows that $\dim(\mathcal{LS}_G \setminus \mathcal{LS}_G^0) < (2g-2)\cdot \dim G + l$. Using 2.11.2 one sees that $\mathcal{LS}_G^0$ is dense in $\mathcal{LS}_G$. So it suffices to prove that the preimage in $\mathcal{LS}_G^0$ of every connected component of $\operatorname{Bun}_G^{0\prime}$ is non-empty and irreducible. This is clear because the morphism $\mathcal{LS}_G^0 \to \operatorname{Bun}_G^{0\prime}$ is a torsor[15] over $T^*\operatorname{Bun}_G^{0\prime}$. □

2.12. **On the Beauville – Laszlo Theorem.** This section is, in fact, an appendix in which we explain a globalized version of the main theorem of [BLa95]. This version is used in 2.3.7 but not in an essential way. So this section can be skipped by the reader.

2.12.1. *Theorem.* Let $p : \widetilde{S} \to S$ be a morphism of schemes, $D \subset S$ an effective Cartier divisor. Suppose that $\widetilde{D} := p^{-1}(D)$ is a Cartier divisor in $\widetilde{S}$ and the morphism $\widetilde{D} \to D$ is an isomorphism. Set $U := S \setminus D$, $\widetilde{U} := \widetilde{S} \setminus \widetilde{D}$. Denote by C the category of quasi-coherent $\mathcal{O}_S$-modules that have no non-zero local sections supported at D. Denote by $\widetilde{C}$ the similar category for $(\widetilde{S}, \widetilde{D})$. Denote by C' the category of triples $(\mathcal{M}_1, \mathcal{M}_2, \varphi)$ where $\mathcal{M}_1$ is a quasi-coherent $\mathcal{O}_U$-module, $\mathcal{M}_2 \in \widetilde{C}$, φ is an isomorphism between the pullbacks of $\mathcal{M}_1$ and $\mathcal{M}_2$ to $\widetilde{U}$.

1) p^* maps C to $\widetilde{C}$, so we have the functor $F : C \to C'$ that sends $\mathcal{M} \in C$ to $(\mathcal{M}|_U, p^*\mathcal{M}, \varphi)$ where φ is the natural isomorphism between the pullbacks of $\mathcal{M}|_U$ and $p^*\mathcal{M}$ to $\widetilde{U}$.
2) $F : C \to C'$ is an equivalence.
3) $\mathcal{M} \in C$ is locally of finite type (resp. flat, resp. locally free of finite rank) if and only if $\mathcal{M}|_U$ and $f^*\mathcal{M}$ have this property.

[15]The torsor structure depends on the choice of an invariant scalar product on $\mathfrak{g}$.

This theorem is easily reduced to the case where S and $\widetilde{S}$ are affine[16] and D is *globally* defined by one equation (so $S = \operatorname{Spec} A$, $\widetilde{S} = \operatorname{Spec} \widetilde{A}$, $D = \operatorname{Spec} A/fA$, $f \in A$ is not a zero divisor). This case is treated just as in [BLa95] (in [BLa95] it is supposed that $\widetilde{A} = \widehat{A}$:=the completion of A for the f-adic topology, but the only properties of $\widehat{A}$ used in [BLa95] are the injectivity of $f : \widehat{A} \to \widehat{A}$ and the bijectivity of $A/fA \to \widehat{A}/f\widehat{A}$).

2.12.2. Let D be a closed affine subscheme of a scheme S. Denote by $\widehat{S}$ the completion of S along D and by $\widehat{S}'$ the spectrum of the ring of regular functions on $\widehat{S}$ (so $\widehat{S}$ is an affine formal scheme and $\widehat{S}'$ is the corresponding true scheme). We have the morphisms $\pi : \widehat{S} \to S$ and $i : \widehat{S} \to \widehat{S}'$.

2.12.3. *Proposition.* There is at most one morphism $p : \widehat{S}' \to S$ such that $pi = \pi$.

Proof. Suppose that $\pi = p_1 i = p_2 i$ for some $p_1, p_2 : \widehat{S}' \to S$. Let $Y \subset \widehat{S}'$ be the preimage of the diagonal $\Delta \subset S \times S$ under $(p_1, p_2) : \widehat{S}' \to S \times S$. Then Y is a locally closed subscheme of $\widehat{S}'$ containing the n-th infinitesimal neighbourhood of $D \subset \widehat{S}'$ for every n. So $(\bar{Y} \setminus Y) \cap D = \emptyset$ and therefore $\bar{Y} \setminus Y = \emptyset$, i.e., Y is closed. A closed subscheme of $\widehat{S}'$ containing all infinitesimal neighbourhoods of D equals $\widehat{S}'$. So $Y = \widehat{S}'$ and $p_1 = p_2$. □

2.12.4. Suppose we are in the situation of 2.12.2 and $D \subset S$ is an effective Cartier divisor. If there exists $p : \widehat{S}' \to S$ such that $pi = \pi$ then $p^{-1}(D) \subset \widehat{S}'$ is a Cartier divisor and the morphism $p^{-1}(D) \to D$ is an isomorphism. So Theorem 2.12.1 is applicable.

[16] For any $x \in S$ there is an affine neighbourhood U of x and an open affine $\widetilde{U} \subset \widetilde{S}$ such that $\widetilde{U} \subset p^{-1}(U)$ and $\widetilde{U} \cap \widetilde{D} = p^{-1}(U) \cap \widetilde{D}$. Indeed, we can assume that S is affine and $x \in D$. Let $\widetilde{U}_1 \subset S$ be an affine neighbourhood of the preimage of x in $\widetilde{D}$. Then $p(\widetilde{U}_1 \cap \widetilde{D})$ is an affine neighbourhood of x in D, so it contains $U \cap D$ for some open affine $U \subset S$ such that $x \in U$. Then $\widetilde{U} := \widetilde{U}_1 \times_S U$ has the desired properties.

2.12.5. Suppose we are in the situation of 2.12.2 and S is quasi-separated. Then there exists $p : \widehat{S}' \to S$ such that $pi = \pi$. The proof we know is rather long. We first treat the noetherian case and then use the following fact (Deligne, private communication): for any quasi-compact quasi-separated scheme S there exists an affine morphism from S to some scheme of finite type over $\mathbb{Z}$.

In 2.3.7 we use the existence of $p : \widehat{S}' \to S$ for $S = X \otimes R$ where X is our curve and R is a $\mathbb{C}$-algebra. So the following result suffices.

2.12.6. *Proposition.* Suppose that in the situation of 2.12.2 S is a locally closed subscheme of $\mathbb{P}^n \otimes R$ for some ring R. Then there exists $p : \widehat{S}' \to S$ such that $pi = \pi$.

Proof. We use Jouanolou's device. Let $\mathbb{P}^*$ be the projective space dual to $\mathbb{P} = \mathbb{P}^n$, $Z \subset \mathbb{P} \times \mathbb{P}^*$ the incidence correspondence, $U := (\mathbb{P} \times \mathbb{P}^*) \setminus Z$. Since the morphism $U \to \mathbb{P}$ is a torsor over some vector bundle on $\mathbb{P}$ and $\widehat{S}$ is an affine formal scheme the morphism $\widehat{S} \to \mathbb{P}$ lifts to a morphism $\widehat{S} \to U$. Since U is affine $\mathrm{Mor}(\widehat{S}, U) = \mathrm{Mor}(\widehat{S}', U)$, so we get a morphism $\widehat{S}' \to U$. The composition $\widehat{S}' \to U \to \mathbb{P}$ yields a morphism $f : \widehat{S}' \to \mathbb{P} \otimes R$. The locally closed subscheme $f^{-1}(S) \subset \widehat{S}'$ contains the n-th infinitesimal neighbourhood of $D \subset \widehat{S}'$ for every n, so $f^{-1}(S) = \widehat{S}'$ (cf. 2.12.3) and f induces a morphism $p : \widehat{S}' \to S \subset \mathbb{P} \otimes R$. Clearly $pi = \pi$. □

Remark. One can also prove the proposition interpreting the morphism $\widehat{S} \to \mathbb{P}^n$ as a pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is an invertible sheaf on $\widehat{S}$ and φ is an epimorphism $\mathcal{O}^{n+1} \to \mathcal{M}$. Then one shows that $(\mathcal{M}, \varphi)$ extends to a pair $(\mathcal{M}', \varphi')$ on $\widehat{S}'$. Of course, this proof is essentially equivalent to the one based on Jouanolou's device.

3. Opers

3.1. Definition and first properties.

3.1.1. Let G be a connected reductive group over $\mathbb{C}$ with a fixed Borel subgroup $B = B_G \subset G$. Set $N = [B, B]$, so $H = B/N$ is the Cartan group. Denote by $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$ the corresponding Lie algebras. $\mathfrak{g}$ carries a canonical decreasing Lie algebra filtration $\mathfrak{g}^k$ such that $\mathfrak{g}^0 = \mathfrak{b}$, $\mathfrak{g}^1 = \mathfrak{n}$, and for any $k > 0$ the weights of the action of $\mathfrak{h} = \mathrm{gr}^0\,\mathfrak{g}$ on $\mathrm{gr}^k\,\mathfrak{g}$ (resp. $\mathfrak{g}r^{-k}\mathfrak{g}$) are sums of k simple positive (resp. negative) roots. In particular $\mathrm{gr}^{-1}\,\mathfrak{g} = \oplus\mathfrak{g}^\alpha$, α is a simple negative root. Set $Z = Z_G = \mathrm{Center}G$.

3.1.2. Let X be any smooth (not necessarily complete) curve, $\mathfrak{F}_B$ a B-bundle on X. Denote by $\mathfrak{F}_G$ the induced G-torsor, so $\mathfrak{F}_B \subset \mathfrak{F}_G$. We have the corresponding twisted Lie algebras $\mathfrak{b}_\mathfrak{F} := \mathfrak{b}_{\mathfrak{F}_B}$ and $\mathfrak{g}_\mathfrak{F} := \mathfrak{g}_{\mathfrak{F}_B} = \mathfrak{g}_{\mathfrak{F}_G}$ equipped with the Lie algebra filtration $\mathfrak{g}^k_\mathfrak{F}$. Consider the sheaves of connections $\mathrm{Conn}(\mathfrak{F}_B)$, $\mathrm{Conn}(\mathfrak{F}_G)$; these are $\mathfrak{b}_\mathfrak{F} \otimes \omega_X$- and $\mathfrak{g}_\mathfrak{F} \otimes \omega_X$-torsors. We have the obvious embedding $\mathrm{Conn}(\mathfrak{F}_B) \subset \mathrm{Conn}(\mathfrak{F}_G)$. It defines the projection $c : \mathrm{Conn}(\mathfrak{F}_G) \to (\mathfrak{g}/\mathfrak{b})_\mathfrak{F} \otimes \omega_X$ such that $c^{-1}(0) = \mathrm{Conn}(\mathfrak{F}_B)$ and $c(\nabla + \nu) = c(\nabla) + \nu \mod \mathfrak{b}_\mathfrak{F} \otimes \omega_X$ for any $\nabla \in \mathrm{Conn}(\mathfrak{F}_G)$, $\nu \in \mathfrak{g}_\mathfrak{F} \otimes \omega_X$.

3.1.3. *Definition.* A G-oper on X is a pair $(\mathfrak{F}_B, \nabla)$, $\nabla \in \Gamma(X, \mathrm{Conn}(\mathfrak{F}_G))$ such that

1. $c(\nabla) \in \mathrm{gr}^{-1}\,\mathfrak{g}_\mathfrak{F} \otimes \omega_X \subset (\mathfrak{g}/\mathfrak{b})_\mathfrak{F} \otimes \omega_X$
2. For any simple negative root α the α-component $c(\nabla)^\alpha \in \Gamma(X, \mathfrak{g}^\alpha_\mathfrak{F} \otimes \omega_X)$ does not vanish at any point of X.

If $\mathfrak{g}$ is a semisimple Lie algebra then a $\mathfrak{g}$-*oper* is a G_{ad}-oper where G_{ad} is the adjoint group corresponding to $\mathfrak{g}$.

We will usually consider G-oper as a G-local system $(\mathfrak{F}_G, \nabla)$ equipped with an extra *oper structure* (a B-flag $\mathfrak{F}_B \subset \mathfrak{F}_G$ which satisfies conditions (1) and (2) above).

G-opers on X form a groupoid $\mathcal{O}\mathfrak{p}_G(X)$. The groupoids $\mathcal{O}\mathfrak{p}_G(X')$ for X' étale over X form a sheaf of groupoids $\mathcal{O}\mathfrak{p}_G$ on $X_{\text{ét}}$.

3.1.4. *Proposition.* Let $(\mathfrak{F}_B, \nabla)$ be a G-oper. Then $\mathrm{Aut}(\mathfrak{F}_B, \nabla) = Z$ if X is connected. □

In particular $\mathfrak{g}$-opers have no symmetries, i.e., $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is a set and $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$ is a sheaf of sets.

3.1.5. *Proposition.* Suppose that X is complete and connected of genus $g > 1$. Let $(\mathfrak{F}_G, \nabla)$ be a G-local system on X that has an oper structure. Then

(i) the oper structure on $(\mathfrak{F}_G, \nabla)$ is unique: the corresponding flag $\mathfrak{F}_B \subset \mathfrak{F}_G$ is the Harder-Narasimhan flag;
(ii) $\mathrm{Aut}(\mathfrak{F}_G, \nabla) = Z$;
(iii) $(\mathfrak{F}_G, \nabla)$ cannot be reduced to a non-trivial parabolic subgroup $P \subset G$.

□

Of course ii) follows from i) and 3.1.4.

3.1.6. *Example.* A GL_n-oper can be considered as an $\mathcal{O}_X$-module $\mathcal{E}$ equipped with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \omega_X$ and a filtration $\mathcal{E} = \mathcal{E}_n \supset \mathcal{E}_{n-1} \supset \cdots \supset \mathcal{E}_0 = 0$ such that

(i) The sheaves $\mathrm{gr}_i \mathcal{E}$, $n \geq i \geq 1$, are invertible
(ii) $\nabla(\mathcal{E}_i) \subset \mathcal{E}_{i+1} \otimes \omega_X$ and for $n - 1 \geq i \geq 1$ the morphism $\mathrm{gr}_i \mathcal{E} \to \mathrm{gr}_{i+1} \mathcal{E} \otimes \omega_X$ induced by ∇ is an isomorphism.

One may construct GL_n-opers as follows. Let $\mathcal{A}, \mathcal{B}$ be invertible $\mathcal{O}_X$-modules and $\partial : \mathcal{A} \to \mathcal{B}$ a differential operator of order n whose symbol $\sigma(\partial) \in \Gamma\left(X, \mathcal{B} \bigotimes \mathcal{A}^{\otimes(-1)} \bigotimes \Theta_X^{\otimes n}\right)$ has no zeros. Our ∂ is a section of $\mathcal{B} \bigotimes_{\mathcal{O}_X} \mathcal{D}_X \bigotimes_{\mathcal{O}_X} \mathcal{A}^{\otimes(-1)}$ or, equivalently, an $\mathcal{O}$-linear map $\mathcal{B}^{\otimes(-1)} \to \mathcal{D}_X \otimes \mathcal{A}^{\otimes(-1)}$. Let $I \subset \mathcal{D}_X \otimes A^{\otimes(-1)}$ be the $\mathcal{D}_X$-sub-module generated by the image of this map. Let $\mathcal{E} := \mathcal{D}_X \otimes \mathcal{A}^{\otimes(-1)}/I$; denote by $\mathcal{E}_i$ the filtration on $\mathcal{E}$ induced by the usual filtration of $\mathcal{D}_X$ by degree of an operator. Then $\mathcal{E}$ is a $\mathcal{D}_X$-module, i.e., an $\mathcal{O}_X$-module with a connection,

and the filtration $\mathcal{E}_i$ satisfies the conditions (i), (ii). Therefore $(\mathcal{E}, \{\mathcal{E}_i\}, \nabla)$ is a GL_n-oper. This construction defines an equivalence between the groupoid of GL_n-opers and that of the data $\partial : \mathcal{A} \to \mathcal{B}$ as above. The inverse functor Φ associates to $(\mathcal{E}, \{\mathcal{E}_i\}, \nabla)$ the following differential operator : $\mathcal{A} \longrightarrow \mathcal{B}, \mathcal{A} := \mathcal{E}_1^{\otimes(-1)}, \mathcal{B} := \omega_X \otimes (\mathcal{E}/\mathcal{E}_{n-1})^{\otimes(-1)}$. Consider $\mathcal{E}$ as a $\mathcal{D}_X$-module. Let $\mathcal{D}_X^{(k)} \subset \mathcal{D}_X$ be the subsheaf of operators of order $\leq k$. Then the morphism $\mathcal{D}_X^{(n-1)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \mathcal{E}$ is an isomorphism and therefore the composition $\mathcal{D}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \mathcal{E} \xrightarrow{\sim} \mathcal{D}_X^{(n-1)} \otimes_{\mathcal{O}_X} \mathcal{E}_1$ defines a splitting of the exact sequence $0 \longrightarrow \mathcal{D}_X^{(n-1)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \mathcal{D}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{E}_1 \longrightarrow \omega_X^{\otimes(-n)} \otimes \mathcal{E}_1 \longrightarrow 0$, i.e., a morphism $\omega_X^{\otimes(-n)} \otimes \mathcal{E}_1 \longrightarrow \mathcal{D}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{E}_1$, which is the same as a differential operator $\partial : \mathcal{A} \longrightarrow \mathcal{B}$ (notice that the isomorphisms $\mathrm{gr}_i \mathcal{E} \xrightarrow{\sim} \mathrm{gr}_{i+1} \mathcal{E} \otimes \omega_X$ induce an isomorphism $\mathcal{E}_1 \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_{n-1}) \otimes \omega_X^{\otimes(n-1)}$, so $\omega_X^{\otimes(-n)} \otimes \mathcal{E}_1 = \omega_X^{\otimes(-1)} \otimes (\mathcal{E}/\mathcal{E}_{n-1}) = \mathcal{B}^{\otimes(-1)}$).

