S. Gukov – Link homologies, instantons & BPS invariants

Note Title 2/13/2008

K (knot or link) in Y ($= S^3, \mathbb{R}^3, \ldots$)

R rep of group G (or of $\mathfrak{g}$ - Lie G)
(or of $U_q\mathfrak{g}$)

$\rightsquigarrow$ link homology $\mathcal{H}^{G,R}_{*,*}(K)$

Old link invariants: R rep of $U_q\mathfrak{g} \rightsquigarrow$
$P_{\mathfrak{g},R}(K)$ polynomial in q.

Relation: $\chi_q(H^{\mathfrak{g},R}_{*,*}(K)) = P_{\mathfrak{g},R}(K)$
Poincaré polynomial

Two well-known physical interpretations of polynomial invariants $P_{\mathfrak{g},R}(K)$:

3d TQFT:
Chern-Simons-Witten theory

Topological strings:
enumerative geometry
of Calabi-Yau 3-folds

each admits a
"homological lift"

4d TQFT

<u>3d TQFT</u> is a functor

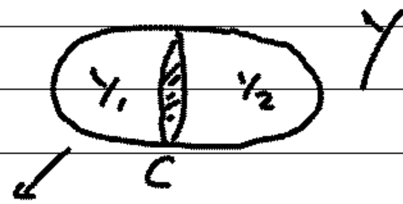
3-manifolds Y $\rightsquigarrow$ number $Z(Y)$ (partition function)

2-manifold C $\rightsquigarrow$ vector space $\mathcal{H}_C$

...

e.g. Heegaard decomposition:

$$Y = Y_1 \cup_C Y_2$$



$Z(Y) = \langle Y_1 | Y_2 \rangle$ $\qquad \langle Y_1 | \in \mathcal{H}_C \ni | Y_2 \rangle$

$\langle\ \rangle$ scalar product on $\mathcal{H}_C$

<u>4d gauge theory</u> M closed 4-manifold $\rightsquigarrow Z(M)$

On $\mathbb{R} \times Y^3$ $\rightsquigarrow$ Hilbert space of groundstates $\mathcal{H}_Y$

On $\mathbb{R}^2 \times C$ $\rightsquigarrow$ category $\mathcal{F}(C)$ of boundary conditions

geometric picture! PDEs on 4-manifolds

eg instanton equations

$Z(M) = \chi(\mathcal{M}(M))$ Euler char of space of solutions of the PDE on M (instantons)

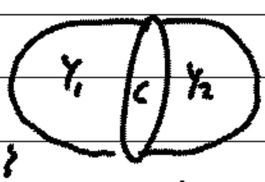
$\mathcal{H}_Y = H^*(\mathcal{M}(Y))$ solutions of reduced PDE on a 3-manifold Y. (monopole eqs)

$\mathcal{F}(C) =$ · Fukaya category of $\mathcal{M}(C)$
or · $D^b(\mathcal{M}(C))$

correspond to A or B type boundary conditions
$\mathcal{M}(C)$ = vortex moduli space

eg <u>Donaldson-Witten theory</u>: moduli of flat G-connections on C, symplectic $\rightsquigarrow$ look at Fukaya category

$\mathcal{H}_Y = HF^{inst}(Y)$ instanton Floer homology of Y.



Heegaard or other "generic" splitting

$\mathcal{B}_1 \in \mathcal{F}(C) \ni \mathcal{B}_2$

$\mathcal{B}_i$: Lagrangian in moduli of flat G-connections on C $\rightsquigarrow$ take their Floer complex:

$\mathcal{H}_Y = \mathrm{Hom}^*_{\mathcal{F}(C)}(\mathcal{B}_1, \mathcal{B}_2) = H^{\text{symplectic}}_{\mathcal{M}(C)}(\mathcal{B}_1, \mathcal{B}_2)$
$= HF^{inst}(Y)$ // Atiyah-Floer conjecture

$\mathcal{M}(C)$ = moduli of flat G-connections on C, <u>symplectic</u>.

- Incorporating knots & links in 3d TQFT
  "line operators": K knot $\rightsquigarrow$ number
  $Z_{Y,K} = P(t)$

- In 4d TQFT: associate to 3-manifold with knot or link a vector space: use
  Surface operators on 2d submanifold

Embedded surface $D \subset M \rightsquigarrow$ number $Z_{M,D}$
... study moduli of "ramified instantons"
(Kronheimer - Mrowka)

- $M = \mathbb{R} \times Y \rightsquigarrow \mathcal{H}_{Y,K}$ knot homology
  $\;\cup \qquad \cup$
  $D = \mathbb{R} \times K$

  $M = \mathbb{R}^2 \times C \rightsquigarrow \mathcal{F}(C,p)$ category associated to a curve and ramification data at p.
  $\;\cup \qquad \cup$
  $D = \mathbb{R}^2 \times p$

Decategorification: What is $\chi(\mathcal{H}_{Y,K})$?
— compactification / reduction on S^1:

$\mathbb{R}\times Y \rightsquigarrow \mathcal{H}_Y \rightsquigarrow \chi(\mathcal{H}_Y) = \mathrm{Tr}_{\mathcal{H}}(-1)^F$
index
$= Z(S^1\times Y)$

So $\chi(\mathcal{H}_{Y,K}) = Z(S^1\times Y, S^1\times K)$

Ex. Donaldson-Witten theory $G = SU_2$
$\chi(\mathcal{H}_{Y,K})$ = Casson invariant of Y
+ equivariant knot signature of K.
— piecewise constant function, jumps at
roots of Alexander polynomial

- Seiberg-Witten theory $\begin{cases} F_A^+ - i(\psi\bar{\psi})^+ = 0 \\ \not{D}_A \psi = 0 \end{cases}$

ψ spinor,
A abelian connection

$\chi(\mathcal{H}_{S^3,K}) = \Delta(K,q)$ Alexander polynomial of K

... q enters into parameters of surface operators
— $\mathcal{H}_K$ is in fact bigraded, electrically

... in fact have grading by $s \in H_1(Y \setminus K)$ / torsion
spin structures

<u>Remark</u> $P_{g,R}(q;K)$ or e.g. Jones polynomial $J(q) = P_{sl_2,V}(q,K)$
from 4d TQFT?

Any 4d topological gauge theory on $M = S^1 \times Y$
has the symmetry corresponding to change of
orientation $Y \longrightarrow \overline{Y}$:
$Z(S^1 \times Y) = Z(S^1 \times \overline{Y})$

Now suppose this $= Z_{CS}(q)$ q primitive root of unity

$\Rightarrow Z(S^1 \times \overline{Y}) = Z_{CS}(q^{-1})$ since

Chern-Simons is based on action

$$S_{CS} = \frac{k}{4\pi} \int_Y \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3 \right)$$

So will only find (from 4d TFT) knot invariants
s.t. $Z(q) = Z(q^{-1})$ ... eg Alexander poly.
BUT Jones polynomial is <u>not</u> of this type!

So to circumvent this study 4d theory on $Y \times$ Interval instead – now must fix boundary conditions $\mathcal{B}_\pm$ at ends

Hence to categorify these invariants need 5d TQFT on $\mathbb{R} \times I \times Y$

4d TQFT	Topological strings
• any G, any Y, ...	• Classical groups on special Y, eg lens spaces, maybe Seifert Manifolds
• doubly graded	• triply graded (package doubly graded knot homologies in efficient way) ... one vector space for each type A, B C, D
• cutting & pasting/functoriality	• hard to see cut/paste
• hard to compute (even for $sl_{n \geq 2}$	• very computable (refined topological vertex)