

# Joel Kamnitzer - Braid group actions on certain derived categories

Note Title

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Fix  $m \geq 2$ .  $V$  rk  $m$  vector bundle on  $\mathbb{P}^1$ .

A Hecke modification of  $V$  at  $x \in \mathbb{P}^1$  of weight  $k$  is a pair  $(V', \varphi)$

$V'$ : rk  $m$  vector bundle

$\varphi: V \hookrightarrow V'$ , cokernel  $\varphi \cong \mathcal{O}_x^{\oplus k}$

$Gr_x(k) = \left\{ \begin{array}{l} \text{Hecke modifications of} \\ \text{the trivial bundle} \end{array} \right\} \cong G(k, \mathbb{C}^m)$   
Grassmannian  
of  $k$ -planes in  $\mathbb{C}^m$

$Gr_x(k_1, \dots, k_n) = \left\{ V_0 \xrightarrow{k_1} V_1 \xrightarrow{k_2} V_2 \dots \xrightarrow{k_n} V_n \right\}$   
trivial all occurring at  $x$

iterated bundle of Grassmannians

$1 \leq k_i \leq m-1$

Problem: Define an action of  $B_n$  on  
 $D\text{Coh}(Gr(k_1, \dots, k_n))$  [weak action]

more precisely  $B_n \rightarrow S_n$  will permute the  $k_i$ 's  
 $\hookrightarrow B_n$  will act by functors between the  
 corresponding categories

Motivation (from Khovanov knot homology)

$$S_n \subset \Lambda^{k_1} \mathbb{C}^m \otimes \dots \otimes \Lambda^{k_n} \mathbb{C}^n \quad \text{slm reps}$$

$$B_n \subset V(\omega_{k_1}) \otimes \dots \otimes V(\omega_{k_n}) \quad U_q \text{ slm reps}$$

$\rightsquigarrow$  leads to knot invariants

Geometric Satake

$$H_x(\text{Gr}(k_1, \dots, k_n)) \simeq \Lambda^{k_1} \mathbb{C}^m \otimes \dots \otimes \Lambda^{k_n} \mathbb{C}^n$$

$$\parallel$$

$$K_0(\text{Dcoh}(\text{Gr}(k_1, \dots, k_n)))$$

$$q \iff \mathbb{C}^* \text{ acting on underlying } \mathbb{P}^1$$

(ie on  $z$ )

Seidel-Thomas  $X$  smooth, projective variety

An object  $E \in D^b(X)$  is spherical if

$$\text{Ext}^i(E, E) \cong H^i(S^d) \quad d = \dim X$$

(as algebras)

$$\& E \otimes \omega_X \cong E.$$

If  $E$  is spherical  $\Rightarrow T_E: D^b(X) \ni$

$$A \mapsto (\text{cone}(\text{Ext}^1(E, A) \otimes E \rightarrow A))$$

is an autoequivalence

"reflection in the object  $E$ "

e.g.  $X$  surface,  $C \subset X$   $(-2)$ -curve

$\Rightarrow \mathcal{O}_C$  is a spherical object

e.g.  $\widetilde{\mathbb{C}^2/\Gamma}$  ALE space has a bunch of exceptional  $\mathbb{P}^1$ 's  $\rightsquigarrow$  autoequivalences,  
An case get braid group action!

Spherical functors (Amo-Bezrukavnikov, Hrbja, Ragnur):

think of  $\mathcal{E} \in \text{Dcoh}(X)$  as a functor

$$\begin{array}{ccc} \text{D}(\text{Vect}) & \longrightarrow & \text{Dcoh } X \\ V & \longmapsto & \mathcal{E} \otimes V \end{array}$$

$\text{Dcoh } Y \xrightarrow{F} \text{Dcoh } X$  functor is spherical!

under some conditions  $\rightsquigarrow$  define twist in  $F$ ,  $T_F: \text{Dcoh}(X) \hookrightarrow$

using right adjoint  $F_R$  of  $F$ :

$$T_F(A) = \text{Cone}(FF_R(A) \rightarrow A)$$

Khovanov-Thomas  $\Rightarrow$  action of  $B_n$  on  $\text{Dcoh } T^*F\ell_n$ .