Applying the above functor Φ to an SL_2-oper one obtains a differential operator $\partial : \mathcal{A} \longrightarrow \omega_X \otimes \mathcal{A}^{\otimes(-1)}$. It is easy to show that one thus obtains an equivalence between the groupoid of SL_2-opers and that of pairs $(\mathcal{A}, \partial)$ consisting of an invertible sheaf $\mathcal{A}$ and a Sturm-Liouville operator $\partial : \mathcal{A} \longrightarrow \omega_X \otimes \mathcal{A}^{\otimes(-1)}$, i.e., a self-adjoint differential operator ∂ of order 2 whose symbol $\sigma(\partial)$ has no zeros. Notice that $\sigma(\partial)$ induces an isomorphism $\omega_X^{\otimes 2} \otimes \mathcal{A} \xrightarrow{\sim} \omega_X \otimes \mathcal{A}^{\otimes(-1)}$, so $\mathcal{A}$ is automatically a square root of $\omega_X^{\otimes(-1)}$.

If $(\mathcal{A}, \partial)$ is a Sturm-Liouville operator and $\mathcal{M}$ is a line bundle equipped with an isomorphism $\mathcal{M}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ then $\mathcal{M}$ has a canonical connection and therefore tensoring $(\mathcal{A}, \partial)$ by $\mathcal{M}$ one obtains a Sturm-Liouville operator $(\widetilde{\mathcal{A}}, \widetilde{\partial})$, $\widetilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{M}$. We say that $(\mathcal{A}, \partial)$ and $(\widetilde{\mathcal{A}}, \widetilde{\partial})$ are *equivalent*. It is easy to see that the natural map $\mathcal{O}\mathfrak{p}_{SL_2}(X) \longrightarrow \mathcal{O}\mathfrak{p}_{sl_2}(X)$ identifies $\mathcal{O}\mathfrak{p}_{sl_2}(X)$ with the set of equivalence classes of Sturm-Liouville operators.

Opers for other classical groups may be described in similar terms (in the local situation this was done in [DS85, section 8]).

3.1.7. Identifying sl_2-opers with equivalence classes of Sturm-Liouville operators (see 3.1.6) one sees that $\mathcal{O}\mathfrak{p}_{sl_2}$ is an $\omega_X^{\otimes 2}$-torsor: a section η of

$\omega_X^{\otimes 2}$ maps a Sturm-Liouville operator $\partial : \mathcal{A} \longrightarrow \mathcal{A} \otimes \omega_X^{\otimes 2}$, $\mathcal{A}^{\otimes(-2)} = \omega_X$, to $\partial - \eta$. Let us describe this action of $\omega_X^{\otimes 2}$ on $\mathcal{O}\mathfrak{p}_{sl_2}$ without using Sturm-Liouville operators.

Identify $\mathfrak{n} \subset sl_2$ with $(sl_2/\mathfrak{b})^*$ using the bilinear form $\mathrm{Tr}(AB)$ on sl_2. If $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$ is an sl_2-oper then according to 3.1.3 the section $c(\nabla)$ trivializes the sheaf $(sl_2/\mathfrak{b})_{\mathfrak{F}_B} \otimes \omega_X$. So $(sl_2/\mathfrak{b})_{\mathfrak{F}_B} = \omega_X^{\otimes(-1)}$, $\mathfrak{n}_{\mathfrak{F}} = \omega_X$, and we have the embedding $\omega_X^{\otimes 2} = \mathfrak{n}_{\mathfrak{F}_B} \otimes \omega_X \hookrightarrow (sl_2)_{\mathfrak{F}_B} \otimes \omega_X$. Translating ∇ by a section μ of $\omega_X^{\otimes 2} \subset (sl_2)_{\mathfrak{F}_B} \otimes \omega_X$ we get a new oper denoted by $\mathfrak{F} + \mu$. This $\omega_X^{\otimes 2}$-action on $\mathcal{O}\mathfrak{p}_{sl_2}$ coincides with the one introduced above, so it makes $\mathcal{O}\mathfrak{p}_{sl_2}$ an $\omega_X^{\otimes 2}$-torsor.

Remark It is well known that this torsor is trivial (even if $H^1(X, \omega_X^{\otimes 2}) \neq 0$, i.e., $g \leq 1$; Sturm-Liouville operators on $\mathbb{P}^1$ or on an elliptic curve do exist). However for families of curves X this torsor may not be trivial.

3.1.8. In 3.1.9 we will use the following notation. Let $B_0 \subset PSL_2$ be the group of upper-triangular matrices. Set $N_0 := [B_0, B_0], \mathfrak{b}_0 := \mathrm{Lie}\, B_0, \mathfrak{n}_0 := \mathrm{Lie}\, N_0$. Identify B_0/N_0 with $\mathbb{G}_m$ via the adjoint action $B_0/N_0 \longrightarrow \mathrm{Aut}\, \mathfrak{n}_0 = \mathbb{G}_m$. Using the matrices $e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ we identify $\mathfrak{n}_0$ and $sl_2/\mathfrak{b}_0$ with $\mathbb{C}$. Then for an sl_2-oper $\mathfrak{F} = (\mathfrak{F}_{B_0}, \nabla)$ the isomorphism $(sl_2/\mathfrak{b}_0)_{\mathfrak{F}_{B_0}} \xrightarrow{\sim} \omega_X^{\otimes(-1)}$ from 3.1.7 (or the isomorphism $\mathfrak{n}_{\mathfrak{F}_{B_0}} \xrightarrow{\sim} \omega_X$) induces an isomorphism between the push-forward of $\mathfrak{F}_{B_0}$ by $B_0 \longrightarrow B_0/N_0 = \mathbb{G}_m$ and the $\mathbb{G}_m$-torsor ω_X.

3.1.9. For any semisimple Lie algebra $\mathfrak{g}$ we will give a rather explicit description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$. In particular we will introduce a "canonical" structure of affine space on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ (for $\mathfrak{g} = sl_2$ it was introduced in 3.1.7).

Let G be the adjoint group corresponding to $\mathfrak{g}$ and B its Borel subgroup. We will use the notation from 3.1.8. Fix a principal embedding $i : sl_2 \hookrightarrow \mathfrak{g}$ such that $i(\mathfrak{b}_0) \subset \mathfrak{b}$; one has the corresponding embeddings $i_G : PSL_2 \hookrightarrow G$, $i_B : B_0 \hookrightarrow B$. Set $V = V_{\mathfrak{g}} := \mathfrak{g}^{N_0}$. Then $\mathfrak{n}_0 \subset V \subset \mathfrak{n}$. One has the

adjoint action Ad of $\mathbb{G}_m = B_0/N_0$ on V. Define a new $\mathbb{G}_m$-action a on V by $a(t)v := t\,\mathrm{Ad}(t)v$, $v \in V$, $t \in \mathbb{G}_m$.

Consider the vector bundle V_{ω_X} i.e., the ω_X-twist of V with respect to the $\mathbb{G}_m$-action a (we consider ω_X as a $\mathbb{G}_m$-torsor on X). Twisting by ω_X the embedding $\mathbb{C} \xrightarrow{\sim} \mathbb{C}e = \mathfrak{n}_0 \hookrightarrow V$ we get an embedding ${\omega_X}^{\otimes 2} \hookrightarrow V_{\omega_X}$.

For any sl_2-oper $\mathfrak{F}_0 = (\mathfrak{F}_{B_0}, \nabla_0)$ its i-push-forward $i\mathfrak{F}_0 = (\mathfrak{F}_B, \nabla)$ is a $\mathfrak{g}$-oper. It follows from 3.1.8 that we have a canonical isomorphism $V_{\omega_X} = V_{\mathfrak{F}_{B_0}} \otimes \omega_X$ and therefore a canonical embedding $V_{\omega_X} \subset \mathfrak{b}_{\mathfrak{F}_0} \otimes \omega_X = \mathfrak{b}_{\mathfrak{F}_B} \otimes \omega_X$. Translating ∇ by a section ν of V_{ω_X} we get a new $\mathfrak{g}$-oper denoted by $i\mathfrak{F}_0 + \nu$.

Let $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}$ be the V_{ω_X}-torsor induced from the ${\omega_X}^{\otimes 2}$-torsor $\mathcal{O}\mathfrak{p}_{sl_2}$ by the embedding ${\omega_X}^{\otimes 2} \subset V_{\omega_X}$. A section of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}$ is a pair $(\mathfrak{F}_0, \nu)$ as above, and we assume that $(\mathfrak{F}_0 + \mu, \nu) = (\mathfrak{F}_0, \mu + \nu)$ for a section μ of ${\omega_X}^{\otimes 2}$. We have a canonical map

$$\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}} \longrightarrow \mathcal{O}\mathfrak{p}_{\mathfrak{g}} \tag{43}$$

which sends $(\mathfrak{F}_0, \nu)$ to $i\mathfrak{F}_0 + \nu$.

3.1.10. *Proposition.* The mapping (43) is bijective. □

Remarks

(i) Though the bijection (43) is canonical we are not sure that it gives a reasonable description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$.

(ii) The space $V = V_{\mathfrak{g}}$ from 3.1.9 depends on the choice of a principal embedding $i : sl_2 \hookrightarrow \mathfrak{g}$ (for such an i there is a unique Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $i(\mathfrak{b}_0)$). But any two principal embeddings $sl_2 \hookrightarrow \mathfrak{g}$ are conjugate by a unique element of $G = G_{\mathrm{ad}}$. So we can identify the V's corresponding to various i's and obtain a vector space (not a subspace of $\mathfrak{g}$!) canonically associated to $\mathfrak{g}$.

(iii) Let G be the adjoint group corresponding to $\mathfrak{g}$, B a Borel subgroup of G. Proposition 3.1.10 implies that for any $\mathfrak{g}$-oper $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$ $\mathfrak{F}_B$ is isomorphic to a certain canonical B-bundle $\mathfrak{F}_B^0$ which does not depend

on $\mathfrak{F}$. Actually $\mathfrak{F}^0_B$ is the push-forward of the canonical $(\mathrm{Aut}^0 O)$-bundle from 2.6.5 by a certain homomorphism $i_B\sigma\pi : \mathrm{Aut}^0 O \longrightarrow B$. Here π is the projection $\mathrm{Aut}^0 O \longrightarrow \mathrm{Aut}(O/m^3)$ where m is the maximal ideal of O, σ is an isomorphism $\mathrm{Aut}(O/m^3) \xrightarrow{\sim} B_0$ where B_0 is a Borel subgroup of PSL_2, and $i_B : B_0 \longrightarrow B$ is induced by a principal embedding $PSL_2 \longrightarrow G$ (σ and i_B are unique up to a unique conjugation).

3.1.11. Assume that X is complete. Then G-opers form a smooth algebraic stack which we again denote as $\mathcal{O}\mathfrak{p}_G(X)$ by abuse of notation. If G is semisimple this is a Deligne-Mumford stack (see 3.1.4); if G is adjoint then $\mathcal{O}\mathfrak{p}_G(X) = \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is a scheme isomorphic to the affine space $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$ via (43).

Remarks

(i) If X is non-complete, then $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is an ind-scheme.

(ii) If X is complete, connected, and of genus $g > 1$, then $\dim \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) = (g-1)\cdot \dim \mathfrak{g}$. Indeed, according to Proposition 3.1.10, $\dim \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) = \dim \underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X) = \dim \Gamma(X, V_{\omega_X})$ and an easy computation due to Hitchin (see Remark 4 from 2.2.4) shows that $\dim \Gamma(X, V_{\omega_X}) = (g-1)\cdot \dim \mathfrak{g}$ if $g > 1$. Actually we will see in 3.1.13 that $\Gamma(X, V_{\omega_X}) = \mathrm{Hitch}_{{}^L\mathfrak{g}}(X)$, so we can just use Hitchin's formula

$$\dim \mathrm{Hitch}_{{}^L\mathfrak{g}}(X) = (g-1)\cdot \dim {}^L\mathfrak{g} = (g-1)\cdot \dim \mathfrak{g}$$

mentioned in 2.2.4(ii).

(iii) Let X be as in Remark ii and G be the adjoint group corresponding to $\mathfrak{g}$. One has the obvious morphism $i : \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathrm{LocSys}_G$ where LocSys_G is the stack of G-local systems on X. One can show that G-local systems which cannot be reduced to a non-trivial parabolic subgroup $P \subset G$ and which have no non-trivial automorphisms form an open substack $U \subset \mathrm{LocSys}_G$ which is actually a smooth variety; U has a canonical symplectic structure. According to 3.1.5 $i(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)) \subset U$

and i is a set-theoretical embedding. In fact i is a closed embedding and $i(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X))$ is a Lagrangian subvariety of U. Besides, $i(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)) = \pi^{-1}(S)$ where $\pi : \mathrm{LocSys}_G \longrightarrow \mathrm{Bun}_G$ corresponds to forgetting the connection and $S \subset \mathrm{Bun}_G$ is the locally closed substack of G-bundles isomorphic to $\mathfrak{F}^0_G$, the G-bundle corresponding to the B-bundle $\mathfrak{F}^0_B$ introduced in Remark iii from 3.1.10 (so S is the classifying stack of the unipotent group $\mathrm{Aut}\,\mathfrak{F}^0_G$).

3.1.12. Denote by $A_{\mathfrak{g}}(X)$ the coordinate ring of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$. We will construct a canonical filtration on $A_{\mathfrak{g}}(X)$ and a canonical isomorphism of graded algebras

$$\sigma_{A(X)} : \mathrm{gr}\, A_{\mathfrak{g}}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(X) \tag{44}$$

where ${}^L\mathfrak{g}$ denotes the Langlands dual of $\mathfrak{g}$ and the r.h.s. of (44) was defined in 2.2.2. We give two equivalent constructions. The one from 3.1.13 is straightforward; it involves the isomorphism (43). The construction from 3.1.14 is more natural.

3.1.13. Using 3.1.8 we identify $A_{\mathfrak{g}}(X)$ with the coordinate ring of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$. Denote by $A^{cl}_{\mathfrak{g}}(X)$ the coordinate ring of the vector space $\Gamma(X, V_{\omega_X})$ corresponding to the affine space $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$. Consider the $\mathbb{G}_m$-action on $A^{cl}_{\mathfrak{g}}(X)$ opposite to that induced by the $\mathbb{G}_m$-action a on V (see 3.1.7); the corresponding grading on $A^{cl}_{\mathfrak{g}}(X)$ is positive. It induces a canonical ring filtration on $A_{\mathfrak{g}}(X)$ and a canonical isomorphism $\mathrm{gr}\, A_{\mathfrak{g}}(X) \xrightarrow{\sim} A^{cl}_{\mathfrak{g}}(X)$.

So to define (44) it remains to construct a graded isomorphism $A^{cl}_{\mathfrak{g}}(X) \xrightarrow{\sim} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(X)$, which is equivalent to constructing a $\mathbb{G}_m$-equivariant isomorphism of schemes $\Gamma(X, V_{\omega_X}) \xrightarrow{\sim} \mathrm{Hitch}_{{}^L\mathfrak{g}}(X)$. According to 2.2.2 $\mathrm{Hitch}_{{}^L\mathfrak{g}}(X) := \Gamma(X, C_{\omega_X})$, $C := C_{{}^L\mathfrak{g}}$. So it suffices to construct a $\mathbb{G}_m$-equivariant isomorphism of schemes $V_{\mathfrak{g}} \xrightarrow{\sim} C_{{}^L\mathfrak{g}}$. ($V_{\mathfrak{g}}$ is equipped with the action a from 3.1.7.)

According to 2.2.1 $C_{{}^L\mathfrak{g}} = \mathrm{Spec}(\mathrm{Sym}\,{}^L\mathfrak{g})^{{}^LG}$ where G is a connected group corresponding to $\mathfrak{g}$. We can identify $(\mathrm{Sym}\,{}^L\mathfrak{g})^{{}^LG}$ with $(\mathrm{Sym}\,\mathfrak{g}^*)^G$ because

both graded algebras are canonically isomorphic to $(\operatorname{Sym}\mathfrak{h}^*)^W$ where W is the Weyl group. So $C_{L_\mathfrak{g}} = C'_\mathfrak{g}$ where

$$C'_\mathfrak{g} = \operatorname{Spec}(\operatorname{Sym}\mathfrak{g}^*)^G\,, \tag{45}$$

i.e., $C'_\mathfrak{g}$ is the affine scheme quotient of $\mathfrak{g}$ with respect to the adjoint action of G. Finally according to Theorem 0.10 from Kostant's work [Ko63] we have the canonical isomorphism $V_\mathfrak{g} \xrightarrow{\sim} C'_\mathfrak{g}$ that sends $v \in V_\mathfrak{g}$ to the image of $v + i(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)) \in \mathfrak{g}$ in $C'_\mathfrak{g}$. It commutes with the $\mathbb{G}_m$-actions.

3.1.14. Here is a more natural way to describe the canonical filtration on $A_\mathfrak{g}(X)$ and the isomorphism (44).

There is a standard way to identify filtered $\mathbb{C}$-algebras with graded flat $\mathbb{C}[\hbar]$-algebras (here $\deg\hbar = 1$). Namely, an algebra A with an increasing filtration $\{A_i\}$ corresponds to the graded $\mathbb{C}[\hbar]$-algebra $A^\sim = \oplus A_i$, the multiplication by $\hbar$ is the embedding $A_i \hookrightarrow A_{i+1}$. Note that $A = A^\sim/(\hbar - 1)A^\sim$, $\operatorname{gr} A = A^\sim/\hbar A^\sim$. Passing to spectra we see that $\operatorname{Spec} A^\sim$ is a flat affine scheme over the line $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[\hbar]$, and the grading on $A^\sim$ is the same as a $\mathbb{G}_m$-action on $\operatorname{Spec} A^\sim$ compatible with the action by homotheties on $\mathbb{A}^1$. We are going to construct the scheme $\operatorname{Spec} A_\mathfrak{g}(X)^\sim$.

