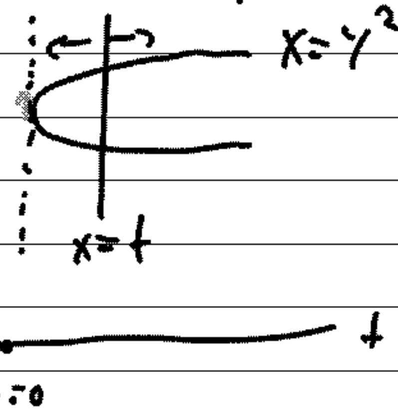
C. Barwick - A Gentle Introduction to Derived Algebraic Geometry

Note Title 2/15/2008

Consider family of intersections

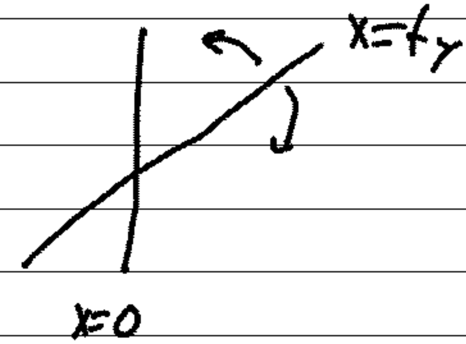


$$\begin{array}{c} \operatorname{Spec} \mathbb{C}[x,y,t]/(x-y^2, x-t) \\ \downarrow \\ \mathbb{A}^1_t \end{array}$$

Classical algebraic geometry tells us how to understand the fiber at 0 using nilpoints

Another family:

$$\begin{array}{c} \operatorname{Spec} \mathbb{C}[x,y,t]/(x, x-ty) \\ \downarrow \\ \mathbb{A}^1_t \end{array}$$



Base change to $t=0$: get $\operatorname{Spec} \mathbb{C}[x,y]/(x,x)$!

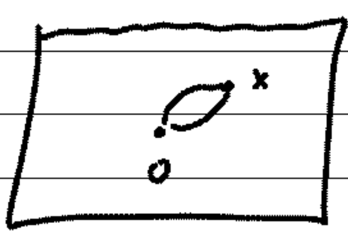
imposing same equation twice!

Claim: systems $\left\{ \begin{matrix} x=0 \\ x=0 \end{matrix} \right\} \neq \{ x=0 \}$

from point of view of deformation theory.

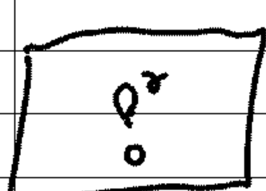
How to make sense of this? want notion of equivalence that can be introduced more than once ---- e.g in groupoids  is not same as  :

think of equations as introducing paths



$\mathbb{C}[x,y]/(x,x)$: connect x to 0 in two different ways! (also need to introduce all paths required by ring structure

$\wr\!\wr$



add a small loop γ: "$0=0$" at 0.

$\simeq \mathbb{C}[y,\gamma]$ with $\deg \gamma = 1$

Consider (model) category $\qquad \begin{array}{c} sAlg(k) \\ \uparrow\downarrow \\ sSet \end{array}$

simplicial k-algebras

(model structure comes from adjoint pair of functors to $sSet$):

$f: R \longrightarrow S$ in $sAlg(k)$ is a weak equivalence if $\pi_* R \xrightarrow{\sim} \pi_* S$:

$\rightsquigarrow$ $\pi_0 R$ commutative ring & $\pi_* R$ graded comm. ring

$$Ho\ sAlg(k) \longrightarrow Ho\ \underset{\text{connective}}{CDGA(k)}$$

an equivalence if k is a $\mathbb{Q}$-algebra

RHS not good for k not a $\mathbb{Q}$-algebra

Let $k = \mathbb{F}_p$: Free algebra on one generator in $sAlg(k)$ is $\mathbb{F}_p[t]$

But the Eilenberg-MacLane space $H(\mathbb{F}_p[t])$ is not the free E_∞ algebra on one generator:

imposing that the free algebra on one generator in the E_∞ world puts you back in $SAlg(k)$ — this characterizes $SAlg(k)$

$X/\mathbb{C}$ smooth projective variety.

Consider the moduli stack $Vect_n(X)$ of rank n vector bundles.

If $\dim X > 1$ then the dimension of the tangent space to $Vect_n(X)$ is not constant:

$$T_{\mathcal{E}} Vect_n(X) \simeq \tau^{\leq 0} C^{\bullet}_{Zar}(X, End\, \mathcal{E})[1]$$

nonpositive truncation of Zariski cochain complex: $\dim = h^1(X, End\, E) - h^0(X, End\, E)$ jumps since there are higher h^i.

<u>Idea</u>: There should exist an object, the derived moduli stack $\mathbb{R}Vect_n X$, whose tangent complex is the <u>untruncated</u> complex $C^{\bullet}_{Zar}(X, End\, \mathcal{E})[1]$

- natural test objects for such moduli functors should be simplicial k-algebras:
  A usual comm. ring we still want
  $RVect_n(X)(A) \simeq Vect_n(X)(A)$
  but have interesting new points for $A \in sAlg(k)$

- analog to introducing nonreduced structure, which is not felt on reduced test objects
  $F(X) \simeq F^{red}(X)$ for X reduced ...

So for A classical expect
$$M^{Der}(A) \simeq (\tau M^{D})(A)$$

<u>Example</u> $X/\mathbb{C}$ smooth

1. $H^1(X, T_X)$ classifies first order deformations
2. obstruction in $H^2(X, T_X)$ to extend a given first order deformation

$H^2(X, TX) \simeq$ equivalence classes of flat families $\mathfrak{X} \to \operatorname{Spec} \mathbb{C}[\gamma]$ ($\deg \gamma = 1$) s.t. $\mathfrak{X}_0 \simeq X$.

