D. Gaitsgory - Kazhdan-Lusztig equivalence between affine algebras & quantum groups

Note Title 3/17/2008

Goal: give new proof of KL equivalence, in which role of Riemann-Hilbert is reduced [being developed by N. Rozenblyum]

For simplicity q not root of unity

$\mathcal{O}_q$ = quantum category $\mathcal{O}$

$\mathcal{O}_{aff}$ = continuous reps of $\hat{g}$ which are Iwahori integrable, $\hat{g}-\mathrm{mod}_\kappa^I$

Here $q : \Lambda \times \Lambda \longrightarrow \mathbb{C}^*$ Λ = weights

$\kappa : h \otimes h \longrightarrow \mathbb{C}$

& q = exponentiate the inverse of $\frac{\kappa}{2}$ (form on h^*). include critical shift: $\kappa = 0$ means critical level

$\mathcal{O}_q \in$ {braided monoidal categories}

$\mathcal{O}_{aff} \in \{\text{chiral categories}\}_{\mathcal{D}\text{-mod}}$: will be discussed today

Comparison:

$$\begin{tikzcd}
\{\text{chiral cat.}\}_{\mathcal{D}\text{-mod}} & & \{\text{braided monoidal}\} \arrow[dl, "\text{equivalence modulo representability}"] \\
& \{\text{chiral categ.}\}_{\text{top}} \arrow[ul, "R\text{-}H"] &
\end{tikzcd}$$

<u>Claim</u>

$$\begin{tikzcd}
\mathcal{O}_{aff} \in \{\text{chiral}\}_{\mathcal{D}\text{-mod}} \arrow[d, "S", "\text{today}"'] & & \mathcal{O}_q \in \{\text{braided mon.}\} \arrow[d, "\text{Friday lecture}"] \\
\underbrace{FS_{\mathcal{D}} \in \{\text{chiral}\}_{\mathcal{D}\text{-mod}}}_{\mathcal{D}\text{-module form of factorizable sheaves}} & & \underbrace{FS_{top} \in \{\text{chiral}\}_{top}}_{\text{topological form of factorizable sheaves}} \arrow[ll, "R\text{-}H"]
\end{tikzcd}$$

X algebraic curve (smooth). <u>Chiral algebra</u>:

$\mathcal{A}$: $\mathcal{D}$-module on X endowed with

$$j_* j^* (\mathcal{A} \boxtimes \mathcal{A}) \longrightarrow \Delta_* \mathcal{A}$$

map of $\mathcal{D}$-modules on $X \overset{\Delta}{\hookrightarrow} X \times X \overset{j}{\hookleftarrow} X \times X \setminus \Delta$

+ skew symmetry & Jacobi.

Unit : $\mathcal{O}_X \hookrightarrow \mathcal{A}$ s.t. $j_* j^*(\mathcal{A} \boxtimes \mathcal{O}) \to \Delta_* \mathcal{A}$
is the canonical map.

Alternative: A factorization algebra is
a collection of $\mathcal{D}$-modules $\mathcal{A}_n$ over X^n $n=1,2,\ldots$
with $\mathcal{A}_2|_{\Delta_X} \xrightarrow{\sim} \mathcal{A}_1$

$$\mathcal{A}_2|_{X \times X - \Delta_X} \simeq \mathcal{A}_1 \boxtimes \mathcal{A}_1|_{X \times X - \Delta}$$

& so on for any n : factorization property.

Equivalence between chiral & factorization algebras:
Cousin short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}_2 & \to & j_* j^* \mathcal{A}_2 & \to & \Delta_! \Delta^! \mathcal{A}_2 & \to & 0 \\ & & & & \| & & \wr\| & & \\ & & & & j_*(\mathcal{A}_1 \boxtimes \mathcal{A}_1) & \to & \Delta_! \mathcal{A}_1 & & \end{array}$$

So $\mathcal{A} := \mathcal{A}_1$ is a chiral algebra,

Conversely $\mathcal{A}$ determines $\mathcal{A}_2$ as the core
of the multiplication etc.

