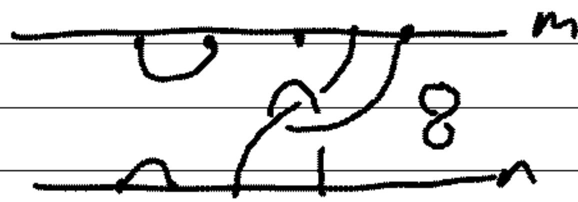
R. Anno - Affine tangles & irreducible exotic sheaves

Note Title 2/12/2008

Tangles: Objects are finite sets of points on the line, & morphisms (n,m) given by arcs & circles in strip times $\mathbb{R}$

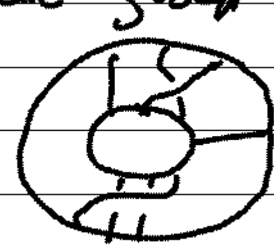


Categorification: objects $(m) \in \mathbb{N}$, morphisms = tangles
2-morphisms = cobordisms of tangles
with isotopies as equivalence relation

Representation of tangles: to $n \in \mathbb{N}$ assign triangulated category $\mathcal{D}_n$
to tangles (n,m) assign functors $\mathcal{D}_n \to \mathcal{D}_m$
to cobordisms define natural transformations.

$$\mathrm{Hom}(n,n) \supset \text{braid group } B_n$$

<u>Affine tangles</u>: draw on annulus $\times\ \mathbb{R}$ instead of strip $\times\ \mathbb{R}$ .... can enumerate elements of affine braid group as braids between two circles



(Similar constructions: Khovanov, Stroppel, Cautis-Kamnitzer)
Conjecture on irreducible exotic sheaves & its relation is due to P. Seidel.

$\mathfrak{g} = \mathfrak{sl}_{2n}$, $Z = \left(\begin{array}{c|c} \ddots & 0 \\ \hline 0 & \ddots \end{array}\right)$ nilpotent with two nonnilpotent blocks.

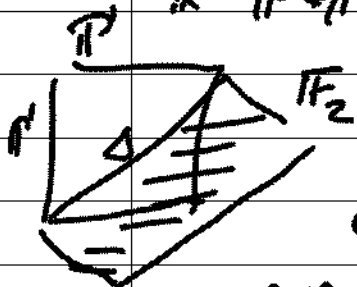
$$\begin{array}{ccccccc} \text{Springer fiber} & \mathcal{B}_Z & \subset & \mathcal{U}_Z & \subset & \widetilde{\mathcal{N}} = T^*\mathcal{B} & \text{cotangent to flags} \\ & \downarrow & & \downarrow & & \downarrow & \\ & Z & \in & \mathcal{S} & \subset & \mathcal{N} & \text{nilpotent cone} \end{array}$$

$\mathcal{S}$ = Slodowy slice, $\mathcal{U}_Z$ is symplectic (smooth) & $\mathcal{B}_Z$ is a (reducible) Lagrangian subvariety.

Ex. 1 $Z = (1,1)$ nilpotent in $\mathfrak{sl}_2$ $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right)$
$\mathcal{U}_Z = \widetilde{\mathcal{N}} = T^*\mathbb{P}^1 \supset \mathbb{P}^1 = \mathcal{B}_Z$

Ex. 2 $Z = (2,2)$ nilpotent: $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$
$\mathcal{B}_Z = \mathbb{P}^1 \times \mathbb{P}^1 \cup \underbrace{\mathbb{F}_2}_{\text{Hirzebruch}} \subset \mathcal{U}_Z$

The two surfaces $\mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{F}_2$ cross transversally along a $\mathbb{P}^1$, diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ & exceptional line in $\mathbb{F}_2$.



$$\mathcal{D}_n = \mathcal{D}^b_{B_z}(U_z) :$$

coherent sheaves on U_z cohomologically supported on B_z.

Affine braid group acting on $D^b(\tilde{N})$ constructed by Khanov-Thomas

... can be linked together to give a tangle category, but for the (n,n) slices as above it can....

For each k we have a projection $\mathcal{B} \xrightarrow{p_k} \mathcal{P}_k$ from G/B to G/P_k, parabolic with k'th index omitted.

$$T^*\mathcal{P}_k \times_{\mathcal{P}_k} \mathcal{B} = D_k \xhookrightarrow[\text{divisor}]{i_k} T^*\mathcal{B}$$

$$\pi_k \downarrow \mathbb{P}^1\text{-bundle}$$

$$\parallel$$

$$p_k^* T^*\mathcal{P}_k \qquad T^*\mathcal{P}_k$$

$$D^b(T^*P_k) \underset{\pi_{k*} i_k^!}{\overset{i_{k*}\pi_k^*}{\rightleftarrows}} D^b(T^*B)$$

right adjoint

These functors are all Fourier-Mukai functors & their kernels form an exact triangle

$$(i_{k*}\pi_k^*)(\pi_{k*} i_k^!) \longrightarrow id \longrightarrow T_k$$

& the functors T_k are autoequivalences of $D^b(T^*B)$, giving generators for a weak braid group action (Khovanov-Thomas).

extra affine generators act by tensor multiplication by line bundles.

