B. Keller - Cluster algebras & Quiver representations

Note Title 3/25/2008

History

Can. bases (Lusztig 1990)

r. Geiss-Leclerc-Schröer

Total positivity (Lusztig)

fd reps of quantum affine algebras (Leclerc)

Cluster algebras Fomin-Zelevinsky 2000

quivers & fd algebras ..categorification

Poisson geometry (Gekhtman-Shapiro-Vainshtein)

Combinatorial polyhedra (F-Z, Chapoton)

Stability conditions (Kontsevich-Soibelman, Chuang-Rouquier)

Higher Teichmüller theory (Fock-Goncharov)

integrable systems (F.-Z.)

(original goal: find combinatorial description of canonical bases..)

Plan:

1. Cluster-finite cluster algebras
2. categorification
3. acyclic cluster algebras

1. Description of cluster algebras:
commutative $\mathbb{Q}$-algebra endowed with a set of distinguished generators (<u>cluster variables</u>), grouped into (overlapping) subsets of fixed finite cardinality (= <u>clusters</u>) constructed inductively via <u>mutation</u>

Classification theorem (F-Z):
Cluster algebras with finitely many cluster variables (=: cluster-finite cluster algebras) are parametrized by the finite root systems

Rk: In particular for each simply-laced Dynkin diagram $\Delta \Longrightarrow$ canonical cluster algebra

$$\mathcal{A}_\Delta \underset{\text{subalgebra}}{\subset} \mathbb{Q}(x_1, \dots x_n) \qquad n = \text{rank of root system}$$

For these algebras, we can directly construct the cluster variables using the <u>knitting algorithm</u>

<u>Examples</u> 1.) $\Delta = A_2 = \bullet\!\!-\!\!\bullet$
Choose orientation: $\vec{A_2} = 1 \longrightarrow 2$

$\mathbb{Z}\vec{A}_2$: $\cdots\cdots \nearrow \searrow \underset{0}{\nearrow} \searrow \underset{1}{\nearrow} \searrow \underset{2}{\nearrow} \searrow \cdots \cdots$

(repetition of $\vec{A}_1$)

add new arrows $\searrow$ to $\mathbb{Z}$ copies of our old diagram

The cluster variables $\longleftrightarrow$ vertices of the repetition, starting from 0^{th} copy

$$\begin{tikzcd}[column sep=small]
& x_2 \arrow[dr] & & x_2' \arrow[dr] & & x_2''=x_1 \arrow[dr] & & \cdots \\
x_1 \arrow[ur] & & x_1' \arrow[ur] & & x_1'' \arrow[ur] & & x_1'''=x_2 &
\end{tikzcd}$$

$x_1' = \dfrac{1+x_2}{x_1}$ simple mutation $\qquad \dfrac{1+\text{predecessor}}{\text{left translate}}$

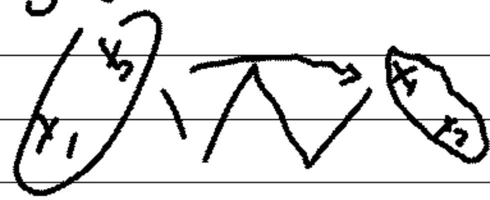
$$x_2' = \frac{1+x_1'}{x_2} = \frac{x_1+1+x_2}{x_1x_2}$$

$$x_1'' = \frac{1+x_2'}{x_1'} = \frac{x_1x_2+x_1+1+x_2}{x_1x_2} \Big/ \frac{1+x_2}{x_1}$$

$$\overset{!!}{=} \frac{1+x_1}{x_2}$$: denominator remains a monomial

$$x_2'' = \frac{1+x_1''}{x_2'} = \frac{x_2+1+x_1}{x_2} \Big/ \frac{x_1+1+x_2}{x_1 x_2} = x_1$$

Get period 5 sequence : periodicity is a glide-reflection



5 cluster variables $x_1 \quad x_2 \quad x_1' \quad x_2' \quad x_1''$

Δ $\mathcal{A}_{A_2}$ = $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, x_2)$ generated by these 5.