[Note: in def. of factorization can replace $\mathcal{D}$-module by $\mathcal{O}$-module: connection comes from factorization via unit axiom]

Suppose $\left\{\begin{array}{c} \mathcal{Y}_n \\ \pi_n \downarrow \uparrow 1_n \\ X^n \end{array}\right\}$ is a factorization ind-scheme (flat connection on $\mathcal{Y}_n$ automatic from unit section)

Take $\mathcal{Y}_n = Gr_{G,n} = \left\{\begin{array}{l} x_1 \cdots x_n \in X, \mathcal{P}\ G\text{-bundle} \\ + \text{triv off } x_1, \cdots x_n \end{array}\right\}$
unit = trivial G-bundle:

$$1_n : X^n \longrightarrow \mathcal{Y}_n .$$

Let's take the δ-function $\mathcal{D}$-module on $\mathcal{Y}_n$ supported on section 1_n, $\delta_{1,n}$.

We'll take its direct image to X^n as a quasicoherent sheaf:

$$\mathcal{A}_n = \pi_{n*}(\delta_{1,n}) \in QCoh(X^n)$$

is a chiral algebra.

Perform this for $\mathcal{Y}_n = Gr_{G,n}$ $\Rightarrow$ obtain Kac-Moody chiral algebra $\mathcal{A}_G$.

Can twist by κ as above: defines TDO on $Gr_{G,n}$, can take κ-twisted $\mathcal{D}$-modules, Γ-functors etc. $\Rightarrow$ $\mathcal{A}_{G,\kappa}$ Kac-Moody vertex algebra of level κ.

Given a chiral algebra $\Rightarrow$ notion of chiral module:

A chiral $\mathcal{A}$-module $\mathcal{M}$ is a $\mathcal{D}$-module on X with $j_* j^*(\mathcal{A} \boxtimes \mathcal{M}) \longrightarrow \Delta_* \mathcal{M}$ satisfying Lie axiom wrt bracket on $\mathcal{A}$.
[$\Leftrightarrow$ square 0 extensions]

$x \in X$ $\Rightarrow$ look at chiral modules supported at x.

<u>Lemma</u> Chiral modules supported at x for $\mathcal{A}_{G,\kappa}$ $\simeq$ $\hat{\mathfrak{g}}\text{-mod}_\kappa$ level κ reps of Kac-Moody algebra (on vector space underlying the $\mathcal{D}$-module)

Chiral modules are analogs of E_2-modules ...

$k \in \mathbb{N}$. We define the notion of chiral $\mathcal{A}$-module on X^k : collection of quasicoherent sheaves $\mathcal{M}^k, \mathcal{M}^{k+1}, \ldots$

$$\begin{array}{cc} \mathcal{M}^k, & \mathcal{M}^{k+1}, \ldots \\ | & | \\ X^k & X^{k+1} \end{array}$$

endowed with factorization isomorphisms –

e.g. $X^{k+n} \supset X^{k+n}_{disj}$ (disjoint k-tuple & n-tuple on X)

require $\mathcal{M}^{k+n}|_{X^{k+n}_{disj}} \simeq \mathcal{M}^k \boxtimes \mathcal{A}_n |_{X^{k+n}_{disj}}$

... ie behavior of $\mathcal{M}^{k+n}$ on remaining n points is governed by $\mathcal{A}$. itself.

$\mathcal{M}^k$ does not need to have the structure of a $\mathcal{D}$-module – can tensor with quasicoherent sheaves on X^k.

The category of such has a flat connection over X^k, so can ask for flat objects ($\mathcal{D}$-objects) in here.

e.g. if $k=1$, $\mathcal{D}$-objects in factorization $\mathcal{A}$-modules on X $\Longleftrightarrow$ chiral $\mathcal{A}$-module as above.

Suppose $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are chiral $\mathcal{A}$-modules.