Let $\mathfrak{F}$ be a G-torsor on X. Denote by $\mathcal{E}_\mathfrak{F}$ the Lie algebroid of infinitesimal symmetries of $\mathfrak{F}$; we have a canonical exact sequence

$$0 \to \mathfrak{g}_\mathfrak{F} \to \mathcal{E}_\mathfrak{F} \xrightarrow{\pi} \Theta_X \to 0.$$

Recall that for $\hbar \in \mathbb{C}$ an *$\hbar$-connection on* $\mathfrak{F}$ is an $\mathcal{O}_X$-linear map $\nabla_\hbar : \Theta_X \to \mathcal{E}_\mathfrak{F}$ such that $\pi\nabla_\hbar = \hbar\,\mathrm{id}_{\Theta_X}$ (usual connections correspond to $\hbar = 1$). One defines a *$G - \hbar$-oper* as in 3.1.3 replacing the connection ∇ by an $\hbar$-connection $\nabla_\hbar$. The above results about G-opers render to $G - \hbar$-opers. In particular $\mathfrak{g} - \hbar$-opers, i.e., $\hbar$-opers for the adjoint group form an affine scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g},\hbar}(X)$. For $\lambda \in \mathbb{C}^*$ we have the isomorphism of schemes

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g},\hbar}(X) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_{\mathfrak{g},\lambda\hbar}(X) \tag{46}$$

defined by $(\mathfrak{F}_B, \nabla_\hbar) \mapsto (\mathfrak{F}_B, \lambda\nabla_\hbar)$. When $\hbar$ varies $\mathcal{O}\mathfrak{p}_{\mathfrak{g},\hbar}(X)$ become fibers of an affine $\mathbb{C}[\hbar]$-scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)^\sim = \operatorname{Spec} A_{\mathfrak{g}}(X)^\sim$. Using an analog of 3.1.9–3.1.10 for $\mathfrak{g} - \hbar$-opers one shows that $A_{\mathfrak{g}}(X)^\sim$ is flat over $\mathbb{C}[h]$. The morphisms (46) define the action of $\mathbb{G}_m$ on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)^\sim$, i.e., the grading of $A_{\mathfrak{g}}(X)^\sim$. The corresponding filtration on $A_{\mathfrak{g}}(X) = A_{\mathfrak{g}}(X)^\sim/(\hbar - 1)A_{\mathfrak{g}}(X)^\sim$ coincides with the filtration from 3.1.13.

To construct (44) is the same as to construct a $\mathbb{G}_m$-equivariant isomorphism between $\mathcal{O}\mathfrak{p}_{\mathfrak{g},0}(X) = \operatorname{Spec}\operatorname{gr} A_{\mathfrak{g}}(X)$ and $\operatorname{Hitch}_{L\mathfrak{g}}(X) = \operatorname{Spec}\mathfrak{z}^{cl}_{L\mathfrak{g}}(X)$. As explained in 3.1.11 $\operatorname{Hitch}_{L\mathfrak{g}}(X) = \Gamma(X, C'_{\omega_X})$ where $C' = C'_{\mathfrak{g}}$ is defined by (45). We have a canonical mapping of sheaves

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g},0} \longrightarrow C'_{\omega_X} \tag{47}$$

which sends $(\mathfrak{F}_B, \nabla_0)$ to the image of $\nabla_0 \in \mathfrak{g}_{\mathfrak{F}} \otimes \omega_X$ by the projection $\mathfrak{g} \longrightarrow C'$. Theorem 0.10 and Proposition 19 from Kostant's work [Ko63] imply that (47) is a bijection. It induces the desired isomorphism $\mathcal{O}\mathfrak{p}_{\mathfrak{g},0}(X) \xrightarrow{\sim} \Gamma(X, C'_{\omega_X})$.

3.2. **Local opers and Feigin-Frenkel isomorphism.**

3.2.1. Let us replace X by the formal disc $\operatorname{Spec} O$, $O \simeq \mathbb{C}[[t]]$. The constructions and results of 3.1 render easily to this situation. $\mathfrak{g}$-opers on $\operatorname{Spec} O$ form a scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ isomorphic to the spectrum of the polynomial ring in a countable number of variables. More precisely, the isomorphism (43) identifies $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ with an affine space corresponding to the vector space $H^0(\operatorname{Spec} O, V_{\omega_O})$, $V := V_{\mathfrak{g}}$. G-opers on $\operatorname{Spec} O$ form an algebraic stack $\mathcal{O}\mathfrak{p}_G(O)$ isomorphic to $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \times B(Z)$ where $B(Z)$ is the classifying stack of the center $Z \subset G$ and $\mathfrak{g} := \operatorname{Lie}(G/Z)$ (the isomorphism is not quite canonical; see (58) for a canonical description of $\mathcal{O}\mathfrak{p}_G(O)$).

Just as in the global situation (see 3.1.12–3.1.14) the coordinate ring $A_{\mathfrak{g}}(O)$ of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ carries a canonical filtration and we have a canonical

isomorphism

$$\sigma_A : \operatorname{gr} A_{\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}^{cl}_{L_{\mathfrak{g}}}(O) \tag{48}$$

(see (44)). Note that $\operatorname{Aut} O$ acts on all the above objects in the obvious way. So $A_{\mathfrak{g}}(O)$ is a filtered $\operatorname{Aut} O$-algebra and σ_A is an isomorphism of graded $\operatorname{Aut} O$-algebras.

3.2.2. *Theorem.* ([FF92]). There is a canonical isomorphism of filtered $\operatorname{Aut} O$-algebras

$$\varphi_O : A_{\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{L_{\mathfrak{g}}}(O) \tag{49}$$

such that $\sigma_{\mathfrak{z}} \operatorname{gr} \varphi_O = \sigma_A$, where $\sigma_{\mathfrak{z}} : \operatorname{gr} \mathfrak{z}_{L_{\mathfrak{g}}}(O) \to \mathfrak{z}^{cl}_{L_{\mathfrak{g}}}(O)$ is the symbol map. □

Remarks

(i) This isomorphism is uniquely determined by some extra compatibilities; see 3.6.7.

(ii) The original construction of Feigin and Frenkel is representation-theoretic and utterly mysterious (for us). A different, geometric construction is given in ???; the two constructions are compared in ???.

(iii) For $\mathfrak{g} = sl_2$ there is a simple explicit description of (49), which is essentially due to Sugawara; see ???.

3.3. **Global version.**

3.3.1. Let us return to the global situation, so our X is a complete curve. We will construct a canonical isomorphism between the algebras $A_{\mathfrak{g}}(X)$ and $\mathfrak{z}_{L_{\mathfrak{g}}}(X)$ (the latter is defined by formula (27) from 2.7.4).

Take $x \in X$. The restriction of a global $\mathfrak{g}$-oper to $\operatorname{Spec} O_x$ defines a morphism of affine schemes

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O_x).$$

This is a closed embedding, so we have the surjective morphism of coordinate rings

$$\theta_x^A : A_{\mathfrak{g}}(O_x) \longrightarrow\!\!\!\!\!\rightarrow \quad A_{\mathfrak{g}}(X)\,. \tag{50}$$

θ_x^A is strictly compatible with the canonical filtrations (to see this use, e.g., the isomorphism (24)).

3.3.2. *Theorem.* There is a unique isomorphism of filtered algebras

$$\varphi_X : A_{\mathfrak{g}}(X) \xrightarrow{\sim} \mathfrak{z}^{L_{\mathfrak{g}}}(X) \tag{51}$$

such that for any $x \in X$ the diagram

$$\begin{array}{ccc} & \overset{\theta_x^A}{} & \\ A_{\mathfrak{g}}(O_x) & \longrightarrow\!\!\!\!\!\rightarrow & A_{\mathfrak{g}}(X) \\ \Big\downarrow{\scriptstyle \varphi_{O_x}} & & \Big\downarrow{\scriptstyle \varphi_X} \\ & \underset{\theta_x^{\mathfrak{z}}}{} & \\ \mathfrak{z}^{L_{\mathfrak{g}}}(O_x) & \longrightarrow\!\!\!\!\!\rightarrow & \mathfrak{z}^{L_{\mathfrak{g}}}(X) \end{array}$$

commutes (here φ_{O_x} is the isomorphism (49) for $O = O_x$). One has $\sigma_{\mathfrak{z}(X)} \cdot \operatorname{gr} \varphi_X = \sigma_{A(X)}$ where $\sigma_{A(X)}$ is the isomorphism (44) and $\sigma_{\mathfrak{z}(X)} : \operatorname{gr} \mathfrak{z}(X) \longrightarrow \mathfrak{z}^{cl}(X)$ was defined at the end of 2.7.4.

Proof Since θ_x^A and $\theta_x^{\mathfrak{z}}$ are surjective and strictly compatible with filtrations it is enough to show the existence of an isomorphism φ_X such that the diagram commutes. According to 2.6.5 we have a $\mathcal{D}_X$-algebra $\mathcal{A}_{\mathfrak{g}} := A_{\mathfrak{g}}(O)_X$ with fibers $A_{\mathfrak{g}}(O_x)$. Any global oper $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ defines a section $\gamma_{\mathfrak{F}} : X \to \operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$, $\gamma_{\mathfrak{F}}(x)$ is the restriction of $\mathfrak{F}$ to $\operatorname{Spec} O_x$. The sections $\gamma_{\mathfrak{F}}$ are horizontal and this way we get an isomorphism between $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ and the scheme of horizontal sections of $\operatorname{Spec} \mathcal{A}_{\mathfrak{g}}$ (the reader who thinks that this requires a proof can find it in 3.3.3). Passing to coordinate rings we get a canonical isomorphism

$$A_{\mathfrak{g}}(X) \xrightarrow{\sim} H_{\nabla}(X, \mathcal{A}_{\mathfrak{g}}) \tag{52}$$

(see 2.6.2 for the definition of H_∇). On the other hand (49) yields the isomorphism of $\mathcal{D}_X$-algebras

$$\varphi : \mathcal{A}_\mathfrak{g} \xrightarrow{\sim} \mathfrak{z}_{L_\mathfrak{g}},$$

hence the isomorphism

$$H_\nabla(X, \mathcal{A}_\mathfrak{g}) \xrightarrow{\sim} H_\nabla(X, \mathfrak{z}_{L_\mathfrak{g}}) = \mathfrak{z}_{L_\mathfrak{g}}(X)\,. \tag{53}$$

Now φ_X is the composition of (52) and (53). □

3.3.3. In this subsection (which can certainly be skipped by the reader) we prove that $\mathfrak{g}$-opers can be identified with horizontal sections of $\operatorname{Spec}\mathcal{A}_\mathfrak{g}$ (this identification was used in 3.3.2).

Denote by $\mathfrak{g}^+$ the set of all $a \in \mathfrak{g}^{-1}$ such that the image of a in $\mathfrak{g}^\alpha$ is nonzero for any simple negative root α (we use the notation of 3.1.1). $\mathfrak{g}^+$ is an affine scheme. Consider the action of $\operatorname{Aut}^0 O$ on $\mathfrak{g}^+$ via the standard character $\operatorname{Aut}^0 O \to \operatorname{Aut}(tO/t^2O) = \mathbb{G}_m$. Denote by B the Borel subgroup of the adjoint group corresponding to $\mathfrak{g}$. Equip B with the trivial action of $\operatorname{Aut}^0 O$. Applying the functor $\mathcal{J} : \{\operatorname{Aut}^0 O\text{-schemes}\} \to \{\operatorname{Aut} O\text{-schemes}\}$ from 2.6.7 we obtain $\mathcal{J}B$ = the scheme of morphisms $\operatorname{Spec} O \to B$ and $\mathcal{J}\mathfrak{g}^+$ = the scheme of $\mathfrak{g}^+$-valued differential forms on $\operatorname{Spec} O$. The group $\mathcal{J}B$ acts on $\mathcal{J}\mathfrak{g}^+$ by gauge transformations and $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)$ is the quotient scheme. The action of $\mathcal{J}B$ on $\mathcal{J}\mathfrak{g}^+$ and the morphism $\mathcal{J}\mathfrak{g}^+ \to \mathcal{O}\mathfrak{p}_\mathfrak{g}(O)$ are $\operatorname{Aut} O$-equivariant. Actually $\mathcal{J}\mathfrak{g}^+$ is a $\mathcal{J}B$-torsor over $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)$. Moreover, a choice of $\eta \in \omega_O^+ := \omega_O \backslash t\omega_O$ defines its section $S_\eta \subset \mathcal{J}\mathfrak{g}^+$, $S_\eta := \eta \cdot i(f) + V \otimes \omega_O$ (here $f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and i, V were defined in 3.1.9). The fact that S_η is a section is just the local form of Proposition 3.1.10. The sections S_η define an $\operatorname{Aut} O$-equivariant section $s : \mathcal{O}\mathfrak{p}_\mathfrak{g}(O) \times \omega_O^+ \to \mathfrak{g}^+ \times \omega_O^+$ of the induced torsor $\mathfrak{g}^+ \times \omega_O^+ \to \mathcal{O}\mathfrak{p}_\mathfrak{g}(O) \times \omega_O^+$.

Now consider the $\mathcal{D}_X$-schemes $(\mathcal{J}\mathfrak{g}^+)_X$, $(\mathcal{J}B)_X$, and $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X = \operatorname{Spec}\mathcal{A}_\mathfrak{g}$. Clearly $(\mathcal{J}B)_X$ is a group $\mathcal{D}_X$-scheme over X and the scheme $(\mathcal{J}\mathfrak{g}^+)_X$ is a $(\mathcal{J}B)_X$-torsor over $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X$. Actually $(\mathcal{J}B)_X = \mathcal{J}(B_X)$

and $(\mathcal{J}\mathfrak{g}^+)_X$ is the scheme of jets of $\mathfrak{g}^+$-valued differential forms on X. Clearly $\mathcal{O}\mathfrak{p}_\mathfrak{g} = \mathrm{Sect}(\mathfrak{g}^+_X)/\mathrm{Sect}(B_X) = \mathrm{Sect}^\nabla((\mathcal{J}\mathfrak{g}^+)_X)/\mathrm{Sect}^\nabla((\mathcal{J}B)_X) \subset \mathrm{Sect}^\nabla(\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X)$. Here Sect denotes the sheaf of sections of an X-scheme and Sect^∇ denotes the sheaf of horizontal sections of a $\mathcal{D}_X$-scheme. To show that $\mathcal{O}\mathfrak{p}_\mathfrak{g} = \mathrm{Sect}^\nabla(\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X)$ it remains to prove the surjectivity of $\mathrm{Sect}^\nabla((\mathcal{J}\mathfrak{g}^+)_X) \to \mathrm{Sect}^\nabla(\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X)$. To this end use the morphism of $\mathcal{D}_X$-schemes $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X \times (\omega_O^+)_X \to (\mathfrak{g}^+)_X$ induced by s and the fact that $(\omega_O^+)_X$ (i.e., the scheme of jets of non-vanishing differential forms) has a horizontal section in a neighborhood of each point of X.

So we have identified $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X)$ with the set of horizontal sections of $\mathcal{O}\mathfrak{p}_\mathfrak{g}(O)_X = \mathrm{Spec}\,\mathcal{A}_\mathfrak{g}$. In the same way one identifies the scheme $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X)$ with the scheme of horizontal sections of $\mathrm{Spec}\,\mathcal{A}_\mathfrak{g}$.

Remark We used s only to simplify the proof ???.

3.4. **G-opers and $\mathfrak{g}$-opers.** In this subsection we assume that G is semisimple (actually the general case can be treated in a similar way; see Remark iii at the end of 3.4.2). We fix a non-zero $y_\alpha \in \mathfrak{g}^\alpha$ for each negative simple root α. Set $G_{\mathrm{ad}} := G/Z$, $B_{\mathrm{ad}} := B/Z$, $H_{\mathrm{ad}} := H/Z$ where Z is the center of G.

3.4.1. There is an obvious projection $\mathcal{O}\mathfrak{p}_G(X) \longrightarrow \mathcal{O}\mathfrak{p}_\mathfrak{g}(X) := \mathcal{O}\mathfrak{p}_{G_{\mathrm{ad}}}(X)$. We will construct a section $\mathcal{O}\mathfrak{p}_\mathfrak{g}(X) \longrightarrow \mathcal{O}\mathfrak{p}_G(X)$ depending on the choice of a square root of ω_X, i.e., a line bundle $\mathcal{L}$ equipped with an isomorphism $\mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega_X$. Let $(\mathfrak{F}_{B_{\mathrm{ad}}}, \nabla)$ be a $\mathfrak{g}$-oper. Lifting it to a G-oper is equivalent to lifting $\mathfrak{F}_{B_{\mathrm{ad}}}$ to a B-bundle, which is equivalent to lifting $\mathfrak{F}_{H_{\mathrm{ad}}}$ to an H-bundle (here $\mathfrak{F}_{H_{\mathrm{ad}}}$ is the push-forward of $\mathfrak{F}_{B_{\mathrm{ad}}}$ by $B_{\mathrm{ad}} \longrightarrow H_{\mathrm{ad}}$). In the particular case $\mathfrak{g} = sl_2$ we constructed in 3.1.8 a canonical isomorphism $\mathfrak{F}_{H_{\mathrm{ad}}} \xrightarrow{\sim} \omega_X$; the construction from 3.1.8 used a fixed element $f \in sl_2/\mathfrak{b}_0$. Quite similarly one constructs in the general case a canonical isomorphism $\mathfrak{F}_{H_{\mathrm{ad}}} \xrightarrow{\sim} \lambda\omega_X :=$ the push-forward of ω_X by the homomorphism $\lambda : \mathbb{G}_m \longrightarrow H_{\mathrm{ad}}$ such that for any simple positive root α, $\lambda(t)$ acts on $\mathfrak{g}^\alpha$ as multiplication by t (the

construction uses the elements y_α fixed at the beginning of 3.4). There is a unique morphism $\lambda^\# : \mathbb{G}_m \longrightarrow H$ such that

$$\lambda^\#(t) \mod Z = \lambda(t)^2 \tag{54}$$

(Indeed, λ corresponds to the coweight $\check{\rho} :=$ the sum of fundamental coweights, and $2\check{\rho}$ belongs to the coroot lattice). We lift $\mathfrak{F}_{H_{\mathrm{ad}}} = \lambda\omega_X$ to the H-bundle $\lambda^\#\mathcal{L}$ where $\mathcal{L}$ is our square root of ω_X.

