A. Braverman - Satake isomorphism for loop groups

Note Title 1/17/2008

Usual Satake isomorphism:

G split reductive, connected

K local non-archimedean field, $\mathcal{O} \subset K$ ring of integers

V spherical representation of $G(K)$: $V^{G(\mathcal{O})} \neq 0$
(irreducible) — can classify:

$\mathcal{H}$ = algebra of $G(\mathcal{O})$-biinvariant functions on $G(K)$ supported on fin. many double cosets

Satake isomorphism: $\mathcal{H} \cong \mathbb{C}[G^\vee]^{G^\vee} = K(\mathrm{Rep}\, G^\vee)$

$G^\vee$ = Langlands dual group

$\rightsquigarrow$ for V irred spherical, $V^{G(\mathcal{O})}$ 1-dim module over $\mathcal{H}$

$\Longleftrightarrow$ semisimple conjugacy classes in $G^\vee$.

General story: $\Gamma \supset \Gamma_0$ group $\supset$ subgroup

$\mathcal{H}(\Gamma, \Gamma_0)$ = vector space with basis $\Gamma_0 \backslash \Gamma / \Gamma_0$

multiplication = convolution

· $X, Y \subset \Gamma$ $\quad X \times_{\Gamma_0} Y \xrightarrow{m_{X,Y}} \Gamma$

$\mathcal{H}$ is an algebra if 1. $m_{X,Y}$ has finite fibers

2. $\mathrm{Im}(m_{X,Y})$ union of fin. many orbits

Satake: $\mathcal{H} = \mathcal{H}(G(K), G(\mathcal{O}))$

$G(\mathcal{O})\backslash G(K)/G(\mathcal{O}) \simeq \Lambda/W$

Λ = coweights of G, W = Weyl group

We're interested in $G(K((t))) \supset G(\mathcal{O}((t)))$

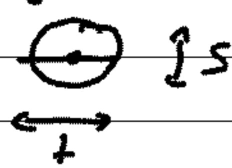
e.g. $K = \mathbb{F}_q((s))$

$G(\mathbb{F}_q((s))((t))) \supset G(\mathbb{F}_q[[s]]((t)))$

Problem: the set of double cosets is huge (e.g. uncountable)

To remedy: choose uniformizer $s \in \mathcal{O}$

($\longleftrightarrow$ transversal direction to our geometric flag)



$A = \mathcal{O}((t))$

$B = \mathcal{O}((t))[s^{-1}] \subset K((t))$

Consider pair $G(B), G(A)$, $G(A)\backslash G(B)/G(A)$

$= \Lambda^+$ again!

But property 1) above fails completely.....

Modify the situation: consider central extension of $G((t))$

Suppose G simply connected, Lie $G = \mathfrak{g}$ simple

$\Rightarrow \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow \hat{G} \longrightarrow G((t)) \longrightarrow 1$

& add loop rotation $\mathbb{G}_m \curvearrowright \hat{G}$

$G_{aff} = \hat{G} \rtimes \mathbb{G}_m$

$$G_{aff}(K) \longrightarrow \underset{\mathbb{G}_m(K)}{\overset{\Vert}{K^*}} \longrightarrow K^*/\mathcal{O}^* = \mathbb{Z}$$

Can define Hecke algebra for $\Gamma_0 \subset \Gamma_+$

subgroup of a semigroup:

replace $K((t))$ by $\mathbb{A}$ everywhere

$\Gamma_+ = G_{aff,+} = \pi^{-1}(\mathbb{Z}_{\geq 0}) \cup \Gamma_0 \cdot \underset{\text{center}}{\underbrace{K^*}}$

$\Gamma_0 = \langle G(\mathbb{A}), \underset{\text{central}}{\mathcal{O}^*}, \underset{\text{loop rotation}}{\mathcal{O}^*} \rangle$

$\mathcal{H}(\Gamma_+, \Gamma_0) \circlearrowleft K^*/\mathcal{O}^* = \mathbb{Z}$: action of center on basis elements

$\Rightarrow \mathcal{H}(\Gamma_+, \Gamma_0)$ is a module over $\mathbb{C}[v, v^{-1}]$

$$\mathcal{H}_{aff} = \mathcal{H}(\Gamma_+, \Gamma_0) \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}((v))$$

Now assume K is a functional field

Theorem a. $\mathcal{H}_{aff}$ is an algebra
[1. holds, 2. fails in a controlled way]

b. Satake isomorphism:

$\mathcal{H}_{aff}$ comm. algebra / $\mathbb{C}((v))$
& graded through $\pi : G_{aff}(K) \longrightarrow \mathbb{Z}$

In fact $\mathcal{H}_{aff}$ defined over $R \subset \mathbb{C}((v))$
series convergent for $|v| < 1$, $v \neq 0$

Example: $G = \mathbb{G}_m$, $\mathcal{H}_{aff}$ = homog. coord ring of $\mathbb{C}^* / v^{\mathbb{Z}}$
elliptic curve (in a projective embedding) $= E_v$

$$\mathcal{H}_{aff} = \bigoplus_{k \geq 0} H^0(E, \mathcal{L}^{\otimes k})$$

$\mathfrak{g}_{aff}$: affine Lie algebra, associated to affine root system

$\mathfrak{g}_{aff}^\vee$: Langlands dual affine Lie algebra, reverse orientations on Dynkin diagram.

in general $(\mathfrak{g}_{aff})^\vee \neq (\mathfrak{g}^\vee)_{aff}$ — can get <u>twisted</u> affine algebras.