$$\text{Gr}(k_1, \dots, k_n) = \left\{ \begin{array}{l} \mathbb{C}[z] \supset L_0 \subset L_1 \subset L_2 \subset \dots \subset L_n \subset \mathbb{C}[z] \\ \dim L_{i+1}/L_i = k_{i+1} \\ \Delta \ni zL_{i+1} \subset L_i \end{array} \right\}$$

$$\text{Let } Z^i(k_1, \dots, k_n) = \{(L_0, L'_0) : L_j = L'_j \text{ for } j \neq i\}$$

ie

$$L_0 - L_1 - \dots - L_{i-1} \begin{array}{c} \xrightarrow{L_i} \\ \searrow L'_i \end{array} L_{i+1} - \dots - L_n$$

$$Z^i \begin{array}{c} \xrightarrow{\pi_1} Gr \\ \xrightarrow{\pi_2} Gr' \end{array}$$

Here  $\sigma_i = \pi_{2*} \pi_1^* : D(G_i(k_i)) \rightarrow D(G_i(k_i))$   
is an equivalence

Theorem (S. Gaudin - Kontsevich)

Assume all  $k_i \in \{1, m-1\}$

Then 1.  $\sigma_i$  is a derived equivalence

2.  $\{\sigma_i\}$  satisfy braid relations

3.  $\Rightarrow$  knot invariant

4. For  $m=2$  get Khovanov homology

Example Consider  $G(1,1)$

$$L_0 \text{ --- } L_1 \text{ --- } L_2$$

$$Z'(1,1): \begin{array}{ccc} L_0 & \text{---} & L_1 & \text{---} & L_2 \\ & & \nearrow & & \searrow \\ & & L_1' & & \\ & & \nwarrow & & \nearrow \end{array}$$

$\Rightarrow$  2 components:

$$Z'(1,1) = \Delta \cup V$$

$$\{L_1 = L_1'\} \quad \{ZL_2 \subset L_0\}$$

$$0 \rightarrow \mathcal{O}_\Delta(-V) \rightarrow \mathcal{O}_{Z'(1,1)} \rightarrow \mathcal{O}_V \rightarrow 0$$

on  $G(1,1) \times G(1,1)$

[ $\Delta \cap V$  is a cluster in  $\Delta$ ]

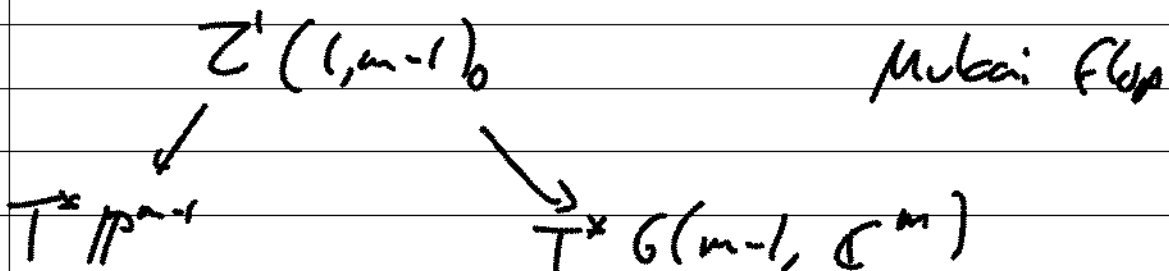
$$\sigma_i(\Lambda) = \text{core} \left( p_{2*} \left( p_1^*(\Lambda) \otimes \mathcal{O}_V(-1) \right) \right) \rightarrow \Lambda \otimes \mathcal{O}_V(-1)$$

$\Rightarrow \sigma_i$  is a spherical functor

Whenever  $k_i = k_{i+1} \in \{1, n-1\} \Rightarrow \sigma_i$  is spherical

$$Gr(1, m-1) \rightarrow T^*P^{m-1} :$$

$$\{V_0 \subset V_1 \subset V_2\} \rightarrow \{V_2 \cong \mathcal{O}(1)^{\oplus m}\}$$

