Sergei Gukov - Link homology & BPS invariants

Note Title 2/21/2008

Three stories:

(K) Homological invariants of 3-manifold $M_3 = S^3/\Gamma$
$\Gamma \subset SU_2 \leftrightarrow ADE$: lens spaces.
eg $\Gamma = \mathbb{Z}/n$

(I) Equivariant instanton counting on $\mathbb{R}^4$
with gauge group G of type Γ ADE
eg $G = U_n$

(C) Curve counting on X CY3
$X = $ " $\mathbb{C}^2/\Gamma$ " $\longrightarrow \mathbb{P}^1$
fibration over $\mathbb{P}^1$ with ADE singular fibers.
e.g $\mathbb{C}^2/\Gamma \times_{\mathbb{C}^*} \mathcal{O}(-1)$ over $\mathbb{P}^1$

Connection between the three:
M-theory on 11-dim space $\mathbb{R} \times \mathbb{R}^4 \times X$
time, instantons, curve counting

(C) POV: counting BPS states on X

Curve counts (GW theory) : integration over (virtual fundamental class on) moduli of instantons $\Sigma_g \longrightarrow X$ $\rightsquigarrow$ $GW_{g,\beta}(X) \in \mathbb{Q}$, $\beta \in H_2(X;\mathbb{Z})$

– these make sense in any dimension, but in dimension three have <u>integer</u> enumerative invariants of X:

- Gopakumar–Vafa invariants
- Donaldson – Thomas invariants (from abelian gauge theory)

M theory POV on G-V invariants:
have M2 branes supported on 3-dimensional submanifolds in spacetime : have Hilbert space for fixed time slice, & $\mathcal{H}_{BPS} = H^*_Q$ Cohomology of BRST charge on Hilbert space

To be time-translation invariant we need our M2 brane to wrap a 3-manifold of the form $\mathbb{R} \times \Sigma$; we'll look at those which appear as point particles in $\mathbb{R}^4$:
so place $\Sigma \subset X$, $p \in \mathbb{R}^4$ $\Rightarrow$

M2 brane on $\mathbb{R} \times p \times \Sigma$: looks like point particle in $\mathbb{R}^4$

$SU(2)_L \times SU(2)_R \cong SO(4) \circlearrowright \mathbb{R}^4 \rightsquigarrow 2$ spins
for this particle, (j_L, j_R)
describing the $SU(2) \times SU(2)$ representation.

$\gamma\beta = \phi_*[\Sigma] \in H_2(X, \mathbb{Z})$ class of Σ.

So on spacetimes $\mathbb{R} \times \mathbb{R}^4 \times X$ we find that
$\mathcal{H}_{BPS}$ is triply graded, by

$$\underset{j_L}{\mathbb{Z}} \oplus \underset{j_R}{\mathbb{Z}} \oplus \underset{\gamma\beta}{\mathbb{Z}^{b_2}} \qquad b_2 = \dim H_2(X, \mathbb{Z})$$

[In fact have also an additional $\mathbb{Z}/2$ grading
by statistics (boson/fermion) which we'll ignore]

We're counting thus embedded curves not
stable maps! $\rightsquigarrow$ get integer invariants ..
also genus g is implicitly contained in
(j_L, j_R)

Let $m = 2j_L + 1$ (dim of $SU(2)$ rep of spin j_L)
$n = 2j_R + 1$

$D_{m,n,\beta} := \mathrm{Tr}\,(-1)^F$ on $\mathcal{H}_{BPS}^{m,n,\beta}$ ($F = \mathbb{Z}/2$ grad'n)

- still not a good invariant [... really count just how many times m,n representation occurs]

For generic X this depends on Kähler structure & complex structure

To get an invariant let $D_{n,\beta} = \mathrm{Tr}_{\mathcal{H}_{BPS}^{*,n,*}} (-1)^F (-1)^m$
— does give an invariant,
the DT or GV invariants

In good cases $D_{n,m,\beta} = \dim H^m(\mathcal{M}_{n,\beta}(X))$
where $\mathcal{M}_{n,\beta}$ = moduli space of
pairs $\{ \Sigma_g \subset X$ & $U(1)$ local system on $\Sigma \}$
with genus g related to m.
eg if Σ is rigid get just Jacobian of Σ.

... so one of the gradings has cohomological origin.

Remark: $D_{n,m,\beta}$ invariants if X is rigid
& $\beta \in H_2(X)$ - no wall crossings possible!

(nothing for ys to pair up with to get wall crossings)

Examples come from toric (local) Calabi-Yau:

ex. $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$,

$H_2(X) = \mathbb{Z}$

or its quotient by Γ ... when resolving singularity find new deformations.