3.4.2. Denote by $\omega^{1/2}(X)$ the groupoid of square roots of ω_X. For a fixed $\mathcal{L} \in \omega^{1/2}(X)$ we have an equivalence

$$\Phi_{\mathcal{L}} : \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \times Z\,\mathrm{tors}(X) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_G(X) \tag{55}$$

where $Z\,\mathrm{tors}(X)$ is the groupoid of Z-torsors on X. $\Phi_{\mathcal{L}}(\mathfrak{F}, \mathcal{E})$ is defined as follows: using $\mathcal{L}$ lift $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ to a G-oper (see 3.4.1) and then twist this G-oper by $\mathcal{E}$. $\Phi_{\mathcal{L}}$ depends on $\mathcal{L}$ in the following way:

$$\Phi_{\mathcal{L}\otimes\mathcal{A}}(\mathfrak{F}, \mathcal{E}) = \Phi_{\mathcal{L}}(\mathfrak{F}, \mathcal{E} \cdot \alpha\mathcal{A})$$

Here $\mathcal{A}$ is a square root of $\mathcal{O}_X$ or, which is the same, a μ_2-torsor on X, while $\alpha\mathcal{A}$ is the push-forward of the μ_2-torsor $\mathcal{A}$ by the morphism

$$\alpha : \mu_2 \longrightarrow Z, \qquad \alpha := \lambda^\#|\mu_2 \tag{56}$$

Recall that $\lambda^\#$ is defined by (54).

Remarks

(i) If one considers $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ as a scheme and $\mathcal{O}\mathfrak{p}_G(X)$ and $Z\,\mathrm{tors}(X)$ as algebraic stacks then (55) becomes an isomorphism of algebraic stacks.

(ii) α is the restriction of "the" principal homomorphism $SL_2 \longrightarrow G$ to the center $\mu_2 \subset SL_2$.

(iii) If G is reductive but not semisimple and $\mathfrak{g} := \mathrm{Lie}(G/Z)$ then one defines the section $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathcal{O}\mathfrak{p}_G(X)$ depending on $\mathcal{L} \in \omega^{1/2}(X)$ as the composition $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X) \longrightarrow \mathcal{O}\mathfrak{p}_{[G,G]}(X) \longrightarrow \mathcal{O}\mathfrak{p}_G(X)$. The results of 3.4.2 remain valid if $Z\,\mathrm{tors}(X)$ is replaced by $Z^\nabla\,\mathrm{tors}(X)$, the groupoid of Z-torsors on X equipped with a connection.

3.4.3. Here is a more natural reformulation of 3.4.2. First let us introduce a groupoid $Z\,\mathrm{tors}_\theta(X)$ (θ should remind the reader about θ-characteristics, i.e., square roots of ω_X). The objects of $Z\,\mathrm{tors}_\theta(\mathcal{L})$ are pairs $(\mathcal{E},\mathcal{L})$, $\mathcal{E}\in Z\,\mathrm{tors}(X)$, $\mathcal{L}\in\omega^{1/2}(X)$, but we prefer to write $\mathcal{E}\cdot\mathcal{L}$ instead of $(\mathcal{E},\mathcal{L})$. We set $\mathrm{Mor}(\mathcal{E}_1\cdot\mathcal{L}_1,\mathcal{E}_2\cdot\mathcal{L}_2):=\mathrm{Mor}(\mathcal{E}_1,\mathcal{E}_2\cdot\alpha(\mathcal{L}_2/\mathcal{L}_1))$ where $\alpha(\mathcal{L}_2/\mathcal{L}_1)$ is the push-forward of the μ_2-torsor $\mathcal{L}_2/\mathcal{L}_1:=\mathcal{L}_2\otimes\mathcal{L}_1^{\otimes(-1)}$ by the homomorphism (56). Composition of morphisms is defined in the obvious way. One can reformulate 3.4.2 as a canonical equivalence:

$$\Phi:\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)\times Z\,\mathrm{tors}_\theta(X)\xrightarrow{\sim}\mathcal{O}\mathfrak{p}_G(X) \tag{57}$$

where $\Phi(\mathfrak{F},\mathcal{L}\cdot\mathcal{E}):=\Phi_{\mathcal{L}}(\mathfrak{F},\mathcal{E})$ and $\Phi_{\mathcal{L}}$ is defined by (55).

In the local situation of 3.2.1 one has a similar canonical equivalence

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)\times Z\,\mathrm{tors}_\theta(O)\xrightarrow{\sim}\mathcal{O}\mathfrak{p}_G(O) \tag{58}$$

where $Z\,\mathrm{tors}_\theta(O)$ is defined as in the global case. Of course all the objects of $Z\,\mathrm{tors}_\theta(O)$ are isomorphic to each other and the group of automorphisms of an object of $Z\,\mathrm{tors}_\theta(O)$ is Z. The same is true for $Z\,\mathrm{tors}(O)$. The difference between $Z\,\mathrm{tors}_\theta(O)$ and $Z\,\mathrm{tors}(O)$ becomes clear if one takes the automorphisms of O into account (see 3.5.2).

3.4.4. To describe an "economical" version of $Z\,\mathrm{tors}_\theta(O)$ we need some abstract nonsense.

Let Z be an abelian group. A *Z-structure* on a category C is a morphism $Z\to\mathrm{Aut}\,\mathrm{id}_C$. Equivalently, a Z-structure on C is an action of Z on $\mathrm{Mor}(c_1,c_2)$, $c_1,c_2\in\mathrm{Ob}C$, such that for any morphisms $c_1\xrightarrow{f}c_2\xrightarrow{g}c_3$ and any $z\in Z$ one has $z(gf)=(zg)f=g(zf)$. A *Z-category* is a category with a Z-structure. If C and C' are Z-categories then a functor $F:C\to C'$ is said to be a *Z-functor* if for any $c_1,c_2\in C$ the map $\mathrm{Mor}(c_1,c_2)\to\mathrm{Mor}(F(c_1),F(c_2))$ is Z-equivariant. If $A\to Z$ is a morphism of abelian groups and C is an A-category we define the *induced* Z-category $C\otimes_A Z$ as follows: $\mathrm{Ob}(C\otimes_A Z)=\mathrm{Ob}C$, the set of $(C\otimes_A Z)$-morphisms $c_1\to c_2$

is $(\mathrm{Mor}_C(c_1,c_2)\times Z)/A = \{\text{the } Z\text{-set induced by the } A\text{-set } \mathrm{Mor}_C(c_1,c_2)\}$, and composition of morphisms in $C\otimes_A Z$ is defined in the obvious way. We have the natural A-functor $C\to C\otimes_A Z$ and for any Z-category C' any A-functor $C\to C'$ has a unique decomposition $C\to C\otimes_A Z\overset{F}{\to}C'$ where F is a Z-functor.

Denote by $\omega^{1/2}(O)$ the groupoid of square roots of ω_O. This is a μ_2-category. $Z\,\mathrm{tors}(O)$ and $Z\,\mathrm{tors}_\theta(O)$ are Z-categories. The canonical μ_2-functor $\omega^{1/2}(O)\to Z\,\mathrm{tors}_\theta(O)$ induces an equivalence $\omega^{1/2}(O)\otimes_{\mu_2} Z\to Z\,\mathrm{tors}_\theta(O)$.

3.4.5. The reader may prefer the following "concrete" versions of $Z\,\mathrm{tors}_\theta(X)$ and $Z\,\mathrm{tors}_\theta(O)$. Define an exact sequence

$$0\to Z\to \widetilde{Z}\to \mathbb{G}_m\to 0 \tag{59}$$

as the push-forward of

$$0\to \mu_2\to \mathbb{G}_m\overset{f}{\to}\mathbb{G}_m\to 0\,,\quad f(x):=x^2 \tag{60}$$

by the morphism (56). Denote by $\widetilde{Z}\,\mathrm{tors}_\omega(X)$ the groupoid of liftings of the $\mathbb{G}_m$-torsor ω_X to a $\widetilde{Z}$-torsor (i.e., an object of $\widetilde{Z}\,\mathrm{tors}_\omega(X)$ is a $\widetilde{Z}$-torsor on X plus an isomorphism between the corresponding $\mathbb{G}_m$-torsor and ω_X). The morphism from (60) to (59) induces a functor $F:\ \omega^{1/2}(X)\to \widetilde{Z}\,\mathrm{tors}_\omega(X)$. The functor $Z\,\mathrm{tors}_\theta(X)\to \widetilde{Z}\,\mathrm{tors}_\omega(X)$ defined by

$$\mathcal{E}\cdot\mathcal{L}\mapsto \mathcal{E}\cdot F(\mathcal{L})\,,\quad \mathcal{E}\in Z\,\mathrm{tors}(X)\,,\quad \mathcal{L}\in\omega^{1/2}(X)$$

is an equivalence.

Quite similarly one defines $\widetilde{Z}\,\mathrm{tors}_\omega(O)$ and a canonical equivalence $Z\,\mathrm{tors}_\theta(O)\overset{\sim}{\longrightarrow}\widetilde{Z}\,\mathrm{tors}_\omega(O)$.

The equivalences (57) and (58) can be easily understood in terms of $\widetilde{Z}\,\mathrm{tors}_\omega(X)$ and $\widetilde{Z}\,\mathrm{tors}_\omega(O)$. Let us, e.g., construct the equivalence

$$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)\times \widetilde{Z}\,\mathrm{tors}_\omega(X)\overset{\sim}{\longrightarrow}\mathcal{O}\mathfrak{p}_G(X)\,.$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda^{\#}}\downarrow & & \downarrow{\scriptstyle\lambda} & & \\ 0 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & H_{\mathrm{ad}} & \longrightarrow & 0 \end{array} \tag{61}$$

Here the upper row is (60); the lower row and the morphisms λ, $\lambda^{\#}$ were defined in 3.4.1. According to 3.4.1 a G-oper on X is the same as a $\mathfrak{g}$-oper on X plus a lifting of the H_{ad}-torsor $\lambda_*(\omega_X)$ to an H-torsor. Such a lifting is the same as an object of $\widetilde{Z}\,\mathrm{tors}_{\omega}(X)$: look at the right (Cartesian) square of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \widetilde{Z} & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \\ 0 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & H_{\mathrm{ad}} & \longrightarrow & 0 \end{array} \tag{62}$$

(the upper row of (62) is (59) and the lower rows of (62) and (61) are the same).

3.4.6. $Z\,\mathrm{tors}(X)$ is a (strictly commutative) Picard category (see Definition 1.4.2 from [Del73]) and $Z\,\mathrm{tors}_{\theta}(X)$ is a Torsor over $Z\,\mathrm{tors}(X)$; actually $Z\,\mathrm{tors}_{\theta}(X)$ is induced from the Torsor $\omega^{1/2}(X)$ over $\mu_2\,\mathrm{tors}(X)$ via the Picard functor $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X)$ corresponding to (56). We will use this language in §4, so let us recall the definitions.

A *Picard category* is a tensor category $\mathcal{A}$ in the sense of [De-Mi] (i.e., a symmetric=commutative monoidal category) such that all the morphisms of $\mathcal{A}$ are invertible (i.e., $\mathcal{A}$ is a groupoid) and all the objects of $\mathcal{A}$ are invertible, i.e., for every $a \in \mathrm{Ob}\,\mathcal{A}$ there is an $a' \in \mathrm{Ob}\,\mathcal{A}$ such that $a \cdot a'$ is a unit object (we denote by $\cdot$ the "tensor product" functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$; in [De-Mi] and [Del73] it is denoted respectively by $\otimes$ and $+$). Strict commutativity means that for every $a \in \mathrm{Ob}\,\mathcal{A}$ the commutativity isomorphism $a \otimes a \xrightarrow{\sim} a \otimes a$ is the identity.

An *Action* of a monoidal category $\mathcal{A}$ on a category C is a functor $\mathcal{A} \times C \to C$ (denoted by $\cdot$) equipped with an associativity constraint, i.e., a functorial

isomorphism $(a_1 \cdot a_2) \cdot c \xrightarrow{\sim} a_1 \cdot (a_2 \cdot c)$, $a_i \in \mathcal{A}$, $c \in C$, satisfying the pentagon axiom analogous to the pentagon axiom for the associativity constraint in $\mathcal{A}$ (see [Del73] and [De-Mi]); we also demand the functor $F : C \to C$ corresponding to a unit object of $\mathcal{A}$ to be fully faithful (then the isomorphism $F^2 \xrightarrow{\sim} F$ yields a canonical isomorphism $F \xrightarrow{\sim} \mathrm{id}$). This definition can be found in [Pa] and §3 from [Yet]. An $\mathcal{A}$-*Module* is a category equipped with an Action of $\mathcal{A}$. If C and $\widetilde{C}$ are $\mathcal{A}$-Modules then an $\mathcal{A}$-*Module functor* $C \to \widetilde{C}$ is a functor $\Phi\colon C \to \widetilde{C}$ equipped with a functorial isomorphism $\Phi(a \cdot c) \xrightarrow{\sim} a \cdot \Phi(c)$ satisfying the natural compatibility condition (the two ways of constructing an isomorphism $\Phi((a_1 \cdot a_2) \cdot c) \xrightarrow{\sim} a_1 \cdot (a_2 \cdot \Phi(c))$ must give the same result; see [Pa], [Yet]). $\mathcal{A}$-Module functors are also called *Morphisms* of $\mathcal{A}$-Modules.

A *Torsor* over a Picard category $\mathcal{A}$ is an $\mathcal{A}$-Module such that for some $c \in \mathrm{Ob}\, C$ the functor $a \mapsto a \cdot c$ is an equivalence between $\mathcal{A}$ and C (then this holds for all $c \in \mathrm{Ob}\, C$).

Let $\mathcal{A}$ and $\mathcal{B}$ be Picard categories. A *Picard functor* $\mathcal{A} \to \mathcal{B}$ is a functor $F\colon \mathcal{A} \to \mathcal{B}$ equipped with a functorial isomorphism $F(a_1 \cdot a_2) \xrightarrow{\sim} F(a_1) \cdot F(a_2)$ compatible with the commutativity and associativity constraints. Then F sends a unit object of $\mathcal{A}$ to a unit object of $\mathcal{B}$, i.e., F is a tensor functor in the sense of [De-Mi]. In [Del73] Picard functors are called additive functors.

Let $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ be a Picard functor and C_i a torsor over $\mathcal{A}_i$, $i = 1, 2$. Then C_2 is equipped with an Action of $\mathcal{A}_1$. In this situation $\mathcal{A}_1$-Module functors $C_1 \to C_2$ are called F-*affine* functors.

Examples. 1) Let A be a commutative algebraic group. Then $A\,\mathrm{tors}(X)$ has a canonical structure of Picard category.

2) A morphism $A \to B$ of commutative algeraic groups induces a Picard functor $A\,\mathrm{tors}(X) \to B\,\mathrm{tors}(X)$.

3) The groupoid $\omega^{1/2}(X)$ from 3.4.2 is a Torsor over the Picard category $\mu_2 \operatorname{tors}(X)$. The groupoid $Z\operatorname{tors}_\theta(X)$ from 3.4.3 is a Torsor over $Z\operatorname{tors}(X)$.

If $F\colon \mathcal{A} \to \mathcal{B}$ is a Picard functor between Picard categories and C is a Torsor over $\mathcal{A}$ then we can form the *induced* Torsor $\mathcal{B} \cdot_{\mathcal{A}} C$ over $\mathcal{B}$. The defintion of $\mathcal{B} \cdot_{\mathcal{A}} C$ can be reconstructed by the reader from the following example (see 3.4.3): if $\mathcal{A} = \mu_2 \operatorname{tors}(X)$, $\mathcal{B} = Z\operatorname{tors}(X)$, F comes from (56), and $C = \omega^{1/2}(X)$ then $\mathcal{B} \cdot_{\mathcal{A}} C = Z\operatorname{tors}_\theta(X)$. The objects of $\mathcal{B} \cdot_{\mathcal{A}} C$ are denoted by $b \cdot c$, $b \in \operatorname{Ob}\mathcal{B}$, $c \in \operatorname{Ob} C$ (see 3.4.3).

The interested reader can formulate the universal property of $\mathcal{B} \cdot_{\mathcal{A}} C$. We need the following weaker property. Given a category $\widetilde{C}$ with an Action of $\mathcal{B}$ and an $\mathcal{A}$-Module functor $\Phi\colon C \to \widetilde{C}$ there is a natural way to construct a $\mathcal{B}$-Module functor $\Psi\colon \mathcal{B} \cdot_{\mathcal{A}} C \to \widetilde{C}$: set $\Psi(b \cdot c) := b \cdot \Phi(c)$, and define Ψ on morphisms in the obvious way (i.e., the map $\operatorname{Mor}(b_1 \cdot c_1, b_2 \cdot c_2) \to \operatorname{Mor}(b_1 \cdot \Phi(c_1), b_2 \cdot \Phi(c_2))$ is the composition $\operatorname{Mor}(b_1 \cdot c_1, b_2 \cdot c_2) = \operatorname{Mor}(b_1, b_2 \cdot c_2/c_1) \to \operatorname{Mor}(b_1 \cdot \Phi(c_1), b_2 \cdot c_2/c_1 \cdot \Phi(c_1)) \xrightarrow{\sim} \operatorname{Mor}(b_1 \cdot \Phi(c_1), b_2 \cdot \Phi(c_2)))$. The isomorphism $\Psi(b_1 \cdot (b_2 \cdot c)) \xrightarrow{\sim} b_1 \cdot \Psi(b_2 \cdot c)$ is the obvious one.

We will use this construction in the following situation. Suppose we have a Picard functor $\ell\colon \mathcal{B} \to \widetilde{\mathcal{B}}$, a Torsor $\widetilde{C}$ over $\widetilde{\mathcal{B}}$, and an ℓ'-affine functor $\Phi\colon C \to \widetilde{C}$ where ℓ' is the composition $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{\ell} \widetilde{\mathcal{B}}$. Then the above construction yields an ℓ-affine functor $\mathcal{B} \cdot_{\mathcal{A}} C \to \widetilde{C}$.

3.4.7. Let Z be an abelian group and $Z\operatorname{tors}$ the Picard category of Z-torsors (over a point). The following remarks will be used in 4.4.9.