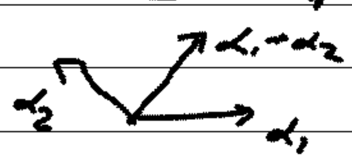
<u>Phenomena</u>:

1. All denominators of cluster variables are monomials (Laurent phenomenon — FZ for <u>all</u> cluster algebras)

2. Periodicity : characterizes Dynkin diagrams

3. Numerology $5 = 2 + 3 = \mathrm{rk}\,\Delta + \#(\text{pos roots})$:

$\underbrace{x_1 \quad x_2}_{\text{trivial ones: rk }\Delta} \qquad \underbrace{x_1' \quad x_2' \quad x_1''}_{\text{pos roots}}$

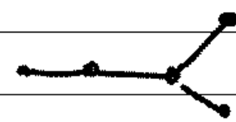


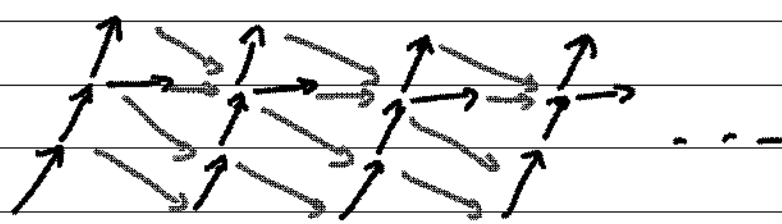
$$x_1' = \frac{\cdots}{x_1} \qquad x_2' = \frac{\cdots}{x_1 x_2} \qquad x_1'' = \frac{\cdots}{x_2}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\alpha_1 \qquad\qquad \alpha_1 + \alpha_2 \qquad\qquad \alpha_2 \quad :$$

read off corresponding positive roots from the denominators..

2) $D_5 =$  $\qquad \vec{D_5} = 1 \to 2 \to 3 \begin{smallmatrix} \nearrow 5 \\ \searrow 4 \end{smallmatrix}$

$\mathbb{Z}\vec{D_5}$: 

$$\begin{array}{cccc}
 & x_5 & & x \\
 & x_3 \quad x_4 & & x_3' \\
 x_2 & & x_2' & \\
 x & & x_1' &
\end{array}$$

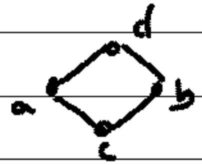
$$x_1' = \frac{1+x_2}{x_1} \qquad x_2' = \frac{1+x_3 x_1'}{x_2} \qquad x_3' = \frac{1+x_2' x_4 x_5}{x_3}$$

Find $25 = 5+20$ cluster variables

Coefficients in numerator are all positive integers
(conjectured for non-Dynkin quivers ...)

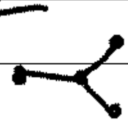
Periodicity here is just a translation + switch of branches

A_n: always a glide reflection, quotient by this symmetry is a Möbius strip.



$a \cdot b = 1 + cd$ recurrence.

2. <u>Categorification</u> Δ a simply laced Dynkin diagram

e.g. , Q a quiver with underlying graph Δ, e.g.

$$\begin{tikzcd}[row sep=small] & & 4 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 3 \end{tikzcd}$$

k an alg. closed field

Representation of Q = diagram of f.d. vector spaces of shape Q, e.g.

$$\begin{tikzcd}[row sep=small] & & V_4 \\ V_1 \arrow[r] & V_2 \arrow[ur] \arrow[dr] & \\ & & V_3 \end{tikzcd}$$

rep(Q) = category of representations

D_Q = bounded derived category of rep (Q^{op})

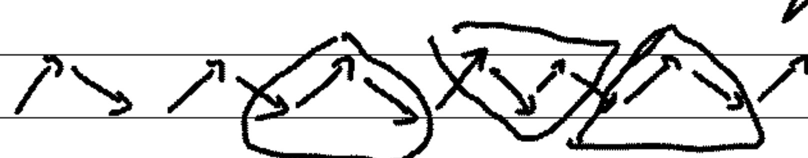
Remark: D_Q is abelian iff Q does not have arrows, but is always triangulated - k-linear & endowed with $\Sigma : V \to V[1]$ & triangles $U \to V \to W \to \Sigma U$

Theorem (Happel, 1986)

(a) We have a natural bijection

$$\left\{ \begin{matrix} \text{indecomp. objects} \\ \text{of } D_Q \end{matrix} \right\} / \text{isom} \longleftrightarrow \left\{ \begin{matrix} \text{vertices of} \\ \mathbb{Z}Q \end{matrix} \right\}$$

... repetition is the Auslander-Reiten quiver of D_Q.



positive roots appearing

(b) D_Q has a Serre functor $S : D_Q \xrightarrow{\sim} D_Q$

s.t. $\mathrm{Hom}(X,Y)^* \xrightarrow{\sim} \mathrm{Hom}(Y, SX) \quad \forall X, Y$

— this lifts periodicity automorphism

(Buan-Marsh-Reineke-Reiten-Todorov)

$\mathcal{C}_Q$ (cluster category) := orbit category of S on D_Q. $D_Q/(S^{-1}\circ\Sigma^2)^{\mathbb{Z}}$

Objects - same as for D_Q

Morphisms: $\mathrm{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{D_Q}(X,(S^{-1}\circ\Sigma^2)^p Y)$

Rks 1. $\mathcal{C}_Q$ also defined by Caldero-Chapoton-Schiffler in type A.