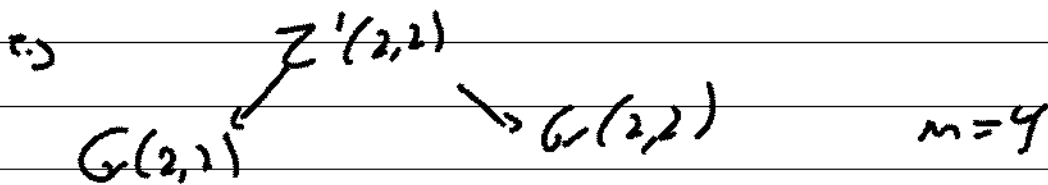


Kawamata:  $\pi_2 \circ \pi_1^*$  is an equivalence

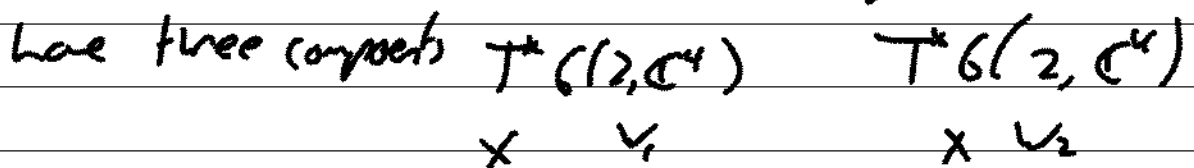
See proof sites equivalence

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Now suppose  $k$ : general



Open subset:  $\{x, v_1, w_2\}$



Theorem (Namikawa)  $\mathbb{T}_2 \times \mathbb{T}_1^*$  is not an equivalence  
(same for completion).

Kawamata: modify to make an equivalence:  
introduce stack supported on  $\mathbb{Z}$   
s.t. corresponding functor is an equivalence.

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How to produce an equivalence, following  
Chang - Raninger:

$SL_2$  categorification

$$\bigoplus_{k+l=M} K(G(k,l)) = \bigwedge^M (\mathbb{C}^M \oplus \mathbb{C}^2)$$

$\bigcup$   
 $S_n = SL_2$

$\Delta$  this is the  $SL_2$  weight decomposition.

$$\bigwedge^k \mathbb{C}^m \otimes \bigwedge^l \mathbb{C}^m \rightarrow \bigwedge^{k+l} \mathbb{C}^m \otimes \bigwedge^k \mathbb{C}^m$$

is achieved by  $sg \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2$

... continues to hold on quantum level

Idea: make " $SL_2$ " act on  $\bigoplus_{k+l=N} D_{\text{coh}}(G(k,l))$



- geometric Satake duality (idea of  
Mirkovic - Vukobratovic, Bruinier - Gaiotto)

Def (Chang-Ragusa) An action of  $sl_2$  on  
an abelian category  $\mathcal{A}$  is a pair of  
adjoint exact functors  $E, F: \mathcal{A} \rightarrow \mathcal{A}$   
along with  $X \in \text{End } E$ ,  $T \in \text{End } F^2$  s.t.

i)  $E, F$  generate  $sl_2 \subset K_0(\mathcal{A})$

ii)  $x := (1 \dots 1 \mid X \mid 1) \in \text{End } E^n$

$t := (1 \dots 1 \mid T \mid 1) \in \text{End } F^{2n}$

generate an affine Hecke algebra condition

Theorem (Chang-Ragusa) For  $\mathcal{A}$  a category  
with  $sl_2$  action,  $A := \bigoplus_{\mathbb{Z}} A_i$  weight decomp.

$\Rightarrow \exists$  equivalence  $T: D(\mathcal{A}) \rightarrow D(A_{-i})$   
categorifying action of  $S$  on  $K_0(\mathcal{A})$ .

- built  $S$  using  $E, F$  above.

$E, F$  is an algebra:

$$\{L_0 \xrightarrow{k} L_1 \xrightarrow{l} L_1' \xrightarrow{l+l'} L_2 = L_2'\} = W(k, l)$$

$$\swarrow$$
$$Gr(k, l)$$

$$\searrow$$
$$Gr(k+1, l-1)$$

only give  $E, F$  on closed algebra

Hope: The complex constructed using  $E, F$

following Chang-Rouquier is an equivalence.