$G_{aff}^\vee$ = connected affine KM group with $Lie(G_{aff}^\vee) = \mathfrak{g}_{aff}^\vee \supset \mathfrak{g}^\vee$
and $\mathfrak{g}^\vee$ integrates to $G^\vee \subset G_{aff}^\vee$

Rep $G_{aff}^\vee$: integrable reps = integrable reps of $\mathfrak{g}_{aff}^\vee$ s.t. $\mathfrak{g}^\vee \oplus \mathbb{C} \oplus \mathbb{C}$ integrates to $G^\vee \times \mathbb{C}^* \times \mathbb{C}^*$
— have central charge ≥ 0
& central charge 0 all come from $G_{aff}^\vee \longrightarrow \mathbb{C}^\times$.

Tensor of integrable reps: V, W irreducible
$\rightsquigarrow V \otimes W$ has infinite length

.... any rep can be tensored by a character of $\mathbb{C}^*$ via $G_{aff}^\vee \longrightarrow \mathbb{C}^*$ (loop rotation)

$\Rightarrow$ K = Grothendieck group of f.d. reps is a module over $\mathbb{C}[v, v^{-1}]$

<u>Lemma</u> $\hat{K} = K \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}((v))$ is an algebra

.... multiplicities in $V \otimes W$ are <u>finite</u>, & length is finite up to $\mathbb{Z}$-action...

<u>Satake</u>: $\mathcal{H}_{aff} \simeq \hat{K}$ as graded $\mathbb{C}((v))$-algebras

$|v|<1 \rightsquigarrow$ projective variety X_v

$\check{G}_{aff} \xrightarrow{\eta} \mathbb{C}^*$ $\quad \eta^{-1}(v)/\hat{G} = X_v$

[semistable G-bundles on elliptic curve]

– elliptic analog of $G^{\vee}/G^{\vee} = T^{\vee}/W$:

$X_v = (E_v \otimes \Lambda^{\vee})/W$

e.g. for $G = GL_n$ $\quad X_v = \mathrm{Sym}^n E_v$.

G semisimple $\rightsquigarrow$ this is a weighted projective space: not very dependent on E_v!

Choice of central extension $\hat{G}$ produces ample line bundle $\mathcal{L}_v$ on X_v

$$\hat{K}_v = \bigoplus_{n \gg 0} H^0(X_v, \mathcal{L}_v^{\otimes n})$$

Combinatorics:

Usual case $\mathcal{H}$ = functions on Λ^+

$\forall \lambda \in \Lambda^+$ $V(\lambda)$ = irreps of $G^\vee$ with highest weight λ

$$\mathcal{H} \underset{S}{\sim} K(\mathrm{Rep}\, G^\vee)$$

Let $f_\lambda = S^{-1}(V(\lambda))$, function on Λ^+

$$f_\lambda(\mu) = \begin{cases} 0 & \mu \text{ not weight of } V(\lambda) \\ q\text{-analog of } \dim V(\lambda)_\mu \end{cases}$$

q = # res field of k

q-analogs .... makes sense in affine case

$\lambda \in$ affine weight lattice $[V(\lambda)] \in \hat{K}(\mathrm{Rep}\, G^\vee_{aff}) \simeq \mathcal{H}_{aff}$

$\mathcal{H}_{aff}$ = functions of Λ^+_{aff}

... usual q-analog formula makes sense, but how to prove??

__Geometry__ Hall can be studied by looking at G-bundles on $S_k = \mathbb{A}^2 /\!/ \mu_k - \{0\}$

μ_k = roots of 1, acting on $\mathbb{A}^2$ antidiagonally

Let $G'_{aff} = G(\mathbb{F}_q(t,s,t^{-1},s^{-1})) \rtimes \mathbb{F}_q(s,s^{-1})^*$ (rotating t)

$$\Gamma = G'_{aff} \supset \Gamma_0 = G(\mathbb{F}_q[t,t^{-1},s]) \rtimes \mathbb{F}_q^*$$

$\Gamma \xrightarrow{\pi} \mathbb{F}_q[s,s^{-1}]^*$, $\quad \Gamma_k = \pi^{-1}(s^k)$

$$\Gamma_0 \backslash \Gamma_k / \Gamma_0 \longleftrightarrow Bun_G(S_k)$$

$$= G[ts^k, t^{-1}s^{-k}, s] \backslash G[t,t^{-1},s,s^{-1}] / G[t,t^{-1},s^{-1}]$$

$= G$ bundles on $U = \operatorname{Spec} \mathbb{F}_q[t,t^{-1},s]$

& $V = \operatorname{Spec} \mathbb{F}_q[ts^k, t^{-1}s^{-k}, s]$

along $\mathbb{G}_m \times \mathbb{G}_m$.