2. $\mathcal{C}_Q \simeq \mathrm{Perf}\,\Gamma/D^b\Gamma$ 2-CY category

where Γ = Ginzburg dga associated to Q with potential $W=0$.

$\Rightarrow$ $\mathcal{C}_Q$ triangulated

- universal 2CY category under D_Q.

Theorem (BMRRT) We have a canonical bijection

$M \in \left\{ \begin{matrix} \text{indecomp objects} \\ \text{of } \mathcal{C}_Q \end{matrix} \right\}_{/\sim} \longleftrightarrow \{\text{cluster variables}\} \ni X_M$

Q: What about the exchange relations?

A: come from triangles in the cluster category

<u>Def</u> For $L, M, N \in \mathcal{C}_Q$, let $\mathrm{Ext}^1(L,M) = \mathrm{Hom}_{\mathcal{C}_Q}(L, \Sigma M)$

$$\mathrm{Ext}^1(L,M)_N = \{ \varepsilon : L \to \Sigma M : \exists \text{ triangle } M \to N \to L \xrightarrow{\varepsilon} \Sigma M \}$$

Constructible subset of $\mathrm{Ext}^1(L,M)$.

<u>Def</u> $X_L := \prod_{i=1}^{s} X_{L_i}$ when $L = L_1 \oplus \dots \oplus L_s$, L_i indecomposable

<u>Theorem</u> (Caldero-Keller)

$L, M \in \mathcal{C}_Q$ s.t. $\mathrm{Ext}^1(L,M) \neq 0 \Rightarrow$

$$X_L X_M = \sum_N \frac{\chi(\mathbb{P}\mathrm{Ext}^1(L,M)_N) + \chi(\mathbb{P}\mathrm{Ext}^1(M,L))_N}{\dim \mathrm{Ext}^1(L,M)} X_N$$

(Euler chars. of projectivizations of Ext spaces)

Remarks 1. $\mathcal{A}_Q$ looks like a dual Ringel-Hall algebra associated with the triangulated category $\mathcal{C}_Q$.

2. Ingredients of proof:
- explicit formula for X_M (Caldero-Chapoton) in terms of Grassmannians of submodules of M
- 2CY property of $\mathcal{C}_Q$.

3. Generalization to acyclic cluster algebras

Fix $n \geq 1$

Def A **seed** is a pair (R, u) where
a) R is a finite quiver w/o loops or 2-cycles & vertex set $\{1, \dots, n\}$

b) $u = \{u_1, \dots u_n\}$ is a free generating set for $\mathbb{Q}(x_1, \dots x_n)$

Fix a vertex k. Then mutation $\mu_k(R,u)$ is the seed (R', u') obtained as follows:

a) $R' = \mu_k(R)$ is obtaind in three steps:

1. For each subquiver $j \xrightarrow{b} k \xrightarrow{a} l$ add a new arrow

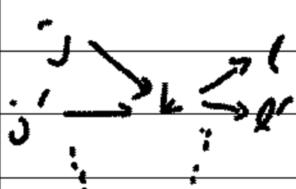
$$\begin{tikzcd} j \arrow[r, "b"] \arrow[rr, bend right, "{[ab]}"'] & k \arrow[r, "a"] & l \end{tikzcd}$$

2. Reverse all arrows incident with k
3. Remove a maximal set of 2-cycles

   eg $\overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \rightsquigarrow \longrightarrow$

b) $u' = u \setminus \{u_k\} \cup \{u_k'\}$ where

$$u_k' = \frac{1}{u_k}\Big(\prod_{\substack{\text{arrows}\\ j \to k}} u_j + \prod_{\substack{\text{arrows}\\ k \to l}} u_l\Big)$$



Fix a quiver Q as above

<u>Def</u> The initial seed $(Q,x) = (Q, \{x_1, \dots x_n\})$

- clusters wrt Q are all u's appearing by mutating (Q,x) iteratively

$\rightsquigarrow$ cluster algebra $\mathcal{A}_Q$ = $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by the cluster variables.

