C. Barwick: ∞-Categories

Note Title 2/27/2008

$$\begin{tikzcd}
Alg/k \arrow[r] \arrow[dr] \arrow[dd] & Set \arrow[d] \\
 & Gpd \arrow[d] \\
sAlg/k \arrow[r] & sSet
\end{tikzcd}$$

Nerve construction: $\Delta \subset Cat$ full subcategory consisting of the categories

$[p]: [0 \to 1 \to \ . \ . \to p]$

$$\begin{tikzcd}
\mathrm{Hom}: Cat^{op} \times Cat \arrow[r] & Set \\
\Delta^{op} \times Cat \arrow[u, phantom, "\cup"] \arrow[ur] &
\end{tikzcd}$$

So categories give rise to simplicial sets:

$$\begin{tikzcd}
\nu: Cat \arrow[r] & sSet \\
Gpd \arrow[u, phantom, "\cup"] \arrow[ur] &
\end{tikzcd}$$

<u>Theorem</u> $C \in Cat$. 1. $\nu C \in sSet$ is fibrant iff $C \in Gpd$
($\Longleftrightarrow$ Kan set)

2. $C, D \in Gpd$ & $f: C \to D$ functor, then
$\nu f: \nu C \to \nu D$ is a weak equivalence iff
$f: C \to D$ is an equivalence

3. $\nu : \mathrm{Ho}\,\mathrm{Gpd} \hookrightarrow \mathrm{Ho}\,s\mathcal{S}et$

inclusion of homotopy categories; X is the nerve of a groupoid iff X is a 1-type

[Note: for Gpd every object is both fibrant & cofibrant for the natural model structure].

n-category $\quad 0 \leq n \leq \infty \quad$ intuitively:

have objects, 1-morphisms ..... n-morphisms

Idea: should be able to give a definition of (weak) n-category so that $\mathrm{Ho}(n\text{-Gpd}) \hookrightarrow \mathrm{Ho}(s\mathcal{S}et)$ with essential image the n-types ($\pi_i = 0 \quad i > n$)

We can instead reverse the logic:

Defn $\quad \infty\mathrm{Gpd} = s\mathcal{S}et_f \quad$ fibrant simplicial sets

Fibrant: can fill in  or 

- gives weak composability: exist lifts (compositions) unique up to contractible space of choices

Idea: (∞, n) Categories : ∞-categories such that the i-morphisms for $i > n$ are all invertible

- try to go up the ladder
  $(\infty,0)\text{-Cat} \hookrightarrow (\infty,1)\text{-Cat} \hookrightarrow (\infty,2)\text{-Cat} \hookrightarrow \cdots\cdots$

Example: SGA 6 : derived categories are "de nature essentiellement non recollables"

... suppose $D(U)$ $U \subset X$ formed a sheaf of categories on X $\rightsquigarrow$ then for a cover $\{U_i\}$ of X we'd have a pseudo lim?

$$D(X) \to \operatorname{Holim}\left[\prod D(U_i) \rightrightarrows \prod D(U_{ij}) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \prod D(U_{ijk}) \right]$$

ie we could get a global complex C on X with given $C|_{U_i} = C_i$ given only

$f_{ij} : C_i|_{U_{ij}} \xrightarrow{\sim} C_j|_{U_{ij}}$

& $\exists\, h : f_{jk} \circ f_{ij} \simeq f_{ik}$

$\Rightarrow$ would need strict commutativity of

$$\begin{tikzcd} & f_{01} \circ f_{12} \circ f_{23} \arrow[dl] \arrow[dr] & \\ f_{02} \circ f_{23} \arrow[dr] & & f_{01} \circ f_{13} \arrow[dl] \\ & f_{03} & \end{tikzcd}$$

Intuition: $\exists$ an ∞-category $L(X)$ s.t.

$$L(X) \longrightarrow \operatorname{holim}\left[\prod_i L(U_i) \rightrightarrows \prod_{ij} L(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right]$$

First approximation: consider only quasi-isomorphisms of complexes, get an ∞-groupoid by taking nerve; and taking fibrant replacement:

Theorem $\nu\, \mathrm{qis}\, \mathrm{Cplx}$ as an ∞-groupoid satisfies

$$\nu\, \mathrm{Cplx}^{\mathrm{quis}}(X) \xrightarrow{\sim} \operatorname{holim}\left[\prod_i \nu\, \mathrm{Cplx}^{\mathrm{quis}}(U_i) \rightrightarrows \cdots\cdots\right]$$

gluing space of objects!
(descent for these ∞-groupoids)

Pass to $(\infty,1)$ categories: different models
S-Cat, Segal Cat, quasi Cat, "$(\infty,1)$ Cat"

S-Category: a category enriched in $s\mathcal{Set}$
$X, Y, \ldots$ objects $\underline{\mathrm{Hom}}(X,Y) \in s\mathcal{Set}$,
compositions are maps of simplicial sets, strictly associative

C category with weak equivalences wC

$\rightsquigarrow$ $L(C, wC) \in S\text{-}Cat$

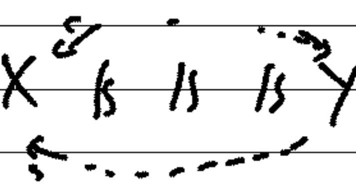
Dwyer-Kan hammock localization:

Obj $L(C,wC) =$ Obj C

$\underline{Hom}(X,Y)$: first make simplicial set of nerve of

zig zags of length n $\quad X \xleftarrow{\sim} X_0 \to X_1 \xleftarrow{\sim} X_2 \to \dots \xleftarrow{\sim} X_n = Y$

& morphisms are hammocks



& take colimit over n of these simplicial sets.

In fact if (C,wC) comes from a model category can stop at $n=2$:

$\underline{Hom}(X,Y) = Hom^2(X,Y) \quad X \xleftarrow{\sim} X_0 \to X_1 \xleftarrow{\sim} Y$

— Comes from fibrant & cofibrant replacements:

$$\begin{tikzcd}
X' \arrow[d, "\sim", "\text{trivial fibration}"'] \arrow[ddrr] & & \\
X \arrow[rr, dashed] & & Y \arrow[d, "\sim"', "\text{cofibration}"] \\
& & Y'
\end{tikzcd}$$

S-Cat has a nice model structure (J. Bergner)

weak equivalences $f : C \longrightarrow D$

1. $\pi_0 C \xrightarrow{\sim} \pi_0(D)$ ($\Rightarrow$ essential surjectivity)
2. $\underline{Hom}_C(X,Y) \xrightarrow{\sim} \underline{Hom}_D(fX, fY)$ (full faithfulness)

Problem with making an enriched version

$\underline{Hom}(A,B) \in S\text{-Cat}$ for $A, B \in S\text{-Cat}$:

$A \in \mathrm{Cat}$ & B, B' $\underline{\text{fibrant}}$ S-categories

$$B \xrightarrow{\sim} B'$$

$$\not\Rightarrow \quad \underline{Hom}(A,B) \xrightarrow{\sim} \underline{Hom}(A,B')$$

... not an internal model category!

Cartesian product doesn't satisfy good compatibilities.

Alternative story

Use Segal's delooping machine . .

Stasheff A loopspace is the same as a grouplike A_∞ space.

Segal: $\Omega X \times \Omega X \longrightarrow \Omega X$

coherence:

$$\begin{array}{c} \Omega X \times \Omega X \\ \uparrow \quad \uparrow \\ \Omega X \times \Omega X \times \Omega X \\ \uparrow\uparrow\uparrow \end{array}$$

Theorem A space Y is a loop space iff $\exists$ simplicial space $Z : \Delta^{op} \longrightarrow sSet$ s.t.