Remarks

(i) A Picard functor from $Z\operatorname{tors}$ to a Picard category $\mathcal{A}$ is "the same as" a morphism $Z \to \operatorname{Aut} 1_{\mathcal{A}}$ where $1_{\mathcal{A}}$ is the unit object of $\mathcal{A}$. More precisely, the natural functor from the category of Picard functors $Z\operatorname{tors} \to \mathcal{A}$ to $\operatorname{Hom}(Z, \operatorname{Aut} 1_{\mathcal{A}})$ is an equivalence. Here the set $\operatorname{Hom}(Z, \operatorname{Aut} 1_{\mathcal{A}})$ is considered as a discrete category.

(ii) The previous remark remains valid if "Picard" is replaced by "monoidal".

(iii) An Action of $Z\,\mathrm{tors}$ on a category C is "the same as" a Z-structure on C, i.e., a morphism $Z \to \operatorname{Aut}\mathrm{id}_C$ (notice that an Action of a monoidal category $\mathcal{A}$ on C is the same as a monoidal functor $\mathcal{A} \to \mathrm{Funct}(C, C)$ and apply the previous remark).

(iv) Let C_1 and C_2 be Modules over $Z\,\mathrm{tors}$. According to the previous remark C_1 and C_2 are Z-categories in the sense of 3.4.4. It is easy to see that a $(Z\,\mathrm{tors})$-Module functor $C_1 \to C_2$ is the same as a Z-functor in the sense of 3.4.4 (i.e., a functor $F : C_1 \to C_2$ has at most one structure of a $(Z\,\mathrm{tors})$-Module functor and such a structure exists if and only if F is a Z-functor).

(v) A Torsor over $Z\,\mathrm{tors}$ is "the same as" a Z-category which is Z-equivalent to $Z\,\mathrm{tors}$. (do we need this???)

3.5. **Local opers II.** For most of the Lie algebras $\mathfrak{g}$ (e.g., $\mathfrak{g} = sl_n$, $n > 5$) the Feigin-Frenkel isomorphism (49) is *not uniquely* determined by the properties mentioned in Theorem 3.2.2 because $A_{\mathfrak{g}}(O)$ has a lot of $\operatorname{Aut} O$-equivariant automorphisms inducing the identity on $\operatorname{gr} A_{\mathfrak{g}}(O)$; this is clear from the geometric description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Spec} A_{\mathfrak{g}}(O)$ in 3.2.1 or from the description of $A_{\mathfrak{g}}(O)$ that will be given in 3.5.6 (see (65)–(68)). The goal of 3.5–3.6 is to formulate the property 3.6.7 that uniquely determines the Feigin-Frenkel isomorphism. This property and also 3.6.11 will be used in the proof of our main theorem 5.2.6. In 3.7 and 3.8 we explain how to extract the properties 3.6.7 and 3.6.11 from [FF92]. One may (or perhaps should) read §4 and (a large part of ?) §5 before 3.5–3.8. We certainly recommend the reader to skip 3.5.16–3.5.18 and 3.6.8–3.6.11 before 3.6.11 is used in ??.

The idea[17] of 3.5 and 3.6 is to "kill" the automorphisms of $A_{\mathfrak{g}}(O)$ and its counterpart $\mathfrak{z}^L{}_{\mathfrak{g}}(O)$ by equipping these algebras with certain additional

[17]Inspired by [Phys]

structures. In the case of $A_{\mathfrak{g}}(O)$ this is the Lie algebroid $\mathfrak{a}_{\mathfrak{g}}$ from 3.5.11. Its counterpart for $\mathfrak{z}^{L_{\mathfrak{g}}}(O)$ is introduced in 3.6.5. The definition of $\mathfrak{a}_{\mathfrak{g}}$ is simple: this is the algebroid of infinitesimal symmetries of the tautological G-bundle $\mathfrak{F}_G^0$ on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$. $\mathfrak{F}_G^0$ and therefore $\mathfrak{a}_{\mathfrak{g}}$ are equipped with an action of $\operatorname{Der} O$. It turns out that the pair $(A_{\mathfrak{g}}(O), \mathfrak{F}_G^0)$ has no nontrivial $\operatorname{Der} O$-equivariant automorphisms (see 3.5.9) and this is "almost" true for $(A_{\mathfrak{g}}(O), \mathfrak{a}_{\mathfrak{g}})$ (see 3.5.13).

3.5.1. We have a universal family of $\mathfrak{g}$-opers on $\operatorname{Spec} O$ parametrized by the scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Spec} A_{\mathfrak{g}}(O)$ from 3.2.1. Fix a one-dimensional free O-module $\omega_O^{1/2}$ equipped with an isomorphism $\omega_O^{1/2} \otimes \omega_O^{1/2} \xrightarrow{\sim} \omega_O$ (of course $\omega_O^{1/2}$ is unique up to isomorphism). Then the above universal family lifts to a family of G-opers; see 3.4.1[18]. So we have a B-bundle $\mathfrak{F}_B$ on $\operatorname{Spec}(A_{\mathfrak{g}}(O)\widehat{\otimes}O) = \operatorname{Spec} A_{\mathfrak{g}}(O)[[t]]$ and a connection ∇ along $\operatorname{Spec} O$ on the associated G-bundle $\mathfrak{F}_G$.

3.5.2. Consider the group ind-scheme $\operatorname{Aut}_2 O := \operatorname{Aut}(O, \omega_O^{1/2})$. We have a canonical exact sequence

$$0 \to \mu_2 \to \operatorname{Aut}_2 O \to \operatorname{Aut} O \to 0 \tag{63}$$

and $\operatorname{Aut}_2 O$ is connected. The exact sequence (63) and the connectedness property can be considered as another definition of $\operatorname{Aut}_2 O$. Denote by $\operatorname{Aut}_2^0 O$ the preimage of $\operatorname{Aut}^0 O$ in $\operatorname{Aut}_2 O$.

$\operatorname{Aut} O$ acts on $A_{\mathfrak{g}}(O)$ and O, so it acts on $\operatorname{Spec}(A_{\mathfrak{g}}(O)\widehat{\otimes}O)$. This action lifts canonically to an action of $\operatorname{Aut}_2 O$ on $(\mathfrak{F}_B, \nabla)$. $\mu_2 \subset \operatorname{Aut}_2 O$ acts on $\mathfrak{F}_B$ via the morphism (56).

3.5.3. *Lemma.* Let L be an algebraic group, A an algebra equipped with an action of $\operatorname{Aut} O$. Consider the action of $\operatorname{Aut} O$ on $A\widehat{\otimes}O$ induced by its actions on A and O. Let $i : \operatorname{Spec} A \hookrightarrow \operatorname{Spec}(A\widehat{\otimes}O)$ be the natural embedding and $\pi : \operatorname{Spec}(A\widehat{\otimes}O) \to \operatorname{Spec} A$ the projection.

[18]To tell the truth we must also choose a non-zero $y_\alpha \in \mathfrak{g}^\alpha$ for each negative simple root α (see 3.4.1)

1) i^* is an equivalence between the category of $\mathrm{Aut}_2 O$-equivariant L-bundles on $\mathrm{Spec}(A\widehat{\otimes}O)$ and that of $\mathrm{Aut}_2^0 O$-equivariant L-bundles on $\mathrm{Spec}\, A$.

2) π^* is an equivalence between the category of $\mathrm{Aut}_2 O$-equivariant L-bundles on $\mathrm{Spec}\, A$ and that of $\mathrm{Aut}_2 O$-equivariant L-bundles on $\mathrm{Spec}(A\widehat{\otimes}O)$ equipped with an $\mathrm{Aut}_2 O$-invariant connection along $\mathrm{Spec}\, O$.

3) These equivalences are compatible with the forgetful functors $\{\mathrm{Aut}_2 O$-equivariant bundles on $\mathrm{Spec}\, A\} \to \{\mathrm{Aut}_2^0 O$-equivariant bundles on $\mathrm{Spec}\, A\}$ and $\{$bundles with connection$\} \to \{$bundles$\}$. □

3.5.4. Denote by $\mathfrak{F}_B^0$ and $\mathfrak{F}_G^0$ the restrictions of $\mathfrak{F}_B$ and $\mathfrak{F}_G$ to $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \mathrm{Spec}\, A_{\mathfrak{g}}(O) \subset \mathrm{Spec}\, A_{\mathfrak{g}}(O)\widehat{\otimes}O$. $\mathfrak{F}_B^0$ is a B-bundle on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ and $\mathfrak{F}_G^0$ is the corresponding G-bundle. $\mathfrak{F}_B^0$ is $\mathrm{Aut}_2^0 O$-equivariant and according to 3.5.3 $\mathfrak{F}_G^0$ is $\mathrm{Aut}_2 O$-equivariant. Since the connection ∇ on $\mathfrak{F}_G$ does not preserve $\mathfrak{F}_B$ the action of $\mathrm{Aut}_2 O$ on $\mathfrak{F}_G^0$ does not preserve $\mathfrak{F}_B^0$. According to 3.5.3 $\mathfrak{F}_G^0$ equipped with the action of $\mathrm{Aut}_2 O$ and the B-structure $\mathfrak{F}_B^0 \subset \mathfrak{F}_G^0$ "remembers"the universal oper $(\mathfrak{F}_B, \nabla)$.

3.5.5. Denote by F_H^0 the H-bundle on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ corresponding to $\mathfrak{F}_B^0$. Since $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is an (infinite dimensional) affine space any H-bundle on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is trivial and the action of H on the set of its trivializations is transitive. In particular this applies to $\mathfrak{F}_H^0$, so $\mathfrak{F}_H$ is the pullback of some H-bundle F_H over $\mathrm{Spec}\, \mathbb{C}$. According to 3.4.1 F_H is the pushforward of the $\mathbb{G}_m$-bundle $\omega_O^{1/2}/t\omega_O^{1/2}$ over $\mathrm{Spec}\, \mathbb{C}$ via the morphism $\lambda^{\#} : \mathbb{G}_m \to H$ defined by (54). In particular the action of $\mathrm{Aut}_2^0 O$ on F_H comes from the composition

$$\mathrm{Aut}_2^0 O \to \mathrm{Aut}(\omega_O^{1/2}/t\omega_O^{1/2}) = \mathbb{G}_m \xrightarrow{\lambda^{\#}} H\,.$$

So the action of $\mathrm{Der}^0 O$ on F_H is the sum of the "obvious" action (the one which preserves any trivialization of F_H) and the morphism $\mathrm{Der}^0 O \to \mathfrak{h}$ defined by $f(t) \cdot t\dfrac{d}{dt} \mapsto f(0)\check{\rho}$. Here $\check{\rho}$ is the sum of fundamental coweights.

3.5.6. Here is an explicit description of $A_{\mathfrak{g}}(O)$ and $\mathfrak{F}_G^0$ in the spirit of 3.1.9–3.1.10. Let $e, f \in sl_2$ be the matrices from 3.1.8. Fix a principal embedding

$i : sl_2 \hookrightarrow \mathfrak{g}$ such that $i(e) \in \mathfrak{b}$. If a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ is chosen so that $i([e,f]) \in \mathfrak{h}$ then $i([e,f])$ can be identified with $2\check{\rho}$. Just as in 3.1.9 set $V := \operatorname{Ker} \operatorname{ad} i(e)$. Choose a basis $e_1, \dots, e_r \in V$ so that $e_1 = i(e)$ and all e_j are eigenvectors of $\operatorname{ad} \check{\rho}$. In fact $[\check{\rho}, e_j] = (d_j - 1)e_j$ where d_j are the degrees of "basic" invariant polynomials on $\mathfrak{g}$ (in particular $d_1 = 2$). The connection

$$\nabla_{\frac{d}{dt}} = \frac{d}{dt} + i(f) + u_1(t)e_1 + \dots u_r(t)e_r \tag{64}$$

on the trivial G-bundle defines a $\mathfrak{g}$-oper and according to 3.1.10 this is a bijection between $\mathfrak{g}$-opers on $\operatorname{Spec} O$, $O := \mathbb{C}[[t]]$, and r-tuples $(u_1(t), \dots, u_r(t))$, $u_j(t) \in \mathbb{C}[[t]]$. Write $u_j(t)$ as $u_{j0} + u_{j1}t + \dots$. Then $A_{\mathfrak{g}}(O)$ is the ring of polynomials in u_{jk}, $1 \le j \le r$, $0 \le k < \infty$. The bundles $\mathfrak{F}_B$, $\mathfrak{F}_G$, $\mathfrak{F}_B^0$, $\mathfrak{F}_G^0$ from 3.5.1 and 3.5.4 are trivial and we have trivialized them by choosing the canonical form (64) for opers.

To describe the action of $\operatorname{Der} O$ on $A_{\mathfrak{g}}(O)$ and $\mathfrak{F}_G^0$ introduce the standard notation $L_n := -t^{n+1}\dfrac{d}{dt} \in \operatorname{Der} \mathbb{C}((t))$ (so $L_n \in \operatorname{Der} O$ for $n \ge -1$). Set $u_j := u_{j0}$. Then

$$u_{jk} = (L_{-1})^k u_j / k! \tag{65}$$

$$L_0 u_j = d_j u_j \tag{66}$$

$$L_n u_j = 0 \quad \text{if } n > 0\,, \quad j \ne 1 \tag{67}$$

$$L_n u_1 = 0 \quad \text{if } n > 0\,,\ n \ne 2\,; \quad L_2 u_1 = -3\,. \tag{68}$$

So $A_{\mathfrak{g}}(O)$ is the commutative $(\operatorname{Der} O)$-algebra with generators $u_1, \dots, u_r$ and defining relations (66)–(68). Denote by L_n^{hor} the vector field on $\mathfrak{F}_G^0$ that comes from our trivialization of $\mathfrak{F}_G^0$ and the action of $\operatorname{Der} O$ on $A_{\mathfrak{g}}(O)$. L_n acts on $\mathfrak{F}_G^0$ as $L_n^{\mathrm{hor}} + M_n$, $M_n \in \mathfrak{g} \otimes A_{\mathfrak{g}}(O)$. One can show that

$$M_0 = -\check{\rho} \tag{69}$$

$$M_1 = -i(e)\,, \quad M_n = 0 \quad \text{for } n > 1 \tag{70}$$

$$M_{-1} = i(f) + u_1 e_1 + \ldots + u_r e_r$$

Only (69) will be used in the sequel (I'm afraid we'll use at least (70)) !???!).

Remark. If $n \geq 0$ then $M_n \in i(\mathfrak{b}_0) \subset \mathfrak{b} \subset \mathfrak{b} \otimes A_{\mathfrak{g}}(O)$ where $\mathfrak{b}_0 := \operatorname{Lie} B_0$ and $B_0 \subset SL_2$ is the group of upper-triangular matrices. This means that we have identified the $\operatorname{Aut}_2^0 O$-equivariant bundle $\mathfrak{F}_B^0$ with the pullback of a certain $\operatorname{Aut}_2^0 O$-equivariant B-bundle on $\operatorname{Spec}\mathbb{C}$ and the latter comes from a certain morphism $\operatorname{Aut}_2^0 O \to B_0 \to B$ (cf. Remark (iii) from 3.1.10).

3.5.7. Let A be an algebra equipped with an action of $\operatorname{Aut} O$. Then $\operatorname{Der} O$ acts on A, the action of L_0 on A is diagonalizable, and the eigenvalues of $L_0 : A \to A$ are integers. Assume that the eigenvalues of $L_0 : A/\mathbb{C} \to A/\mathbb{C}$ are positive. Then $A = \mathbb{C} \oplus A_+$ where A_+ is the sum of all eigenspaces of L_0 in A corresponding to positive eigenvalues. A_+ is the unique L_0-invariant maximal ideal of A. The corresponding point of $\operatorname{Spec} A$ will be denoted by 0. Since $[L_0, L_n] = -nL_n$ we have $L_{-1}A_+ \subset A_+$. Assume that

$$L_1 A_+ \subset A_+\,. \tag{71}$$

In particular (71) is satisfied if the eigenvalues of L_0 on $A/\mathbb{C}$ are greater than 1, e.g., for $A = A_{\mathfrak{g}}(O)$ (see (66) and (65)).

Assume that G is the adjoint group corresponding to $\mathfrak{g}$. Let $\mathcal{E}$ be an $\operatorname{Aut} O$-equivariant G-bundle on $\operatorname{Spec} A$. The algebra $\mathbb{C}L_{-1} + \mathbb{C}L_0 + \mathbb{C}L_1 \simeq sl_2$ stabilizes $0 \in \operatorname{Spec} A$, so it acts on the fiber of $\mathcal{E}$ over 0. Thus we obtain a morphism $\sigma : sl_2 \to \mathfrak{g}$ defined up to conjugation.

Example. The point $0 \in \operatorname{Spec} A_{\mathfrak{g}}(O)) = \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ is the push-forward via the principal embedding $sl_2 \to \mathfrak{g}$ of the sl_2-oper corresponding to the Sturm-Liouville operator $(d/dt)^2$. If $A = A_{\mathfrak{g}}(O)$ and $\mathcal{E} = \mathfrak{F}_G^0$ then σ is the principal embedding.

3.5.8. *Proposition.*

1) The following conditions are equivalent:
   a) the $\operatorname{Aut} O$-equivariant G-bundle $\mathcal{E}$ is isomorphic to $\varphi^*\mathfrak{F}^0_G$ for some $\operatorname{Aut} O$-equivariant $\varphi : \operatorname{Spec} A \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$;
   b) there is an $\operatorname{Aut}^0 O$-invariant B-structure on $\mathcal{E}$ such that the corresponding $\operatorname{Aut}^0 O$-equivariant H-bundle is isomorphic to the pullback of the $\operatorname{Aut}^0 O$-equivariant H-bundle F_H on $\operatorname{Spec}\mathbb{C}$ defined in 3.5.5[19];
   c) $\sigma : sl_2 \to \mathfrak{g}$ is the principal embedding.
2) The morphism φ and the isomorphism $\mathcal{E} \xrightarrow{\sim} \varphi^*\mathfrak{F}^0_G$ mentioned in a) are unique.
3) The B-structure mentioned in b) is unique.

