Joel Kamnitzer - Braid group actions on certain derived categories

Note Title 3/6/2008

Fix $m \geq 2$. V rk m vector bundle on $\mathbb{P}^1$.
A Hecke modification of V at $x \in \mathbb{P}^1$ of weight k is a pair (V', φ)

V' : rk m vector bundle
$\varphi : V \hookrightarrow V'$, $\operatorname{coker} \varphi \cong \mathcal{O}_x^{\oplus k}$

$$Gr_x(k) = \left\{ \begin{array}{c} \text{Hecke modifications of} \\ \text{the trivial bundle} \end{array} \right\} \cong G(k, \mathbb{C}^m)$$

Grassmannian of k-planes in $\mathbb{C}^m$

$$Gr_x(k_1, \dots, k_n) = \{ \underset{\substack{\| \\ \text{trivial}}}{V_0} \overset{k_1}{\hookrightarrow} V_1 \hookrightarrow V_2 \cdots \overset{k_n}{\hookrightarrow} V_n \}$$

all occurring at x

iterated bundle of Grassmannians
$1 \leq k_i \leq m-1$

Problem: Define an action of B_n on $D\mathrm{Coh}(Gr(k_1, \dots k_n))$ [weak action] more precisely $B_n \twoheadrightarrow S_n$ will permute the k_i's & B_n will act by functors between the corresponding categories

<u>Motivation</u> (from Khovanov knot homology)

$S_n \circlearrowright \Lambda^{k_1}\mathbb{C}^m \otimes \dots \otimes \Lambda^{k_n}\mathbb{C}^m$ sl_m reps

$B_n \circlearrowright V(\omega_{k_1}) \otimes \dots \otimes V(\omega_{k_n})$ $U_q sl_m$ reps

$\rightsquigarrow$ leads to knot invariants

Geometric Schur

$$H_*(Gr(k_1, \dots k_n)) \simeq \Lambda^{k_1}\mathbb{C}^m \otimes \dots \otimes \Lambda^{k_n}\mathbb{C}^m$$
$$\parallel$$
$$K_0(DCoh(Gr(k_1, \dots k_n)))$$

$q \longleftrightarrow \mathbb{C}^*$ acting on underlying $\mathbb{P}^1$ (ie on z)

<u>Seidel - Thomas</u> X smooth, projective variety
An object $\mathcal{E} \in DCoh(X)$ is <u>spherical</u> if
$\mathrm{Ext}^\bullet(\mathcal{E},\mathcal{E}) \cong H^*(S^d)$ $d = \dim X$
(as algebras)
& $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$.

If $\mathcal{E}$ is spherical $\Rightarrow$ $T_\mathcal{E} : DCoh(X) \circlearrowleft$
$A \longmapsto \mathrm{Cone}(\mathrm{Ext}^\bullet(\mathcal{E}, A) \otimes \mathcal{E} \longrightarrow A)$
is an autoequivalence
"reflection in the object $\mathcal{E}$"

<u>e.g.</u> X surface, $C \subset X$ (-2)-curve
$\Rightarrow$ $\mathcal{O}_C$ is a spherical object
e.g. $\widetilde{\mathbb{C}^2/\Gamma}$ ALE space has a bunch of exceptional $\mathbb{P}^1$'s $\rightsquigarrow$ autoequivalences,
A_n case get braid group action!

Spherical functors (Anno-Bezrukavnikov, Horja, Rouquier):

think of $\mathcal{E} \in DCoh(X)$ as a functor

$$D(\mathrm{Vect}) \longrightarrow DCoh\, X$$
$$V \longmapsto \mathcal{E} \otimes V.$$

$DCoh\, Y \xrightarrow{F} DCoh\, X$ functor is spherical

under some conditions $\rightsquigarrow$ define

twist in F, $T_F : DCoh(X) \circlearrowleft$

using right adjoint F_R of F:

$$T_F(A) = \mathrm{Cone}(FF_R(A) \longrightarrow A)$$

Khovanov-Thomas $\Rightarrow$ action of B_n

on $DCoh\, T^* Fl_{\mathbb{C}^n}$.

$$Gr(k_1, \dots k_n) = \Big\{ \mathbb{C}[z]^m = L_0 \subset L_1 \subset L_2 \subset \dots \subset L_n \subset \mathbb{C}[z, z^{-1}]^m$$
$$\dim L_{i+1}/L_i = k_{i+1}$$
$$\& \quad z L_{i+1} \subset L_i \Big\}$$

Let $Z^i(k_1,\dots,k_n) = \{(L_\bullet, L'_\bullet) : L_j = L_j' \text{ for } j \neq i\}$

ie
$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & L_i \arrow[dr, no head] & & & \\
L_0 \arrow[r, no head] & L_1 \arrow[r, no head] & \cdots L_{i-1} \arrow[ur, no head] \arrow[dr, no head] & & L_{i+1} \arrow[r, no head] & \cdots \arrow[r, no head] & L_n \\
 & & & L_i' \arrow[ur, no head] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
Z^i \arrow[r, "\pi_1"] \arrow[r, swap, shift right=2, "\pi_2"] & Gr \quad Gr'
\end{tikzcd}$$

Here $\sigma_i = \pi_{2*}\pi_1^* : DCoh(Gr(k)) \to DCoh(G(s_i k))$
is an equivalence