1. $Z_0 \simeq *$ contractible

Segal condition [2. $Z_p \xrightarrow{\sim} Z_1 \times \cdots \times Z_1$ (map coming from the simplicial structure is an equivalence)

3. $Z_1 \simeq Y$

ie we have not a product but

$$\begin{array}{ccc} Z_1 \times Z_1 & \xleftarrow{\sim} & Z_2 \\ & & \downarrow \\ & & Z_1 \end{array}$$

Let's do the same for $(\infty,1)$-categories

$$\begin{tikzcd} \underline{Hom}(X,Y) \times \underline{Hom}(Y,Z) & \underline{Hom}(X,Y,Z) \arrow[l, "\sim"'] \arrow[d] \\ & \underline{Hom}(X,Z) \end{tikzcd}$$

E category:

$$\begin{array}{ccc} \mathrm{Colax}(\underline{O}, E) & \simeq & sE \\ \cup & & \cup \\ \mathrm{Monoid}(\underline{O}, E) & \simeq & \text{Segal condition} \end{array}$$

Segal categories: consider simplicial space

$$A : \Delta^{op} \longrightarrow s\,Set$$

with A_0 = set (discrete! objects)

$$A_p \xrightarrow{\sim} A_1 \times_{A_0} \cdots \times_{A_0} A_1 \qquad p\text{-composable arrows}$$

These are the fibrant objects of a natural model category left adjoint: invert arrows

$$\infty\text{-Gpd} \rightleftarrows (\infty,1)\,\mathrm{Cat}$$

throw away noninvertible morphisms

The Segal picture is better wrt inversion but not wrt throwing away morphisms.

Completeness condition: change POV:
have A_0 play the role of the "interior"
– the category with forgotten noninvertible morphisms.

$$A_0 = \mathbb{R}\mathrm{Hom}(\bar{1}, A) \quad \text{where } \bar{1} = \bullet \rightleftarrows \bullet$$

More precisely $\displaystyle A_0 \xrightarrow{\sim} \operatorname*{holim}_{p \in (\Delta/\bar{1})^{op}} A_p$

... p varies over nerve of category $\bar{1}$.

$\Rightarrow$ better definition (Reedy style) of Dwyer-Kan

$$L(C,W) : \Delta^{op} \longrightarrow s\mathrm{Set}$$
$$p \longmapsto N.\,w(C^{[p]})$$

- nerve of category of weak equivalences of $\mathcal{C}^{[p]}$.   0-simplices: nerve of category of quasi isomorphisms.

[Charles Rezk]

This gives an <u>internal</u> model category, CSS, of complete Segal spaces: simplicial spaces st fibrants satisfy Segal & completeness properties & Quillen equivalence $CSS \rightleftarrows S\text{-}Cat$ & a Quillen adjunction $sSet \underset{(-)_0}{\rightleftarrows} \underline{CSS}$

We'll define $(\infty,1)$-categories as complete Segal spaces.

Consider pseudofunctor $Mod: X^{op}_{Zar} \to Cat$

$$U \mapsto Mod(\mathcal{O}_U)$$
$$f \mapsto f^*$$

can make into a functor...

Homotopic version: replace modules by complexes

$$X_{Zar}^{op} \longrightarrow (\infty,1)\ Cat$$
$$U \longmapsto LC_{pbr}(\mathcal{O}_U)$$
$$f \longmapsto f^*$$

gives a sheaf of $(\infty,1)$ categories:

$$LC_{pbr}(X) \xrightarrow{\sim} \mathrm{holim}\left[\prod_i LC_{pbr}(U_i) \rightrightarrows \prod_{i,j} LC_{pbr}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \right]$$

Strictification Theorem Suppose $F: I^{op} \to \mathrm{Mod\,Cat}$ is a left Quillen presheaf. Then $\mathrm{holim}\ LF$ is equivalent to $L\,\mathrm{Sect}_{hc}(F)$ homotopy Cartesian sections of F

(ie model structure on sections where cofibrants are homotopy Cartesian sections)

~ allows explicit computations of mapping spaces

Enlarged version: uses multi-∞ categories.

Consequence: $A \in E_n$ ring spectra $\Rightarrow$ $K\,A \in E_{n-1}$-ring spectrum (algebraic K theory – same for THH, TC etc)

$F : I^{op} \longrightarrow Cat$

$Sect(F)$:

- $(x_i)_{i \in I}$ $\quad x_i \in Obj\, F_i$
- $f^* x_j \longrightarrow x_i \quad \forall f : i \to j$

Consequence: weak diagrams $RHom(I, LM)$ can be strictified $\simeq L(M^I)$

... false for $(\infty, 2)$ categories, can't always strictify diagrams.