Proof. According to 3.5.5 b) follows from a). To deduce c) from b) just look what happens over $0 \in \operatorname{Spec} A$. Let us deduce a) from b) and show that 2) follows from 3). To do this it suffices to show that if a B-structure $\mathcal{E}_B \subset \mathcal{E}$ with the property mentioned in b) is fixed there is exactly one way to construct $\operatorname{Aut} O$-equivariant $\varphi : \operatorname{Spec} A \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ and $f : \mathcal{E} \xrightarrow{\sim} \varphi^*\mathfrak{F}^0_G$ so that $f(\mathcal{E}_B) = \varphi^*\mathfrak{F}^0_B$. According to 3.5.3 $\mathcal{E}$ and $\mathcal{E}_B$ yield a G-bundle $\widetilde{\mathcal{E}}_G$ on $\operatorname{Spec}(A\widehat{\otimes}O)$ with a B-structure $\widetilde{\mathcal{E}}_B \subset \widetilde{\mathcal{E}}_G$, a connection ∇ on $\widetilde{\mathcal{E}}_G$ along $\operatorname{Spec} O$, and an action of $\operatorname{Aut} O$ on $(\widetilde{\mathcal{E}}_G, \widetilde{\mathcal{E}}_B, \nabla)$. Now the uniqueness of φ and f is clear and to prove their existence we must show that $(\widetilde{\mathcal{E}}_G, \widetilde{\mathcal{E}}_B, \nabla)$ is a family of opers, i.e., we must prove that $c(\nabla)$ defined in 3.1.2 satisfies conditions 1 and 2 from Definition 3.1.3. In our situation $c(\nabla)$ is an $\operatorname{Aut} O$-invariant section of $(\mathfrak{g}/\mathfrak{b})_{\widetilde{\mathcal{E}}} \bigotimes_O \omega_O$ and it is enough to verify conditions 1 and 2 for its restriction $c_0(\nabla)$ to $\operatorname{Spec} A \subset \operatorname{Spec} A\widehat{\otimes}O$. $c_0(\nabla)$ is an $\operatorname{Aut}^0 O$-invariant element of $H^0(\operatorname{Spec} A, (\mathfrak{g}/\mathfrak{b})_{\mathcal{E}_B}) \otimes \omega_O/t\omega_O$. Let $\operatorname{gr}^k \mathfrak{g}$ have the same meaning as in 3.1.1. Since we know the H-bundle corresponding to $\mathcal{E}_B$ we

[19]Since G is the adjoint group the action of $\operatorname{Aut}^0_2 O$ on F_H factors through $\operatorname{Aut}^0 O$

see that there is an $\mathrm{Aut}^0 O$-equivariant isomorphism

$$H^0(\operatorname{Spec} A, \operatorname{gr}^k \mathfrak{g}_{\mathcal{E}_B}) \otimes \omega_O/t\omega_O \xrightarrow{\sim} A \otimes (\omega_O/t\omega_O)^{\otimes(k+1)} \otimes \operatorname{gr}^k \mathfrak{g}\,. \tag{72}$$

Since L_0 acts on $(\omega_O/t\omega_O)^{\otimes(k+1)}$ as multiplication by $-k-1$ the $\mathrm{Aut}^0 O$-invariant part of $A \otimes (\omega_O/t\omega_O)^{\otimes(k+1)}$ equals 0 if $k < -1$ and $\mathbb{C}$ if $k = -1$. Therefore

$$c_0(\nabla) \in \operatorname{gr}^{-1} \mathfrak{g} \subset A \otimes \operatorname{gr}^{-1} \mathfrak{g} = H^0(\operatorname{Spec} A, \operatorname{gr}^{-1} \mathfrak{g}_{\mathcal{E}_B}) \otimes \omega_O/t\omega_O\,.$$

So we have checked condition 1 from 3.1.3 and it remains to check condition 2 over some point of $\operatorname{Spec} A$, e.g., over $0 \in \operatorname{Spec} A$. Denote by $(\widetilde{\mathcal{E}}^0_G, \widetilde{\mathcal{E}}^0_B, \nabla)$ the restriction of $(\widetilde{\mathcal{E}}_G, \widetilde{\mathcal{E}}_B, \nabla)$ to $\{0\} \times \operatorname{Spec} O \subset \operatorname{Spec}(A \widehat{\otimes} O)$. Then $\widetilde{\mathcal{E}}^0_G$ is the trivial G-bundle, ∇ is the trivial connection, sl_2 acts on $(\widetilde{\mathcal{E}}^0_G, \nabla)$ via the morphism $\sigma : sl_2 \to \mathfrak{g}$ mentioned in 3.5.7 and the embedding $sl_2 = \mathbb{C}L_{-1} + \mathbb{C}L_0 + \mathbb{C}L_1 \hookrightarrow \operatorname{Der} O$, $\widetilde{\mathcal{E}}^0_B$ is invariant with respect to sl_2. Since σ is the principal embedding $(\widetilde{\mathcal{E}}^0_G, \widetilde{\mathcal{E}}^0_B, \nabla)$ is the oper corresponding to $0 \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$.

Let us prove 3). Set $\mathfrak{a} = H^0(\operatorname{Spec} A, \mathfrak{g}_{\mathcal{E}})$, $\mathfrak{a}_k := \{a \in \mathfrak{a} | L_0 a = ka\}$. If a B-structure on $\mathcal{E}$ is fixed then the filtration $\mathfrak{g}^k$ from 3.1.1 induces a filtration $\mathfrak{a}^k$ on $\mathfrak{a}$. If the B-structure has the property mentioned in b) then $\mathfrak{a}^k$ is $\mathrm{Aut}^0 O$-invariant and $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ is $\mathrm{Aut}^0 O$-isomorphic to $A \otimes (\omega_O/t\omega_O)^{\otimes k} \otimes \operatorname{gr}^k \mathfrak{g}$ (see (72)). Therefore the eigenvalues of L_0 on $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ are $\geq -k$ and the A-module $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ is generated by its L_0-eigenvectors corresponding to the eigenvalue $-k$. So

$$\mathfrak{a}^k = \sum_{i \leq -k} A\mathfrak{a}_i\,. \tag{73}$$

The B-structure on $\mathcal{E}$ is reconstructed from the Borel subalgebra $\mathfrak{a}^0 \subset \mathfrak{a}$.

It remains to deduce b) from c). Define $\mathfrak{a}^k$ by (73). Since $\mathfrak{a}$ is a free L_0-graded A-module of finite type so are $\mathfrak{a}^k/\mathfrak{a}^{k+1}$. Therefore $\mathfrak{a}^k$ defines a vector subbundle of $\mathfrak{g}_{\mathcal{E}}$. If $k = 0$ this subbundle is a Lie subalgebra, so it defines a section $s : \operatorname{Spec} A \to S_{\mathcal{E}}$ where S is the scheme of subalgebras of $\mathfrak{g}$. An infinitesimal calculation shows that the morphism $G/B \to S$, $g \mapsto g\mathfrak{b}g^{-1}$, is

an open embedding and since G/B is projective it is also a closed embedding. According to c) $s(0) \in (G/B)_{\mathcal{E}} \subset S_{\mathcal{E}}$, so $s(\operatorname{Spec} A) \subset (G/B)_{\mathcal{E}}$ and s defines a B-structure on $\mathcal{E}$. Clearly it is $\operatorname{Aut}^0 O$-invariant. The corresponding $\operatorname{Aut}^0 O$-equivariant H-bundle on $\operatorname{Spec} A$ is the pullback of some $\operatorname{Aut}^0 O$-equivariant H-bundle F on $\operatorname{Spec} \mathbb{C}$ (this is true for *any* $\operatorname{Aut}^0 O$-equivariant H-bundle on $\operatorname{Spec} A$ and *any* torus H; indeed, one can assume that $H = \mathbb{G}_m$, interpret a $\mathbb{G}_m$-bundle as a line bundle and use the fact that a graded projective A-module of finite type is free). To find F look what happens over $0 \in \operatorname{Spec} A$. □

Remark. The proof of Proposition 3.5.8 shows that if c) is satisfied then there is a unique $\operatorname{Aut}^0 O$-invariant B-structure on $\mathcal{E}$.

3.5.9. *Corollary.* If G is the adjoint group then the pair $(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O), \mathfrak{F}^0_G)$ has no nontrivial $\operatorname{Aut} O$-equivariant automorphisms.

This is statement 2) of Proposition 3.5.8 for $A = A_{\mathfrak{g}}(O)$.

3.5.10. Recall that a *Lie algebroid* over a commutative $\mathbb{C}$-algebra R is a Lie $\mathbb{C}$-algebra $\mathfrak{a}$ equipped with an R-module structure and a map $\varphi : \mathfrak{a} \to \operatorname{Der} R$ such that 1) φ is a Lie algebra morphism and an R-module morphism, 2) for $a_1, a_2 \in \mathfrak{a}$ and $f \in R$ one has $[a_1, fa_2] = f[a_1, a_2] + v(f)a_2$, $v := \varphi(a_1)$.

Remarks

(i) [Ma87] and [Ma96] are standard references on Lie algebroids and Lie groupoids. See also [We] and [BB93]. In this paper we need only the definition of Lie algebroid.

(ii) Lie algebroids are also known under the name of ($\mathbb{C}$, R)-Lie algebras (see [R]) and under a variety of other names (see [Ma96]).

3.5.11. Denote by $\mathfrak{a}_{\mathfrak{g}}$ the space of (global) infinitesimal symmetries of $\mathfrak{F}^0_G$. Elements of $\mathfrak{a}_{\mathfrak{g}}$ are pairs consisting of a vector field on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Spec} A_{\mathfrak{g}}(O)$ (i.e., a derivation of $A_{\mathfrak{g}}(O)$) and its lifting to a G-invariant vector field on the principal G-bundle $\mathfrak{F}^0_G$. $\mathfrak{a}_{\mathfrak{g}}$ is a Lie algebroid over $A_{\mathfrak{g}}(O)$. We have a

canonical exact sequence.

$$0 \to \mathfrak{g}_{\mathrm{univ}} \to \mathfrak{a}_{\mathfrak{g}} \to \operatorname{Der} A_{\mathfrak{g}}(O) \to 0$$

where $\mathfrak{g}_{\mathrm{univ}}$ is the space of global sections of the $\mathfrak{F}_G^0$-twist of $\mathfrak{g}$. Of course $\mathfrak{a}_{\mathfrak{g}}$ and $\mathfrak{g}_{\mathrm{univ}}$ do not change if G is replaced by the adjoint group G_{ad}. So $\mathfrak{a}_{\mathfrak{g}}$ and $\mathfrak{g}_{\mathrm{univ}}$ do not depend on the choice of $\omega_O^{1/2}$.

The action of $\operatorname{Der} O$ on $\mathfrak{F}_G^0$ induces a Lie algebra morphism $\operatorname{Der} O \to \mathfrak{a}_{\mathfrak{g}}$. In particular $\operatorname{Der} O$ acts on $\mathfrak{a}_{\mathfrak{g}}$.

3.5.12. *Lemma.* The adjoint representation of $\mathfrak{a}_{\mathfrak{g}}$ on $\mathfrak{g}_{\mathrm{univ}}$ defines an isomorphism between $\mathfrak{a}_{\mathfrak{g}}$ and the algebroid of infinitesimal symmetries of $\mathfrak{g}_{\mathrm{univ}}$. □

3.5.13. *Proposition.* The group of $\operatorname{Der} O$-equivariant automorphisms of the pair $(A_{\mathfrak{g}}(O), \mathfrak{a}_{\mathfrak{g}})$ equals $\operatorname{Aut}\Gamma$ where Γ is the Dynkin graph of $\mathfrak{g}$.

Proof. Let G be the adjoint group corresponding to $\mathfrak{g}$. Denote by L the group of $\operatorname{Der} O$-equivariant automorphisms of $(A_{\mathfrak{g}}(O), \mathfrak{g}_{\mathrm{univ}})$. According to 3.5.12 we have to show that $L = \operatorname{Aut}\Gamma$. We have the obvious morphisms $i : \operatorname{Aut}\Gamma = \operatorname{Aut}(G, B)/B \to L$ and $\pi : L \to \operatorname{Aut}\Gamma$ such that $\pi i = \mathrm{id}$. $\operatorname{Ker}\pi$ is the group of $\operatorname{Der} O$-equivariant automorphisms of $(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O), \mathfrak{F}_G^0)$, so $\operatorname{Ker}\pi$ is trivial according to 3.5.9. □

3.5.14. *Proposition.* The pair $(A_{\mathfrak{g}}(O), \mathfrak{a}_{\mathfrak{g}})$ does not have nontrivial $\operatorname{Der} O$-equivariant automorphisms inducing the trivial automorphism of $\operatorname{gr} A_{\mathfrak{g}}(O)$ (gr corresponds to the filtration from 3.2.1).

Proof. Let Γ be the Dynkin graph of $\mathfrak{g}$. According to 3.5.13 and (48) we have to show that the action of $\operatorname{Aut}\Gamma$ on the algebra $\mathfrak{z}^{cl}_{L_{\mathfrak{g}}}(O)$ from 2.7.1 is exact. So it suffices to show that the action of $\operatorname{Aut}\Gamma$ on $W\backslash\mathfrak{h}$ is exact (W denotes the Weyl group). Let $C \subset \operatorname{Aut}\mathfrak{h}$ be the automorphism group of the root system. There is an $a \in \mathfrak{h}$ whose stabilizer in C is trivial. So the action of $\operatorname{Aut}\Gamma = C/W$ on $W\backslash\mathfrak{h}$ is exact. □

3.5.15. We equip $\mathfrak{a}_\mathfrak{g}$ with the weakest translation-invariant topology such that the stabilizer of any regular function on the total space of $\mathfrak{F}^0_G$ is open (recall that $\mathfrak{a}_\mathfrak{g}$ acts on $\mathfrak{F}^0_G$). This is the weakest translation-invariant topology such that the $\mathfrak{a}_\mathfrak{g}$-centralizer of every element of $\mathfrak{g}_{\mathrm{univ}}$ is open. So the topology is reconstructed from the Lie algebroid structure on $\mathfrak{a}_\mathfrak{g}$.

Clearly the canonical morphism $\operatorname{Der} O \to \mathfrak{a}_\mathfrak{g}$ is continuous.

3.5.16. Denote by $\mathfrak{a}_\mathfrak{b}$ the Lie algebroid of (global) infinitesimal symmetries of $\mathfrak{F}^0_B$. Let $\mathfrak{b}_{\mathrm{univ}}$ (resp. $\mathfrak{n}_{\mathrm{univ}}$) denote the space of global sections of the $\mathfrak{F}^0_B$-twist of $\mathfrak{b}$ (resp. $\mathfrak{n}$). There is a canonical exact sequence

$$0 \to \mathfrak{b}_{\mathrm{univ}} \to \mathfrak{a}_\mathfrak{b} \to \operatorname{Der} A_\mathfrak{g}(O) \to 0\,.$$

$\mathfrak{a}_\mathfrak{b}$ is a subalgebroid of $\mathfrak{a}_\mathfrak{g}$; in fact $\mathfrak{a}_\mathfrak{b}$ is the normalizer of $\mathfrak{b}_{\mathrm{univ}} \subset \mathfrak{a}_\mathfrak{g}$. The image of $\operatorname{Der}^0 O$ in $\mathfrak{a}_\mathfrak{g}$ is contained in $\mathfrak{a}_\mathfrak{b}$.

$\mathfrak{n}_{\mathrm{univ}}$ is an ideal in $\mathfrak{a}_\mathfrak{b}$ and $\mathfrak{a}_\mathfrak{b}/\mathfrak{n}_{\mathrm{univ}}$ is the algebroid of (global) infinitesimal symmetries of $\mathfrak{F}^0_H$. Since $\mathfrak{F}^0_H$ is trivial and its trivialization is "almost" unique (see 3.5.5) $\mathfrak{a}_\mathfrak{b}/\mathfrak{n}_{\mathrm{univ}}$ is canonically isomorphic to the semidirect sum of $\operatorname{Der} A_\mathfrak{g}(O)$ and $A_\mathfrak{g}(O) \otimes \mathfrak{h}$. Denote by $\mathfrak{a}_\mathfrak{n}$ the preimage of $\operatorname{Der} A_\mathfrak{g}(O) \subset \mathfrak{a}_\mathfrak{b}/\mathfrak{n}_{\mathrm{univ}}$ in $\mathfrak{a}_\mathfrak{b}$.

Remark. According to 3.5.5 the composition $\operatorname{Der}^0 O \to \mathfrak{a}_\mathfrak{b}/\mathfrak{n}_{\mathrm{univ}} = \operatorname{Der} A_\mathfrak{g}(O) \oplus (A_\mathfrak{g}(O) \otimes \mathfrak{h})$ is contained in $\operatorname{Der} A_\mathfrak{g}(O) \oplus \mathfrak{h}$; it is equal to the sum of the natural morphism $\operatorname{Der}^0 O \to \operatorname{Der} A_\mathfrak{g}(O)$ and the morphism $\operatorname{Der}^0 O \to \mathfrak{h}$ such that $L_0 \mapsto -\check{\rho}$, $L_n \mapsto 0$ for $n > 0$.

3.5.17. We are going to describe $\mathfrak{a}_\mathfrak{b}$, $\mathfrak{b}_{\mathrm{univ}}$, etc. in terms of the action of L_0 on $\mathfrak{a}_\mathfrak{g}$. The following notation will be used. If $\operatorname{Der} O$ acts on a topological vector space V so that the eigenvalues of $L_0 : V \to V$ are integers denote by $V^{\leq k}$ the smallest closed subspace of V containing all $v \in V$ such that $L_0 v = nv$, $n \leq k$. Set $V^{<k} := V^{\leq k-1}$. If V is a topological module over some algebra A and W is a subspace of V we denote by $A \cdot W$ the smallest closed subspace of V containing aw for every $a \in A$ and $w \in W$.