Restrict to Springer fibers for Z (n,n)-nilpotent:

$$\begin{array}{ccc} D_k \cap U_{Z_n} & \xhookrightarrow{\text{divisor}} & U_{Z_n} \\ \downarrow & & \\ \boxed{\begin{array}{c} U_{k,Z_n} \\ \| \\ U_{Z_{n-2}} \end{array}} & & \end{array}$$

P_k analog of Springer/Slodowy picture.

- partial flag Springer for Z_n gives full Springer for Z_{n-2}.

$\Rightarrow$ sequence of derived categories & functors

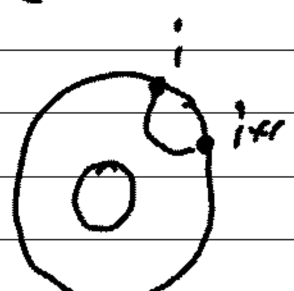
$$\begin{tikzcd} D_{2n-4} \arrow[r, bend left, "1"] \arrow[r, bend right, "2n-3"'] & D_{2n-2} \arrow[r, bend left, "F^1"] \arrow[r, bend right, "F^{2n-1}"'] & D_{2n} \end{tikzcd}$$

with exact triangles $F^k R^{F^k} \to \mathrm{id} \to T_{2n}^k : D_{2n} \circlearrowleft$

with T_{2n}^k generating weak braid group action on D_{2n} (here R^{F^k} is the right adjoint to F^k). $\to$ assign to generators of affine tangle category:

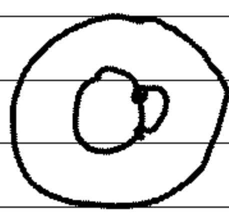
[Cautis - Kamnitzer]

...



$\rightsquigarrow g^i_{2n} = \mathcal{E}_k \otimes i_{k*} \pi_k^*$

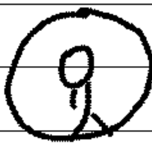
$\mathcal{E}_k$ = tautological line bundle



$\rightsquigarrow$ adjoint $R^{g^i_{2n}} = L^{g^i_{2n}}[-2]$

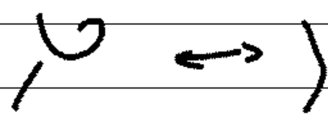
$f^i_{2n} = R^{g^i_{2n}}[1] = L^{g^i_{2n}}[-1]$

 $\rightsquigarrow t^i_{2n} = T^i_{2n}$

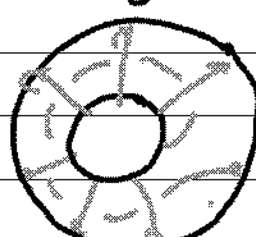
 $\rightsquigarrow (T^i_{2n})^{-1}$ inverse functor.

... unoriented version of Cautis-Kamnitzer.

$\rightsquigarrow$ extends to framed tangles: satisfy 1^{st} Reidemeister

 (tangles with normal vector)

$\rightsquigarrow$ get an extension
to full affine braid group (not in Cautis-Kamnitzer)

- add generator  $\rightsquigarrow \otimes \mathcal{E}^\vee_{2n}$: tensor with tautological line bundle on G/B - dual to last quotient line V_{2n}/V_{2n-1} in flag. $(n=1: \otimes \mathcal{O}_{\mathbb{P}^1}(-1).\ n=2: \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1))$

Exotic t-structure : characterized in terms of affine braid actn.

$\alpha \in D_n$ lives in $D^{\geq 0}$ $\iff$

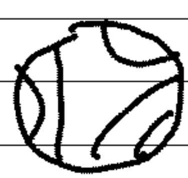
$\forall\, b^+ \in ABr_{2n}$ a <u>positive</u> affine braid

... ie all intersections are positive not negative .... we have

$$R\Gamma \, \underbrace{\psi(b^+)\,\alpha}_{\text{action of braid on } \alpha} \in D^{\geq 0}(Vect)$$

<u>Claim</u> The functors g^i_{2n} are t-exact & send irreducibles to irreducibles [second part follows from first by Bezrukavnikov-Mirkovic]

<u>Corollary</u> The functors corresponding to crossingless matchings



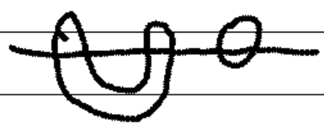
$: D_0 \longrightarrow D_{2n}$

$D_0 = D^b(Vect) \ni k$

define irreducible exotic sheaves for each crossingless matching, which by dim K_0 argument form a basis $\rightsquigarrow$ complete set of irreducibles

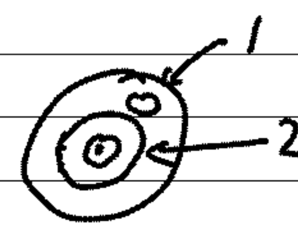
$\Rightarrow$ can describe Exts of irreducible exotic sheaves in terms of crossingless matchings [currently up to a sign] ...

Khovanov description of sl_2 homology: vector space labelled by crossingless matchings:



$\longrightarrow$ 2 copies of $\mathbb{C}[x]/x^2$ (labelled by loops)

Here have two kinds of loops:



Type 1: don't wrap origin, assign $\Lambda^0 V \oplus \Lambda^2 V$ for V a symplectic vector space / Frobenius algebra

Type 2: assign $\Lambda^1 V$.

- Ext algebra fully described as vector space with $\pm$ basis, & multiplication described up to sign.

$\rightsquigarrow$ describe D_{2n} as modules over a version of Khovanov algebra.