A chiral pairing of $\mathcal{M}_1, \mathcal{M}_2 \longrightarrow \mathcal{M}_3$ is a map of $\mathcal{D}$-modules

$$j_* j^* (\mathcal{M}_1 \boxtimes \mathcal{M}_2) \longrightarrow \Delta_! \mathcal{M}_3$$

compatible with $\mathcal{A}$ actions:

on X^3 $\quad j_* j^* (\mathcal{A} \boxtimes \mathcal{M}_1 \boxtimes \mathcal{M}_2) \rightrightarrows \Delta_! \mathcal{M}_3$

add up to zero.

– Kazhdan Lusztig binary operation on Lie algebra modules (defined via coinvariants on $\mathbb{P}^1$) is roughly this — this is part of K-L tensor product corresponding to coinvariants of square of R-matrix.

Problem: associativity is on the other side of Riemann Hilbert. Also in general this "tensor product" is not representable.

— we'll discuss an alternative.

<u>Lemma</u> To give a chiral pairing $\mathcal{M}_1, \mathcal{M}_2 \to \mathcal{M}_3$ is equivalent to specifying an object

$\widetilde{\mathcal{M}}$, a chiral $\mathcal{A}$-module on X^2 with connection s.t. $\widetilde{\mathcal{M}}|_{X^2-\Delta} \simeq \mathcal{M}_1 \boxtimes \mathcal{M}_2|_{X^2-\Delta}$ & $\widetilde{\mathcal{M}}|_\Delta \simeq \mathcal{M}_3$

— as the cone of $j_*j^*(M_1 \boxtimes M_2) \longrightarrow \Delta_! \mathcal{M}_3$ & vice versa,

$\Rightarrow$ discuss chiral $\mathcal{A}$-modules $\forall k$ instead of trying to directly formulate tensor product.

Category over Y scheme (say affine)
$\Rightarrow$ module category over $QC(Y)$

$Y' \longrightarrow Y$ can base change $C_{Y'} = Y' \underset{Y}{\times} C_Y$ as $D^{perf}(Y') \underset{D^{perf}(Y)}{\otimes} C_Y$.

$\exists$ notion of category over Y with a connection:
$Y \rightrightarrows Y$ infinitesimally close give data of identifications etc.

Chiral category over X ($\mathcal{D}$-mod sense) is the data of categories $\mathcal{C}_n \longrightarrow X^n$ with factorization:

e.g. $X \longrightarrow X^2$ $\quad \mathcal{C}_2|_X \simeq \mathcal{C}_1$

$\mathcal{C}_2|_{X \times X \setminus \Delta} \simeq \mathcal{C}_1 \boxtimes \mathcal{C}_1|_{X \times X \setminus \Delta}$

+ unit object $\mathbb{1}_n \in \mathcal{C}_n$.

e.g. unit example $\mathcal{C}_n = QC(X^n)$.

e.g. $\mathcal{Y}_n$ factorization scheme $\Rightarrow$ take

$\downarrow \mathcal{I}_n$ $\quad \mathcal{C}_n = QC(\mathcal{Y}_n)$

X^n

$\mathcal{C}_n' = \mathcal{D}\text{-mod}(\mathcal{Y}_n)^{\text{vert}}$ connection along fiber

($\mathcal{Y}_n$ of ind finite type)

Units $\Rightarrow$ any $\mathcal{C}_n$ automatically acquires connection over X^n, compatible with factorization.

e.g. A chiral algebra $/X$,

$\{\mathcal{C}_k = \text{chiral modules over } X^k\}$ is a chiral category

— reps of a chiral algebra form a chiral category.