3.5.18. *Proposition.* i) The following equalities hold:

$$\mathfrak{b}_{\mathrm{univ}} = A_{\mathfrak{g}}(O) \cdot (\mathfrak{g}_{\mathrm{univ}})^{\leq 0} \tag{74}$$

$$\mathfrak{n}_{\mathrm{univ}} = A_{\mathfrak{g}}(O) \cdot \mathfrak{g}_{\mathrm{univ}}^{<0} \tag{75}$$

$$\mathfrak{a}_{\mathfrak{b}} = A_{\mathfrak{g}}(O) \cdot (\mathfrak{a}_{\mathfrak{g}})^{\leq 0} \tag{76}$$

$$\mathfrak{a}_{\mathfrak{n}} = A_{\mathfrak{g}}(O) \cdot \mathfrak{a}_{\mathfrak{g}}^{<0} \tag{77}$$

ii) The image of the morphism

$$(\mathfrak{a}_{\mathfrak{g}})^{\leq 0} \to A_{\mathfrak{g}}(O)(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}/A_{\mathfrak{g}}(O)\mathfrak{a}_{\mathfrak{g}}^{<0} = \mathfrak{a}_{\mathfrak{b}}/\mathfrak{a}_{\mathfrak{n}} = A_{\mathfrak{g}}(O) \otimes \mathfrak{h}$$

equals $\mathfrak{h}$, so we have a canonical isomorphism

$$(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}/(A_{\mathfrak{g}}(O) \cdot \mathfrak{a}_{\mathfrak{g}}^{<0} \cap (\mathfrak{a}_{\mathfrak{g}})^{\leq 0}) \xrightarrow{\sim} \mathfrak{h} \tag{78}$$

Proof. i) (74)–(77) follow from (69). Or one can notice that (74) and (75) are particular cases of (73) and prove, e.g., (76) as follows. According to (74) $A_{\mathfrak{g}}(O) \cdot (\mathfrak{a}_{\mathfrak{g}})^{\leq 0} \supset \mathfrak{b}_{\mathrm{univ}}$ and $A_{\mathfrak{g}}(O) \cdot (\operatorname{Der} A_{\mathfrak{g}}(O))^{\leq 0} = \operatorname{Der} A_{\mathfrak{g}}(O)$, so $A_{\mathfrak{g}}(O) \cdot (\mathfrak{a}_{\mathfrak{g}})^{\leq 0} \supset \mathfrak{a}_{\mathfrak{b}}$. $A_{\mathfrak{g}}(O) \cdot (\mathfrak{a}_{\mathfrak{g}})^{\leq 0} \subset \mathfrak{a}_{\mathfrak{b}}$ because $(\mathfrak{a}_{\mathfrak{g}}/\mathfrak{a}_{\mathfrak{b}})^{\leq 0} = (\mathfrak{g}_{\mathrm{univ}}/\mathfrak{b}_{\mathrm{univ}})^{\leq 0} = (\mathfrak{g}_{\mathrm{univ}})^{\leq 0}/(\mathfrak{b}_{\mathrm{univ}})^{\leq 0} = 0$ according to (74).

ii) The image of $(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}$ in $A_{\mathfrak{g}}(O) \otimes \mathfrak{h}$ equals $(A_{\mathfrak{g}}(O) \otimes \mathfrak{h})^{\leq 0} = \mathfrak{h}$. □

3.6. **Feigin-Frenkel isomorphism II.**

3.6.1. Let A be an associative algebra over $\mathbb{C}[h]$ flat as a $\mathbb{C}[h]$-module. Set $A_0 := A/hA$. Denote by $\mathfrak{Z}$ the center of A_0. If $\mathfrak{Z} = A_0$, i.e., if A_0 is commutative, then $\mathfrak{Z}$ is equipped with the standard Poisson bracket

$$\{z_1, z_2\} := [\widetilde{z}_1, \widetilde{z}_2]/h \bmod h \tag{79}$$

where $z_1, z_2 \in \mathfrak{Z}$ and $\widetilde{z}_i$ is a preimage of z_i in A. Hayashi noticed in [Ha88] that even without the assumption $\mathfrak{Z} = A_0$ (79) is a well-defined Poisson bracket on $\mathfrak{Z}$ (in particular the r.h.s. of (79) belongs to $\mathfrak{Z}$).

Remarks

(i) In the above situation there is a canonical Lie algebra morphism $\varphi : \mathfrak{Z} \to \operatorname{Der} A_0/\operatorname{Int} A_0$ where $\operatorname{Int} A_0$ is the space of inner derivations. φ is defined by $\varphi(z) = D_z$, $D_z(a) := [\widetilde{z}, \widetilde{a}]/h \bmod h$ where $\widetilde{z}, \widetilde{a} \in A$ are preimages of $z \in \mathfrak{Z}$ and $a \in A_0$. If $z' \in \mathfrak{Z}$ then $D_z(z') = \{z, z'\}$. $\operatorname{Der} A_0/\operatorname{Int} A_0$ is a $\mathfrak{Z}$-module and $\varphi(z_1 z_2) = z_1\varphi(z_2) + z_2\varphi(z_1)$. So φ induces a $\mathfrak{Z}$-module morphism $\Phi : \Omega^1_{\mathfrak{Z}} \to \operatorname{Der} A_0/\operatorname{Int} A_0$. In fact Φ is a morphism of Lie algebroids over $\mathfrak{Z}$ (see 3.5.10 for the definition of Lie algebroid); the Lie algebroid structure on $\operatorname{Der} A_0/\operatorname{Int} A_0$ is defined in the obvious way and the one on $\Omega^1_{\mathfrak{Z}}$ is the standard algebroid structure induced by the Poisson bracket on $\mathfrak{Z}$ (cf. [We88]), i.e., $[dz, dz'] := d\{z, z'\}$ for $z, z' \in \mathfrak{Z}$ and the morphism $\Omega^1_{\mathfrak{Z}} \to \operatorname{Der}\mathfrak{Z}$ maps dz to $\operatorname{grad} z$, $(\operatorname{grad} z)(z') := \{z, z'\}$.

(ii) The above constructions make sense if $\mathbb{C}[h]$ is replaced by $\mathbb{C}[h]/(h^3)$.

3.6.2. Now let $\mathfrak{g}$ be a semisimple Lie algebra and $K := \mathbb{C}((t))$. Denote by A the completed universal enveloping algebra of the Lie algebra $\widetilde{\mathfrak{g} \otimes K}$ from 2.5.1, i.e., $A := \varprojlim_n (U\widetilde{\mathfrak{g} \otimes K})/J_n$ where $J_n \subset U\widetilde{\mathfrak{g} \otimes K}$ is the left ideal generated by $\mathfrak{g} \otimes t^n\mathbb{C}[[t]] \subset \mathfrak{g} \otimes K \subset \widetilde{\mathfrak{g} \otimes K}$, $n \geq 0$. Consider the $\mathbb{C}[h]$-algebra structure on A defined by $ha = \mathbf{1} \cdot a - a$, $a \in A$, where $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K} \subset A$. A is flat over $\mathbb{C}[h]$ and A/hA is the completed twisted universal enveloping algebra $\overline{U}' = \overline{U}'(\mathfrak{g} \otimes K)$ from 2.5.2 and 2.9.4. So (79) defines a Poisson bracket on the center $\mathfrak{Z}$ of $\overline{U}'$. It was introduced in [Ha88], so we call it the *Hayashi bracket*.

3.6.3. For an open Lie subalgebra $\mathfrak{a} \subset \mathfrak{g} \otimes O$ denote by $\mathcal{I}_{\mathfrak{a}}$ (resp. $\widetilde{\mathcal{I}}_{\mathfrak{a}}$) the closure of the left ideal of $\overline{U}'$ (resp. of $A = \overline{U}\widetilde{\mathfrak{g} \otimes K}$) generated by $\mathfrak{a} \subset \mathfrak{g} \otimes O \subset \widetilde{\mathfrak{g} \otimes K}$. Clearly $\mathcal{I}_{\mathfrak{a}}$ is the image of $\widetilde{\mathcal{I}}_{\mathfrak{a}}$ in $\overline{U}'$. Set $I_{\mathfrak{a}} := \mathcal{I}_{\mathfrak{a}} \cap \mathfrak{Z}$. We equip $\mathfrak{Z}$ with the topology induced from $\overline{U}'$. The ideals $\mathcal{I}_{\mathfrak{a}}$ (resp. $I_{\mathfrak{a}}$) form a base of neighbourhoods of zero in $\overline{U}'$ (resp. in $\mathfrak{Z}$).

3.6.4. *Lemma.*

(i) $\{I_{\mathfrak{a}}, I_{\mathfrak{a}}\} \subset I_{\mathfrak{a}}$.

(ii) The Hayashi bracket on $\mathfrak{Z}$ is continuous.

Proof. Use the fact that $A/\widetilde{\mathcal{I}}_{\mathfrak{a}}$ equipped with the $\mathbb{C}[h]$-module structure from 3.6.2 is flat. □

3.6.5. Set $I := I_{\mathfrak{g}\otimes O}$. The canonical morphism $\mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O)$ is surjective (see 2.9.3–2.9.5) and its kernel equals I. So $\mathfrak{z}_{\mathfrak{g}}(O) = \mathfrak{Z}/I$.

Denote by I^2 the closed ideal of $\mathfrak{Z}$ generated by elements of the form ab where $a, b \in I$. Then I/I^2 is a Lie algebroid over $\mathfrak{z}_{\mathfrak{g}}(O)$ (the commutator $I/I^2 \times I/I^2 \to I/I^2$ and the mapping $I/I^2 \to \operatorname{Der} \mathfrak{z}_{\mathfrak{g}}(O)$ are induced by the Hayashi bracket). The Lie algebra $\operatorname{Der} O$ acts on I/I^2 and $\mathfrak{z}_{\mathfrak{g}}(O)$. These actions are continuous (I/I^2 is equipped with the topology induced from $\mathfrak{Z}$ and $\mathfrak{z}_{\mathfrak{g}}(O)$ is discrete).

3.6.6. Let us formulate a more precise version of Theorem 3.2.2. We have the algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ and the Lie algebroid I/I^2 over $\mathfrak{z}_{\mathfrak{g}}(O)$. On the other hand denote by ${}^L\mathfrak{g}$ the Langlands dual and consider the algebra $A_{{}^L\mathfrak{g}}(O)$ (see 3.2.1) and the Lie algebroid $\mathfrak{a}_{{}^L\mathfrak{g}}$ over it (see 3.5.11). I/I^2 and $\mathfrak{a}_{{}^L\mathfrak{g}}$ are equipped with topologies (see 3.6.5 and 3.5.15). The Lie algebra $\operatorname{Der} O$ acts on all these objects. $\mathfrak{z}_{\mathfrak{g}}(O)$ and $A_{{}^L\mathfrak{g}}(O)$ are equipped with filtrations (see 1.2.5 and 3.2.1), and we have the morphism $\sigma_A^{-1}\sigma_{\mathfrak{z}} : \operatorname{gr} \mathfrak{z}_{\mathfrak{g}}(O) \to \operatorname{gr} A_{{}^L\mathfrak{g}}(O)$ where $\sigma_{\mathfrak{z}} : \operatorname{gr} \mathfrak{z}_{\mathfrak{g}}(O) \to \mathfrak{z}_{\mathfrak{g}}^{cl}(O)$ is the symbol map and σ_A is the isomorphism (48) with $\mathfrak{g}$ replaced by ${}^L\mathfrak{g}$.

3.6.7. *Theorem.* There is an isomorphism of filtered $\operatorname{Der} O$-algebras

$$\varphi_O : A_{{}^L\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O) \tag{80}$$

such that $\operatorname{gr} \varphi_O^{-1} = \sigma_A^{-1}\sigma_{\mathfrak{z}}$ and φ_O extends to a topological $\operatorname{Der} O$-equivariant isomorphism of Lie algebroids

$$\mathfrak{a}_{{}^L\mathfrak{g}} \xrightarrow{\sim} I/I^2\,. \tag{81}$$

This theorem can be extracted from [FF92] (see 3.7.12–3.7.17).

Remark. According to 3.5.14 the isomorphisms (80) and (81) are unique.

In 3.6.11 we will formulate an additional property of the isomorphism (81). But first we must define an analog of (78) for the algebroid I/I^2.

3.6.8. We will use the notation from 3.5.17.

Lemma. Set $\mathcal{I}_- := (\overline{U}')^{\leq 0} \cap \mathcal{I}_{\mathfrak{a}}$ where $\mathfrak{a} = t\mathfrak{g}[[t]]$ and $\mathcal{I}_{\mathfrak{a}}$ was defined in 3.6.3. Then $\mathcal{I}_-$ is a two-sided ideal in $(\overline{U}')^{\leq 0}$ and

$$(\overline{U}')^{\leq 0} = U\mathfrak{g} \oplus \mathcal{I}_- \,. \tag{82}$$

Proof. (82) is clear. Since $\mathcal{I}_-$ is a left ideal and $[\mathfrak{g}, \mathcal{I}_-] \subset \mathcal{I}_-$ (82) implies that $\mathcal{I}_-$ is a two-sided ideal. □

Define $\pi : (\overline{U}')^{\leq 0} \to U\mathfrak{g}$ to be the morphism such that $\pi(\mathcal{I}_-) = 0$ and $\pi(a) = a$ for $a \in U\mathfrak{g}$.

Here is an equivalent definition of π. Set $Vac'_{\mathfrak{a}} := \overline{U}'/\mathcal{I}_{\mathfrak{a}}$, $\mathfrak{a} = t\mathfrak{g}[[t]]$. Then $Vac'_{\mathfrak{a}}$ is a left $\overline{U}'$-module and a right $U\mathfrak{g}$-module. The eigenvalues of L_0 on $Vac'_{\mathfrak{a}}$ are non-negative and $\mathrm{Ker}(L_0 : Vac'_{\mathfrak{a}} \to Vac'_{\mathfrak{a}}) = U\mathfrak{g}$. So $U\mathfrak{g} \subset Vac'_{\mathfrak{a}}$ is invariant with respect to the left action of $(\overline{U}')^{\leq 0}$. The left action of $(\overline{U}')^{\leq 0}$ commutes with the right action of $U\mathfrak{g}$, so it defines a morphism $(\overline{U}')^{\leq 0} \to U\mathfrak{g}$. This is π.

3.6.9. Denote by C the center of $U\mathfrak{g}$. Then

$$\pi(\mathfrak{Z}^{\leq 0}) \subset C\,, \qquad \pi(\mathfrak{Z} \cdot \mathfrak{Z}^{<0} \cap \mathfrak{Z}^{\leq 0}) = 0\,.$$

Let $m \subset C$ be the maximal ideal corresponding to the unit representation of $U\mathfrak{g}$. Recall that $I := \mathrm{Ker}(\mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O))$. Then $\pi(I^{\leq 0}) \subset m$. Since $(I^2)^{\leq 0} \subset I^{\leq 0} \cdot I^{\leq 0} + (\mathfrak{Z} \cdot \mathfrak{Z}^{<0} \cap \mathfrak{Z}^{\leq 0})$ one has $\pi((I^2)^{\leq 0}) \subset m^2$. So π induces a $\mathbb{C}$-linear map $d : (I/I^2)^{\leq 0} \to m/m^2$ such that $\mathfrak{z}_{\mathfrak{g}}(O) \cdot (I/I^2)^{<0} \cap (I/I^2)^{\leq 0} \subset \mathrm{Ker}\, d$.

Exercise. $\pi(\{z_1, z_2\}) = 0$ for $z_1, z_2 \in \mathfrak{Z}^{\leq 0}$ (so d is a Lie algebra morphism).

3.6.10. Identify C with the algebra of W-invariant polynomials on $\mathfrak{h}^*$ where W is the Weyl group. Then m consists of W-invariant polynomials on $\mathfrak{h}^*$ vanishing at $\rho :=$ the sum of fundamental weights. Since $\rho \in \mathfrak{h}^*$ is regular we can identify m/m^2 with $\mathfrak{h}$ by associating to a W-invariant polynomial from m its differential at ρ. So we have constructed a map

$$d:\ (I/I^2)^{\leq 0}/(\mathfrak{z}_{\mathfrak{g}}(O)\cdot(I/I^2)^{<0}\cap(I/I^2)^{\leq 0})\to\mathfrak{h} \tag{83}$$

3.6.11. *Theorem.* The diagram

$$\begin{array}{ccc} (\mathfrak{a}_{L_{\mathfrak{g}}})^{\leq 0}/(A_{L_{\mathfrak{g}}}(O)\cdot\mathfrak{a}_{L_{\mathfrak{g}}}^{<0}\cap(\mathfrak{a}_{L_{\mathfrak{g}}})^{\leq 0}) & \xrightarrow{\sim} & \mathfrak{h}^* \\ \downarrow\wr & & \uparrow\wr \\ d:\ (I/I^2)^{\leq 0}/(\mathfrak{z}_{\mathfrak{g}}(O)\cdot(I/I^2)^{<0}\cap(I/I^2)^{\leq 0}) & \longrightarrow & \mathfrak{h} \end{array} \tag{84}$$

anticommutes. Here the upper arrow is the isomorphism (78) with $\mathfrak{g}$ replaced by ${}^L\mathfrak{g}$, the left one is induced by (81), and the right one comes from the scalar product (18).

This theorem can be extracted from [FF92] (see 3.8.15–3.8.22).

3.6.12. The reason why the "critical" scalar product (18) appears in 3.6.11 is not very serious. The reader may prefer the following point of view. Denote by B the set of invariant bilinear forms on $\mathfrak{g}$. For each $b \in B$ we have the completed twisted universal enveloping algebra $\overline{U}'_b = \overline{U}'_b(\mathfrak{g}\otimes K)$ corresponding to the cocycle $(u,v)\mapsto \operatorname{Res} b(du,v)$, $u,v\in\mathfrak{g}\otimes K$ (so $\overline{U}' = \overline{U}'_c$ where c is defined by (18)). One can associate to $b \in B$ a Poisson bracket $\{\ \}_b$ on $\mathfrak{Z}$ by applying the general construction from 3.6.1 to the family of algebras $\overline{U}'_{c+hb}$ depending on the parameter h (the bracket from 3.6.2 corresponds to $b = c$). The Lie algebroid structure on I/I^2 depends on b. Then 3.6.7 and 3.6.11 hold for every nondegenerate $b \in B$ (notice that in (84) *both* vertical arrows depend on b).