(I) In string or M theory on $\mathbb{C}^2/\Gamma$ one finds ADE gauge theory supported on the singular locus

In our case $\mathbb{C}^2/\Gamma \otimes \mathcal{O}(-1) \times \mathbb{P}^1$

get G gauge theory on $\mathbb{R} \times \mathbb{R}^4 \times \mathbb{P}^1$
(we'll ignore the compact factor)

BPS states on $\mathbb{R} \times \mathbb{R}^4 \times X$:

$\mathbb{R} \times p \times \Sigma$ completely localized in $\mathbb{R}^4$ $\rightsquigarrow$ get instanton in $\mathbb{R}^4 \simeq \mathbb{C}^2$

Consider $\mathbb{T} = S^1 \times S^1 \circlearrowright \mathbb{C}^2 \longleftrightarrow j_L, j_R$
(torus of $SO(4)$) $\quad (z_1, z_2) \mapsto (e^{i\varepsilon_1} z_1, e^{i\varepsilon_2} z_2)$

$\mathcal{M}(k, G_p) =$ moduli space of ASD connections on $\mathbb{R}^4$, $c_2 = k$ instanton number
$\circlearrowleft$

$\widetilde{\mathbb{T}} = S^1 \times S^1 \times \mathbb{T}_G \qquad \mathbb{T}_G =$ torus of G

$$Z_{inst} = \sum_k \Lambda^k \int_{\mathcal{M}_{inst}(k, G)} 1 \in \mathbb{Q}(\varepsilon_1, \varepsilon_2, \vec{a})[\![\Lambda]\!]$$

weights of $\widetilde{\mathbb{T}}$

Nekrasov partition function

Parameters $\varepsilon_1, \varepsilon_2, \vec{a}$ are analogs of $\beta \in H_2(X)$ in the $\mathbb{C}$ description.

Ex. $\Gamma = 1$ trivial $\quad M_3 = S^3$

$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\quad G_p = U(1)$

[in general G_p would have a factor $U(1)^{b_2}$ on a CY3 with bigger H_2]

$$Z_{inst} = \sum_k \Lambda^k \int_{S^k\mathbb{R}^4} 1 = \sum_k \frac{\Lambda^k}{k!\,(\varepsilon_1\varepsilon_2)^k}$$
$$= e^{\Lambda/\varepsilon_1\varepsilon_2}$$

K-theoretic version ... comes from 5-dim gauge theory:

$$Z_K = \exp\left(\sum_d \frac{\Lambda^d}{d(1-e^{\varepsilon_1 d})(1-e^{\varepsilon_2 d})} \right)$$

.... factor in GW theory corresponding to an isolated curve

||| To avoid complications we'll continue to assume Γ is trivial .....

(K) let's consider knot or link $K \subset M_3$ (& maybe later representations of quantum groups..

(I) introduce surface operators on $\mathbb{R}^2 \subset \mathbb{R}^4$

(C) Count also curves with boundary on a Lagrangian submanifold $L_K \subset X$

BPS invariants: $\mathbb{R}\times\mathbb{R}^4\times X$

now look at M5 branes, on 6-dimensional submanifolds.

$$\begin{array}{rccccc} & \mathbb{R} & \times & \mathbb{R}^4 & \times & X \\ & \cong & & \cup & & \cup \\ M5 = & \mathbb{R} & \times & \mathbb{R}^2 & \times & L_K \\ & \cong & & \cup & & \cup \\ M2 = & \mathbb{R} & \times & p & \times & \Sigma \end{array}$$

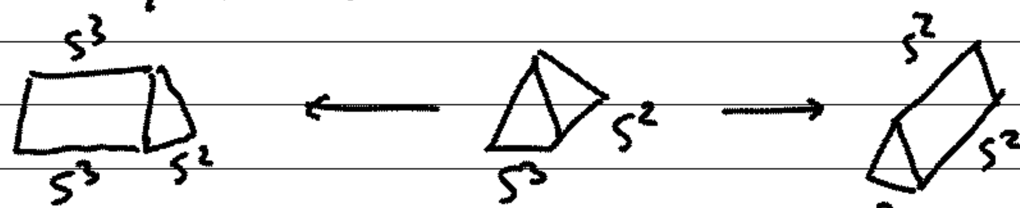
[$\mathbb{R}^2$ preserved by two $SU(2)$s]

can now end on the M5: $\partial\Sigma \subset L_K$

$[\Sigma] = \beta \in H_2(X, L_K)$ relative homology class

Assume $\Gamma = 1$ trivial, $X = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$:

asymptotically a cone on $S^3\times S^2$

$T^*S^3 =$  $= X$

$\cup \qquad\qquad\qquad\qquad \cup$

L_K = conormal to $K \subset S^3$ $\dashrightarrow$ L_K

"same" Lagrangian

transition doesn't change scarcely asymptotically,

C. Taubes says can be extended inwards?

produces Lagrangian $\mathcal{L}_K$ for every knot diagram.

$$T^*S^3 \xleftarrow{\text{deform}} \quad \text{[cone]} \quad \xrightarrow{\text{resolve}} X$$

$$\begin{array}{ccc} T^*S^3 = \{\sum z_i^2 = \mu\} & \sum z_i^2 = 0 & \text{resolved con. fold} \end{array}$$

$\mathcal{L}_K$: equator in S^2 $\times$ knot in S^3 + cone — cone on $K \times$ equator — cone on $K \times$ equator

equator is the generator of $H_2 = \mathbb{Z}$

$\Rightarrow$ can have discs covering either hemisphere of core $\mathbb{P}^1$ & ending on equator inside $\mathcal{L}_K$.

So $\mathcal{H}_{BPS}$ is $\mathbb{Z} \oplus \mathbb{Z} \oplus \underbrace{\mathbb{Z}}_{= H_2(X, \mathcal{L}_K)}$ graded