Theorem (S. Cautis - Kamnitzer)
Assume all $k_i \in \{1, m-1\}$
Then
1. σ_i is a derived equivalence
2. $\{\sigma_i\}$ satisfy braid relation
3. $\rightsquigarrow$ knot invariant
4. For $m=2$ get Khovanov homology

Example Consider $Gr(1,1)$

i.e. $L_0 \underset{1}{—} L_1 \underset{1}{—} L_2$

$Z'(1,1)$:
$$\begin{tikzcd} & L_1 \arrow[dr, "1"] & \\ L_0 \arrow[ur, "1"] \arrow[dr, "1"'] & & L_2 \\ & L_1' \arrow[ur, "1"'] & \end{tikzcd}$$

$\Rightarrow$ 2 components:

$Z'(1,1) = \Delta \cup V$

$\{L_1 = L_1'\}$ $\{zL_2 \subset L_0\}$

$$0 \to \mathcal{O}_\Delta(-V) \to \mathcal{O}_{Z'(1,1)} \to Q \to 0$$

on $Gr(1,1) \times Gr(1,1)$

[$\Delta \cap V$ is a divisor in Δ]

$\sigma_1(\Lambda) = \mathrm{Cone}\left(p_{2*}(p_1^*(\ ) \otimes Q[-1]) \to (\_ \otimes \mathcal{O}(-V))\right)$

$\Rightarrow$ σ_1 is a spherical functor

Whenever $k_i = k_{i+1} \in \{1, n-1\} \Rightarrow \sigma_i$ is spherical

$Gr(1, m-1) \supset T^* \mathbb{P}^{m-1}$:

$\{V_0 \hookrightarrow V_1 \hookrightarrow V_2\} \supset \{V_2 = \mathcal{O}(1)^{\oplus m}\}$

$$\begin{tikzcd} & Z'(1,m-1)_0 \arrow[dl] \arrow[dr] & \\ T^*\mathbb{P}^{m-1} & & T^*G(m-1, \mathbb{C}^m) \end{tikzcd}$$

Mukai Flop

Kawamata: $\pi_{2*}\pi_1^*$ is an equivalence

Same proof gives equivalence

Now suppose k: general

e.g.

$$\begin{tikzcd} & Z'(2,2) \arrow[dl] \arrow[dr] & \\ G(2,2) & & G(2,2) \end{tikzcd}$$

$m=4$

Open subset:

$$\begin{tikzcd} & \{X, W_1, W_2\} \arrow[dl] \arrow[dr] & \\ T^*G(2,\mathbb{C}^4) & & T^*G(2,\mathbb{C}^4) \\ X \quad W_1 & & X \quad W_2 \end{tikzcd}$$

have three components

__Theorem__ (Namikawa) $T_{2*}T_1^*$ is not an equivalence
(save for completion).

Kawamata: modify to make an equivalence:
introduce sheaf supported on Z
s.t. corresponding functor is an equivalence.

How to produce an equivalence, following
Chuang – Rouquier:
$\mathfrak{sl}_2$ __categorification__

$$\bigoplus_{k+l=M} K(Gr(k,l)) = \Lambda^N(\mathbb{C}^M \otimes \mathbb{C}^2) \circlearrowleft S_m \times SL_2$$

& this is the SL_2 weight decomposition.

$$\Lambda^k \mathbb{C}^m \otimes \Lambda^l \mathbb{C}^m \to \Lambda^l \mathbb{C}^m \otimes \Lambda^k \mathbb{C}^m$$

is achieved by $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2$

... continues to hold on quantum level

__Idea__: make "SL_2" act on $\bigoplus_{k+l=N} DCoh(Gr(k,l))$

- geometric Skew Howe duality (ideas of Mirković - Vybornov, Braverman - Gaitsgory)

Def (Chuang - Rouquier) An action of sl_2 on an abelian category A is a pair of adjoint exact functors $E, F : \mathcal{C} \circlearrowleft$ along with $X \in \mathrm{End}\, E$, $T \in \mathrm{End}\, E^2$ s.t.

i) E, F generate $sl_2 \curvearrowright K_0(A)$

ii) $X_i := 1 1 \cdots 1 X 1 1 1 \in \mathrm{End}\, E^n$

$t_i = 1 \cdots 1 T 1 \cdots 1 \in \mathrm{End}\, E^{2n}$

generate an affine Hecke algebra condition

Theorem (Chuang-Rouquier) For A a category with sl_2 action, $A = \bigoplus_{\mathbb{Z}} A_i$ weight decomp.

$\Rightarrow \exists$ equivalence $T : D(A_i) \longrightarrow D(A_{-i})$ categorifying action of s on $K_0(A)$.

-- build s using E, F above.

E, F in our situation:

$$\begin{tikzcd}[column sep=tiny] & \{ L_0 \overset{k}{\subset} L_1 \overset{1}{\subset} L_1' \overset{l-1}{\subset} L_2 = L_2' \} = W(k,l) \arrow[dl] \arrow[dr] & \\ Gr(k,l) & & G(k+1,l-1) \end{tikzcd}$$

only give E, F on derived category

<u>Hope</u>: The complex constructed using E, F following Chuang-Rouquier is an equivalence.