$\mathcal{A}_{G,\kappa} \Rightarrow$ {Chiral $\mathcal{A}_{G,\kappa}$-mod} $\leftarrow \mathcal{O}_{aff}(G)$

$\mathcal{A}_{H,\kappa} \Rightarrow$ {chiral $\mathcal{A}_{H,\kappa}$-mod} $\leftarrow \mathcal{O}_{aff}(H)$

[normalization for H: centered at $0 = crit_H$]

$\mathcal{O}_{aff}(H)$ - category of modules over $\hat{\mathfrak{h}}((t))$ which are integrable wrt $H[[t]]$.
It's semisimple & is equivalent to H-modules as long as κ is nondegenerate, via

$$V \longmapsto V^{\mathfrak{h} \otimes t\, \mathbb{C}[[t]]}$$

Let $\{\mathcal{C}_n\}$ be a chiral category.
$\exists$ notion of a chiral algebra in $\mathcal{C}$
— $\mathcal{A}_n \in \mathcal{C}_n$ with factorization

— and given such $\mathcal{A}_\bullet$ have chiral category of chiral $\mathcal{A}_\bullet$-modules in $\mathcal{C}$.

$\mathcal{C}^1, \mathcal{C}^2$ chiral categories have notion of functor $F: \mathcal{C}^1 \to \mathcal{C}^2$: compatible with factorization; DON'T require unital.

Then $F(\mathbb{1}_{\mathcal{C}_i}) = \mathcal{A}_n$ form a chiral algebra in $\mathcal{C}^2$, & since all of $\mathcal{C}^1$ are modules for the unit get lifting

$$F : \mathcal{C}^1 \longrightarrow \{\mathcal{A}\text{-mod in } \mathcal{C}^2\}$$

Now take $\mathcal{C}^1 = \mathcal{A}_{G,\kappa}$ -mod

$\mathcal{C}^2 = \mathcal{A}_{H,\kappa}$ -mod

$F : \mathcal{C}^1 \longrightarrow \mathcal{C}^2$: BRST cohomology wrt $\mathfrak{n}$

... implements the critical shift.

$$\mathrm{BRST}_{\mathfrak{n}} : \mathcal{A}_{G,\kappa}\text{-mod} \longrightarrow \mathcal{A}_{H,\kappa}\text{-mod}$$

$x \in X$: on fibers this is

$$\begin{array}{ccc} \hat{\mathfrak{g}}\text{-mod}_\kappa & \longrightarrow & \mathfrak{h}\text{-mod}_\kappa \\ \| & & \| \\ \mathcal{C}^1|_x & & \mathcal{C}^2|_x \end{array}$$

$M \longmapsto H^{\infty/2}(\mathfrak{n}((t)), M)$ semi-infinite cohomology

---- tensor by a Clifford algebra, which is a chiral algebra, have total differential.

Let $\mathcal{B} = BRST_{\mathfrak{n}}(\mathcal{A}_{G,\kappa})$ unit object in $\mathcal{A}_{G,\kappa}$-mod

--- this is a chiral algebra in $\mathcal{A}_{H,\kappa}$-mod, which in fact lives in $\mathcal{O}_{aff}(H)$ - integrable for Taylor loops in H.

<u>Def</u> $FS_{D\text{-mod}}$:= the chiral category of $\mathcal{B}$-modules in $\mathcal{O}_{aff}(H)$.

Claim: this is same $\mathcal{B}$ (or $\mathcal{D}$-module version thereof) as in the previous talk.

$$BRST_{\mathfrak{n}} : \mathcal{O}_{aff}(G) \longrightarrow FS_{D\text{-mod}}$$

<u>Theorem</u> For κ irrational this is an equivalence of chiral categories. [NOT true for κ rational - it's <u>conservative</u>, not an equivalence]

Let's think of $\mathcal{V}$ as a chiral algebra in dg vect ... nontrivial in many degrees ($0 \mapsto -|\omega|$).

$\mathcal{B}$ is lattice gradd, $\mathcal{B}_{Ran}$ over the Ran space is actually a perverse sheaf (single degree) $\mathcal{L}$ is the $\mathcal{B}$ from the previous talk.

$$H \qquad\qquad \check{H}$$
$$x \qquad\qquad x^{-1}$$

$\mathcal{O}_{aff}(H) \doteq$ $\mathcal{D}$-mod $(Gr_{\check{H}})$ of level x^{-1}.

$\hat{h}\text{-mod}_x^{H[[t]]} \simeq$ as chiral categories

Easier to relate RHS to factorizable sheaves – λ component factorizes as X^{λ}.