&

$$\begin{tikzcd}
\mathbb{C}[\varepsilon]/\varepsilon^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{C}[\gamma] \arrow[d] \\
\mathbb{C} \arrow[r] & \mathbb{C}[\varepsilon]/\varepsilon^2
\end{tikzcd}$$

pullback diagram (homotopy fiber product)

ie second order deformation is same as lifting to odd direction

ie $\operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^3$ is pushout of $\operatorname{Spec} \mathbb{C}[\gamma]$ & $\operatorname{Spec} \mathbb{C}$ over $\operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$.

Let $\mathcal{A}ff = sAlg(k)^{op}$ derived affines:

for $A \in sAlg(k)$, $\operatorname{Spec} A$ is topologically the same as $\operatorname{Spec} \pi_0 A$, difference lies in structure sheaf

$\mathcal{O}_{\operatorname{Spec} A}$ sheaf in $sAlg(k)$.

Have notion of Zariski site:

a Zariski monomorphism is $f: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

s.t. $\operatorname{Mod} B \underset{\text{fully faithful on homotopy categories}}{\leftrightarrows} \operatorname{Mod} A$

$\iff$ Zariski open $\operatorname{Spec} \pi_0 B \hookrightarrow \operatorname{Spec} \pi_0 A$

$\rightsquigarrow$ Zariski site of $\operatorname{Spec} A$.

Notion of sheaf: $\mathcal{F}: \underset{\substack{\text{ordinary category} \\ \text{with topology } \tau}}{C^{op}} \longrightarrow sAlg(k)$

$\{U_\alpha \longrightarrow X\}_{\alpha \in \Lambda}$ covering $\Rightarrow$

$$\mathcal{F} X \xrightarrow{\sim} \operatorname{holim}_{\Delta^{\to}} \left(\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha\beta} \mathcal{F}(U_{\alpha\beta}) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \right)$$

only for ordinary coverings, not hypercoverings

Category of sheaves in this case is still related to presheaves in the same way as usual:

$$\text{Sheaves on } C \underset{\longrightarrow}{\xleftarrow{\text{sheafify}}} \text{presheaves on } C.$$

Cotangent complex:

$f : X \longrightarrow Y \rightsquigarrow$ Kähler differentials $\Omega^1_{X/Y}$

$X \xrightarrow{f} Y \to Z \Rightarrow$ exact sequence

$$f^* \Omega^1_{Y/Z} \longrightarrow \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

Not short exact...

Illusie: introduce higher terms to make this long exact.

A a k-algebra, resolve by simplicial free k-algebra $P_\bullet \to A$, apply Kähler differentials term by term $\uparrow_k$ & totalize $\rightsquigarrow \mathbb{L}_{A/k}$

$\Rightarrow$ get distinguished triangle

$$f^* \mathbb{L}_{Y/Z} \to \mathbb{L}_{X/Z} \to \mathbb{L}_{X/Y}$$

$F : \mathrm{Aff}^{op} \longrightarrow s\mathrm{Set}$ moduli functor

define absolute cotangent complex $\in$ quasicoh module / F

... for category $\mathrm{Tot}\, F$ "total ∞-category of F"

$$(X, \eta) \quad X \in \mathrm{Aff} \quad \eta \in F(X)$$

A module M over F is a functorial assignment $(X, \eta) \longmapsto M(X, \eta) \in \mathrm{Mod}(X)$

M is quasicoherent if all transition maps are equivalences.

(module is a section
qc module is a homotopy cartesian section)

$(X, \eta) \in \mathrm{Tot}\ F \quad X = \mathrm{Spec}\, A$, $M \in \mathrm{Mod}\, A$

$\rightsquigarrow$ can define $X_M = \mathrm{Spec}\,(A \oplus M)$

let
$$\begin{tikzcd} \Omega(X,\eta,M) \arrow[r] \arrow[d] & F(X_M) \arrow[d] \\ * \arrow[r, "\eta"'] & F(X) \end{tikzcd}$$

$\Omega(X,\eta)$: Mod $X \longrightarrow$ sSet

$M \longmapsto \Omega(X,\eta,M)$

<u>Def</u> $\mathbb{L}_F$ is the corepresenting object of $\Omega(X,\eta)$, i.e. $\Omega(X,\eta,M) \simeq \mathrm{Hom}(\mathbb{L}_F, M)$.

may not be representable by an ordinary module, might need a <u>stable</u> module

Relative version: $f: F \longrightarrow G \longrightarrow \mathrm{Spec}\, k$

$\Rightarrow$ define $\mathbb{L}_{F/G}$ as <u>cofiber</u>

$$f^* \mathbb{L}_G \longrightarrow \mathbb{L}_F \longrightarrow \mathbb{L}_{F/G}$$

Can now make sense of smooth / étale / lci maps between functors with cotangent complex

$\mathbb{R}\mathrm{Vect}_n(X) : s\mathrm{Alg}(\mathbb{C}) \longrightarrow s\mathrm{Set}$

$$A \longmapsto \mathrm{Vect}_n(X \times_{\mathbb{C}} \mathrm{Spec} A)$$

– represented by derived Artin stack

Cotangent complex (dualized)

$$\mathbb{L}^\vee_{\mathbb{R}\mathrm{Vect}_n(X)/\mathbb{C},\ \mathcal{E}} \simeq C^\bullet_{\mathrm{Zar}}(X, \mathrm{End}\ \mathcal{E})[1]$$