3.6.13. In fact, the action of $\operatorname{Der} O$ on I/I^2 mentioned in 3.6.6–3.6.7 comes from a canonical morphism $\operatorname{Der} O \to I$, which is essentially due to Sugawara. We will explain this in 3.6.16 after a brief overview of Sugawara formulas in 3.6.14–3.6.15. These formulas also yield elements of $\mathfrak{z}_{\mathfrak{g}}(O)$; in the case $\mathfrak{g} = sl_2$ they generate $\mathfrak{z}_{\mathfrak{g}}(O)$. We remind this in 3.6.18. Both 3.6.18 and 3.6.19 are not used in the sequel (?).

3.6.14. In this subsection we remind the general Sugawara formulas. In 3.6.15 we remind their consequences for the critical level.

Let A be the completed universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes K}$. As a vector space $\widetilde{\mathfrak{g} \otimes K}$ is the direct sum of $\mathfrak{g} \otimes K$ and $\mathbb{C} = \mathbb{C} \cdot \mathbf{1}$. The Sugawara elements $\widetilde{\mathfrak{L}}_n \in A$ are defined by

$$\widetilde{\mathfrak{L}}_n := \frac{1}{2} \sum_{r+l=n} g^{\lambda\mu} : e_\lambda^{(r)} e_\mu^{(l)} : \tag{85}$$

Here $\{e_\lambda\}$ is a basis of $\mathfrak{g}$, $e_\lambda^{(r)} := e_\lambda t^r \in \mathfrak{g}((t)) = \mathfrak{g} \otimes K \subset \widetilde{\mathfrak{g} \otimes K}$, $(g^{\lambda\mu})$ is inverse to the Gram matrix (e_λ, e_μ) with respect to the "critical" scalar product (18) and

$$: e_\lambda^{(r)} e_\mu^{(l)} := \begin{cases} e_\lambda^{(r)} e_\mu^{(l)} & \text{if} \quad r \le l \\ e_\mu^{(l)} e_\lambda^{(r)} & \text{if} \quad r > l \end{cases} \tag{86}$$

Of course summation over λ and μ is implicit in (85). Clearly the infinite series (85) converges and $\widetilde{\mathfrak{L}}_n \to 0$ for $n \to \infty$.

Remark. If $n \neq 0$ then $: e_\lambda^{(r)} e_\mu^{(l)} :$ can be replaced in (85) by $e_\lambda^{(r)} e_\mu^{(l)}$. Indeed, since $g^{\lambda\mu}$ is symmetric $g^{\lambda\mu}[e_\lambda^{(r)}, e_\mu^{(l)}] = 0$ unless $r + l = 0$, $r \neq 0$.

The proof of the following formulas can be found[20], e.g., in Lecture 10 from [KR] and § 12.8 from [Kac90] :

$$\operatorname{ad}\widetilde{\mathfrak{L}}_n = hL_n \tag{87}$$

$$L_m(\widetilde{\mathfrak{L}}_n) = (m-n)\widetilde{\mathfrak{L}}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot(\dim\mathfrak{g})\cdot\mathbf{1}\,. \tag{88}$$

In (87) $\operatorname{ad}\widetilde{\mathfrak{L}}_n$ is an operator $A\to A$, $L_n := -t^{n+1}\frac{d}{dt}\in \operatorname{Der}K$ is also considered as an operator $A\to A$ (the Lie algebra $\operatorname{Der}K$ acts on A in the obvious way), and h has the same meaning as in 3.6.2, i.e., $h: A\to A$ is multiplication by $\mathbf{1}-1$.

Using (87) one can rewrite (88) in the Virasoro form:

$$[\widetilde{\mathfrak{L}}_m,\widetilde{\mathfrak{L}}_n] = h((m-n)\widetilde{\mathfrak{L}}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot(\dim\mathfrak{g})\cdot\mathbf{1})\,. \tag{89}$$

3.6.15. The image of $\widetilde{\mathfrak{L}}_n$ in $A/hA=\overline{U}'$ will be denoted by $\mathfrak{L}_n$. According to (87) $\mathfrak{L}_n$ belongs to the center $\mathfrak{Z}\subset\overline{U}'$ and

$$\{\mathfrak{L}_n, z\} = L_n(z), \qquad z\in\mathfrak{Z} \tag{90}$$

where $\{\ \}$ denotes the Hayashi bracket on $\mathfrak{Z}$. According to (88) and (89)

$$L_m(\mathfrak{L}_n) = (m-n)\mathfrak{L}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot\dim\mathfrak{g} \tag{91}$$

$$\{\mathfrak{L}_m,\mathfrak{L}_n\} = (m-n)\mathfrak{L}_{m+n} + \delta_{m,-n}\cdot\frac{m^3-m}{12}\cdot\dim\mathfrak{g}\,. \tag{92}$$

3.6.16. If $n\geq -1$ then $\mathfrak{L}_n\in I := \operatorname{Ker}(\mathfrak{Z}\to\mathfrak{z}_{\mathfrak{g}}(O))$ (indeed, a glance at (85) shows that $\mathfrak{L}_n$ annihilates the vacuum vector from Vac'). If $m,n\geq -1$ then the "Virasoro term" $\delta_{m,-n}(m^3-m)$ vanishes, so one has the continuous Lie algebra morphism $\operatorname{Der}O\to I$ defined by $L_n\mapsto\mathfrak{L}_n$, $n\geq -1$. It induces a continuous algebra morphism

$$\operatorname{Der}O\to I/I^2\,. \tag{93}$$

[20]The reader should take in account that experts in Kac – Moody algebras usually equip $\mathfrak{g}$ with the scalar product obtained by dividing (18) by minus the dual Coxeter number.

Remark. According to (90) the action of $\operatorname{Der} O$ on I/I^2 induced by (93) coincides with the action considered in 3.6.6–3.6.7.

3.6.17. *Lemma.* The composition of (93) and the isomorphism $I/I^2 \xrightarrow{\sim} \mathfrak{a}_{L_\mathfrak{g}}$ inverse to (81) is equal to the morphism $\operatorname{Der} O \to \mathfrak{a}_{L_\mathfrak{g}}$ from 3.5.11.

Proof The two morphisms $\operatorname{Der} O \to \mathfrak{a}_{L_\mathfrak{g}}$ induce the same action of $\operatorname{Der} O$ on $\mathfrak{a}_{L_\mathfrak{g}}$. So they are equal by 3.5.12. □

3.6.18. Denote by $\overline{\mathfrak{L}}_n$ the image of $\mathfrak{L}_n$ in $\mathfrak{Z}/I = \mathfrak{z}_\mathfrak{g}(O)$. If $n \geq -1$ then $\overline{\mathfrak{L}}_n = 0$. The natural morphism $\mathbb{C}[\overline{\mathfrak{L}}_{-2}, \overline{\mathfrak{L}}_{-3}, \dots] \to \mathfrak{z}_\mathfrak{g}(O)$ is injective and if $\mathfrak{g} = sl_2$ it is an isomorphism. To show this it is enough to compute the principal symbol of $\overline{\mathfrak{L}}_n$ and to use the description of $\mathfrak{z}_\mathfrak{g}^{cl}(O)$ from 2.4.1. If $\mathfrak{z}_\mathfrak{g}^{cl}(O)$ is identified with the space of $G(O)$-invariant polynomials on $\mathfrak{g}^* \otimes \omega_O$ (see 2.4.1) then the principal symbol of $\overline{\mathfrak{L}}_n$ is the polynomial $\ell_n : \mathfrak{g}^* \otimes \omega_O \to \mathbb{C}$ defined by $\ell_n(\eta) = \frac{1}{2}\operatorname{Res}(\eta, \eta) L_n$; here $(\eta, \eta) \in \omega_O^{\otimes 2}$, $L_n \in \omega_K^{\otimes(-1)}$, $(\eta, \eta) L_n \in \omega_K$, so the residue makes sense. Clearly the mapping $\mathbb{C}[\ell_{-2}, \ell_{-3}, \dots] \to \mathfrak{z}_\mathfrak{g}^{cl}(O)$ is injective and if $\mathfrak{g} = sl_2$ it is an isomorphism.

For $\mathfrak{g} = sl_2$ the Feigin – Frenkel isomorphism is the unique $\operatorname{Der} O$-equivariant isomorphism $A_{L_\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_\mathfrak{g}(O)$. An sl_2-oper over $\operatorname{Spec} O$ can be represented as a connection $\frac{d}{dt} + \left(\begin{smallmatrix} 0 & u \\ 1 & 0 \end{smallmatrix}\right)$, $u = u(t) = u_0 + u_1 t + \dots$, or as a Sturm – Liouville operator $\left(\frac{d}{dt}\right)^2 - u(t) : \omega_O^{-1/2} \to \omega_O^{3/2}$. One has $A_{sl_2}(O) = \mathbb{C}[u_0, u_1, \dots]$ and the Feigin – Frenkel isomorphism maps u_j to $-2\overline{\mathfrak{L}}_{-2-j}$.

For any semisimple $\mathfrak{g}$ we gave in 3.5.6 a description of $A_{L_\mathfrak{g}}(O)$ as an algebra with an action of $\operatorname{Der} O$; see (64)–(68). Using the $\operatorname{Der} O$-equivariance property of the Feigin – Frenkel isomorphism one sees that if $\mathfrak{g}$ is simple then $\overline{\mathfrak{L}}_{-2-j} \in \mathfrak{z}_\mathfrak{g}(O)$ corresponds to $cu_{1j} \in A_{L_\mathfrak{g}}(O)$, $c = -(\dim \mathfrak{g})/6$ (???).

3.6.19. Consider the vacuum module $Vac_\lambda := Vac_A / (h - \lambda)\, Vac_A$, where Vac_A is the quotient of A modulo the closed left ideal generated by $\mathfrak{g} \otimes O$. In 2.9.3 we mentioned that $\operatorname{End}_A Vac_\lambda = \mathbb{C}$ for $\lambda \neq 0$. The following proof

of this statement was told us by E. Frenkel. As explained in 2.9.3–2.9.5 any endomorphism $f : Vac_\lambda \to Vac_\lambda$ comes from some central element z of $A/(h-\lambda)A$. In fact the center of $A/(h-\lambda)A$ equals $\mathbb{C}$ if $\lambda \neq 0$, but instead of proving this let us notice that $[\widetilde{\mathfrak{L}}_0, z] = 0$ and therefore $L_0(z) = 0$ (see (87)). So $[L_0, f] = 0$ where L_0 is considered as an operator in Vac_λ. Therefore f preserves the space $\mathrm{Ker}(L_0 : Vac_\lambda \to Vac_\lambda)$, which is generated by the vacuum vector. Since the A-module Vac_λ is generated by this space f is a scalar operator.

3.7. **The center and the Gelfand - Dikii bracket.**

3.7.1. Set $Y := \operatorname{Spec} O$, $Y' := \operatorname{Spec} K$ where, as usual, $O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. Let A be a (commutative) $\operatorname{Aut} O$-algebra. Then for any smooth curve X one obtains a $\mathcal{D}_X$-algebra A_X (see 2.6.5). Though Y and Y' are not curves in the literal sense the construction from 2.6.5 works for them (with a minor change explained below). So one gets a $\mathcal{D}_Y$-algebra A_Y and a $\mathcal{D}_{Y'}$-algebra $A_{Y'}$, which is the restriction of A_Y to Y'. The fiber of A_Y at the origin $0 \in Y$ equals A.

Let us explain some details. The definition of A_X from 2.6.5 used a certain scheme $X^\wedge$. Since Y is not a curve in the literal sense the definition of $Y^\wedge$ should be modified as follows. Denote by Δ_n the n-th infinitesimal neighbourhood of the diagonal $\Delta \subset \operatorname{Spec} O\widehat{\otimes}O$. The morphism $\operatorname{Spec} O\widehat{\otimes}O \to \operatorname{Spec} O \otimes O = Y \times Y$ induces an embedding $\Delta_n \hookrightarrow Y \times Y$ (if $n > 0$ then Δ_n is *smaller* than the n-th infinitesimal neighbourhood of the diagonal $\Delta \subset Y \times Y$). Now in the definition of an R-point of $Y^\wedge$ one should consider only R-morphisms $\gamma : \operatorname{Spec} R\widehat{\otimes}O \to Y$ with the following property: for any n there is an N such that the morphism $\operatorname{Spec} O/t^nO \times \operatorname{Spec} O/t^nO \times \operatorname{Spec} R \to Y \times Y$ induced by γ factors through Δ_N (then one can set $N = 2n-2$).

3.7.2. Sometimes we will use the section

$$Y \to Y^\wedge \tag{94}$$

corresponding to the morphism $\gamma : \operatorname{Spec} O\widehat{\otimes}O \to Y = \operatorname{Spec} O$ defined by

$$\gamma^*(t) = t \otimes 1 + 1 \otimes t\,. \tag{95}$$

The section (94) yields an isomorphism

$$A_Y \xrightarrow{\sim} A \otimes \mathcal{O}_Y\,. \tag{96}$$

Of course (94) and (96) are not canonical: they depend on the choice of a local parameter $t \in O$.

3.7.3. In the situation of 3.7.1 consider the functor $F : \{\mathbb{C}\text{-algebras}\} \to \{\text{Sets}\}$ such that $F(R)$ is the set of horizontal Y'-morphisms $\operatorname{Spec} R\widehat{\otimes}K \to \operatorname{Spec} A_{Y'}$ or, which is the same, the set of horizontal K-morphisms $H^0(Y', A_{Y'}) \to R\widehat{\otimes}K$. F is representable by an ind-affine ind-scheme S (which may be called the ind-scheme of horizontal sections of $\operatorname{Spec} A_{Y'}$). Indeed, F is a closed subfunctor of the functor $R \mapsto \operatorname{Hom}(V, R\widehat{\otimes}K)$ where $V = H^0(Y', A_{Y'})$ and Hom means the set of K-linear maps.

Denote by A_K the ring of regular functions on S. This is a complete topological algebra (the ideals of A_K corresponding to closed subschemes of S form a base of neighbourhoods of 0).

A_K is equipped with an action of the group ind-scheme $\operatorname{Aut} K$ (an R-point of $\operatorname{Aut} K$ is an automorphism of the topological R-algebra $R\widehat{\otimes}K$).

The scheme of horizontal sections of $\operatorname{Spec} A_Y$ is canonically isomorphic to $\operatorname{Spec} A$ (to a horizontal section $s : Y \to \operatorname{Spec} A_Y$ one associates $s(0) \in \operatorname{Spec} A$). This is a closed subscheme of $S = \operatorname{Spec} A_K$, so we get a canonical epimorphism

$$A_K \to A\,. \tag{97}$$

Clearly it is $\operatorname{Aut} O$-equivariant.

Example. Suppose that $A = \mathbb{C}[u_0, u_1, u_2, \dots]$ and $u_k = (L_{-1})^k u_0/k!$, $L_0 u_0 = du_0$, $d \in \mathbb{Z}$ (as usual, $L_n := -t^{n+1}\frac{d}{dt} \in \operatorname{Der} O$). Then one has the obvious isomorphism f between the $\mathcal{D}_Y$-scheme $\operatorname{Spec} A_Y$ and the scheme of jets of d-differentials on Y. Clearly $\operatorname{Aut} O = \operatorname{Aut} Y$ acts on both schemes by functoriality. f is equivariant with respect to the group ind-scheme of $\operatorname{Aut} O$ generated by L_0 and L_{-1}. Using f we identify horizontal sections of $\operatorname{Spec} A_{Y'}$ with d-differentials on Y', i.e., sections of $\omega_{Y'}^{\otimes d}$. A d-differential on Y' can be written as $\sum_i \widetilde{u}_i t^i (dt)^{\otimes d}$, so $A_K = \mathbb{C}[[\dots \widetilde{u}_{-1}, \widetilde{u}_0, \widetilde{u}_1, \dots]$ where

$$\mathbb{C}[[\dots \widetilde{u}_{-1}, \widetilde{u}_0, \widetilde{u}_1, \dots] := \varprojlim_n \mathbb{C}[\dots \widetilde{u}_{-1}, , \widetilde{u}_0, \widetilde{u}_1, \dots]/(u_{-n}, u_{-n-1}, \dots)\,. \tag{98}$$

Clearly $L_0 \widetilde{u}_k = (d+k)\widetilde{u}_k$, $L_{-1}\widetilde{u}_k = (k+1)\widetilde{u}_{k+1}$, and the morphism (97) maps $\widetilde{u}_k$ to u_k if $k \geq 0$ and to 0 if $k < 0$.

3.7.4. Denote by $\mathfrak{z}_{\mathfrak{g}}(K)$ the algebra A_K from 3.7.3 in the particular case $A = \mathfrak{z}_{\mathfrak{g}}(O)$ (see 2.5.1 or 2.7.2 for the definition of $\mathfrak{z}_{\mathfrak{g}}(O)$). We are going to define a canonical morphism from $\mathfrak{z}_{\mathfrak{g}}(K)$ to the center $\mathfrak{Z}$ of the completed twisted universal enveloping algebra $\overline{U}' = \overline{U}'(\mathfrak{g} \otimes K)$. To this end rewrite (34) as a K-linear map $\mathfrak{z}_{\mathfrak{g}}(O) \otimes_O K \to \mathfrak{Z} \widehat{\otimes} K$. Using the noncanonical isomorphism $\mathfrak{z}_{\mathfrak{g}}(O)_Y \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O) \otimes \mathcal{O}_Y$ (see (96)) one gets a map

$$H^0(Y', \mathfrak{z}_{\mathfrak{g}}(O)_{Y'}) \to \mathfrak{Z} \widehat{\otimes} K\,, \tag{99}$$

which is easily shown to be canonical, i.e., independent of the choice of a local parameter $t \in O$ (in fact, (34) is a noncanonical version of (99); (34) depends on the choice of t because (32) involves $\zeta + t$, which is nothing but the noncanonical section $Y' \to Y'^{\wedge}$ defined by (95)).

3.7.5. *Theorem.*

(i) The map (99) is a horizontal morphism of K-algebras. Therefore (99) defines a continuous morphism

$$\mathfrak{z}_{\mathfrak{g}}(K) \to \mathfrak{Z}\,. \tag{100}$$