Exchange graph: vertices $\longleftrightarrow$ seeds up to renumbering
edges $\longleftrightarrow$ mutations

<u>Example</u> $Q = \vec{A_3} = (1 \to 2 \to 3)$

$$\begin{array}{ccc} 1 \longrightarrow 2 \longrightarrow 3 \\ x_1 \quad x_2 \quad x_3 \\ (1) \end{array}$$

μ_2:

$$\begin{tikzcd} 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[ll, bend right] \end{tikzcd} \quad (2)$$

$$x_1 \qquad \frac{x_1 + x_3}{x_2} \qquad x_3$$

μ_1:

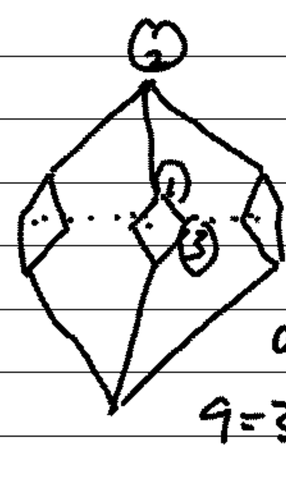
$$1 \longleftarrow 2 \longrightarrow 3 \quad (3)$$

$$\frac{1 + x_2}{x_1} \qquad x_2 \qquad x_3$$

μ_1 ... μ_3 ...

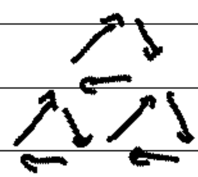
... ...

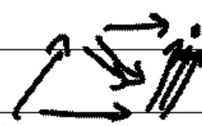
Exchange graph:



3rd Stasheff associahedron
14 vertices $\longleftrightarrow$ 14 clusters
9 faces $\longleftrightarrow$
9 = 3 + 6 cluster variables

Theorem (FZ) Q connected, no loops or 2-cycles
$\mathcal{A}_Q$ has fin. many cluster variables iff
Q is mutation equivalent to an orientation
of a Dynkin quiver.
In this case Δ is unique $=$ cluster type of Q
& # cluster variables $=$ rk Δ + # pos roots

e.g.  $\underset{\text{mutation}}{\sim}$ $\vec{D}_6$

Categorification Q a finite quiver w/o oriented
cycles e.g. 

k alg. closed field
D_Q bounded derived category of rep(Q^{op})

$\mathcal{C}_Q$ = cluster category = orbit category under $\Sigma^{-2}\circ S$

Def $L\in\mathcal{C}_Q$ is *rigid* if $\mathrm{Ext}^1(L,L)=0$

Theorem (Caldero-Keller) a) We have an explicit bijection $\left\{\begin{matrix}\text{indecomp.} \\ \text{rigid obj of } \mathcal{C}_Q\end{matrix}\right\}/\simeq \xrightarrow{\sim} \left\{\begin{matrix}\text{cluster variables} \\ \text{of } \mathcal{A}_Q\end{matrix}\right\}$

b. The clusters correspond exactly to the cluster-tilting sets $\{T_1, \dots T_n\}$:

$\mathrm{Ext}^1(T_i, T_j) = 0 \ \forall i, j$

c. (Xiao-Xu) The multiplication formula holds if we replace $\sum_N$ by $\int_N d\chi$ - integration against Euler characteristic.

Adjacency of clusters:

$\{T_1, \dots T_n\}$ — $\{T_1^*, T_2, \dots T_n\}$

with $\dim \mathrm{Ext}^1(T_1, T_1^*) = 1$:

$\exists$ exactly 2 indecomposables T_1, T_1^* completing a given $\{T_2, \dots T_n\}$

Consequence (Caldero-Reineke)

Q without oriented cycles $\Rightarrow$ all cluster variables $\in \mathbb{N}[x_1^{\pm 1}, \dots x_n^{\pm 1}]$

Exchange relations: suppose $\mathrm{Ext}^1(T_1, T_1^*) = k$

$\Rightarrow \exists! /\sim \quad T_1^* \to B \to T_1 \to \Sigma T_1^*$

but we have a CY2 so $\mathrm{Ext}^1(T_1^*, T_1) = k$

$\Rightarrow \exists! /\sim \quad T_1 \to B' \to T^* \to \Sigma T_1$

Relation: $X_{T_1} X_{T_1'} = X_B + X_{B'}$

-- B may be decomposable $B = \oplus B_i$, $B' = \oplus B_i'$
rigid indecomposables

$\Rightarrow X_{T_1} X_{T_1'} = \prod X_{B_i} + \prod X_{B_i'}$

For $M \in \mathrm{rep}\, Q^{op} \to \mathcal{C}_Q$ rigid & indecomposable

denominator $(X_M) = \prod x^{d_i}$

$(d_1, \dots d_n) = \underline{\dim}\ M.$: $\mathrm{denom}(X_1) = \frac{1}{x_1} \Leftrightarrow$ negative simple roots